CS and SWF ENG 4-6TE3: Final

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THIS EXAMINATION PAPER INCLUDES **2** PAGES AND **4** QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.
DURATION OF THE EXAM: 3 hours

Special instructions:
You might use a letter-size paper, both sides, with your hand-written notes.
The use of the standard McMaster (Casio FX-991) calculatoris allowed.

Questions:

1. **Concepts, duality:**

   (a) What is the rate of convergence of the sequence:
   $$x^i = (\tfrac{1}{i^2}, 1 + \tfrac{1}{i}) \qquad i = 1, 2, \ldots$$

   (b) Show that if the function $f(x): \ R^n \to R$ is convex and the function $g(x): \ R \to R$ is monotone non-decreasing convex function, then the function $h(x) = g(f(x))$ is a convex function as well.

   (c) Consider the nonlinear optimization problem:
   $$\begin{array}{llll} \text{(NLO)} & \min & f(x) & \\ & \text{s.t.} & g_1(x) & \geq 0 \\ & & g_2(x) & = 0 \\ & & g_3(x) & \leq 0 \\ & & x & \in \mathcal{C}, \end{array}$$
   where the functions are defined in appropriate spaces and $\mathcal{C}$ is a set.
   Define when this problem satisfies the Slater condition.

   (d) Give the Karush-Kuhn-Tucker condition for the NLO problem presented in (c).

   (e) Let $f(x) = x^2, \ \ g_1(x) = -x^3, \ \ g_2(x) = x, \ \ g_3(x) = x, \ \ \mathcal{C} = R_{\oplus}$.
   Does this NLO problem satisfies the Slater condition?

   (f) Derive and present the dual of this specific NLO. (You can decide if you want to use the Lagrange or the Wolfe dual. Simplify your dual as much as possible.)

   14p

2. **Unconstrained Optimization:** Consider the function $f(x) = -x_1 - x_2 + \exp(x_1 + x_2)$.

   (a) Is this a convex function?

   (b) Let $\bar{x}^T = (0.5, -0.5)$. Which algorithm would be the best to minimize the function $f(x)$ starting from the given point $\bar{x}$. Give at least two arguments why would you choose that algorithm

   (c) Apply a step of the Trust region algorithm with $\alpha = 1, \ \mu = 1/4, \ \ , \eta = 3/4, \ \ \gamma_1 = 1/2, \ \ \gamma_2 = 2$ and give next iterate.

   (d) Let $x^1 = (0.5, -0.5)^T$, $x^2 = (0, 0)^T$ and $x^3 = (0.5, 0)^T$. Apply one step of the Nelder-Mead Simplex Method with $\alpha = 1/2, \ \ \beta = 3/4, \ \ \gamma = 2$.

   (e) Give the linear interpolation polynom $Q(x)$ of the function $f(x)$ through the points $x^1, \ x^2, \ x^3$.

   12p

SEE THE OTHER SIDE FOR QUESTIONS 3, AND 4!

3. Reduction, duality, : Consider the following linearly constrained non-linear optimization (NLO) problem:

$$\text{(NLO)} \qquad \min \quad x_1 + x_2 - x_1 \log|x_1| - x_2 \log|x_2| \qquad \text{s.t.} \quad x_1 + x_2 = 0.$$

a. Reduce this problem to an unconstrained optimization problem.

b. Give the Lagrange function of (NLO) and define the saddle point.

c. Give the SQP quadratic subproblem for the (NLO) problem at the point $x^1 = (2, -2)^T$ and Lagrange multiplier value $y^1 = 1$.

d. Consider the following NLO problem

$$\text{(NLO)} \qquad \min \quad -x_1 \qquad \text{s.t.} \quad \sqrt{x_1^2 + x_2^2} \leq x_2.$$

Prove that for this problem the duality gap at optimality is infinity.

10p

4. Linearly Constrained NLO: Consider the following linearly constrained non-linear optimization (NLO) problem:

$$\begin{array}{llll} \text{(NLO)} & \min & 3x_1^2 + x_2^2 + 2x_3^2 & \\ & \text{s.t.} & x_1 + x_2 + x_3 & = 1 \\ & & x_1, x_2, x_3 & \geq 0. \end{array}$$

a. Is this a convex optimization problem? Justify your answer.

b. Apply one step of the Reduced Gradient method for this problem starting from the point $x = (0, 1, 0)$ with the initial basis given by x_2.

c. Using barrier functions reformulate this linearly constrained NLO problem as a parameterized optimization problem with a linear equality constraint.

d. Using penalty functions reformulate this linearly constrained NLO problem as a parameterized unconstrained optimization problem.

14p

THE END