CS and SWF ENG 4-6TE3: Final

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THIS EXAMINATION PAPER INCLUDES **2** PAGES AND **4** QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.
DURATION OF THE EXAM: 3 hours

Special instructions:
You might use a letter-size paper, both sides, with your hand-written notes.
The use of the standard McMaster (Casio FX-991) calculatoris allowed.

Questions:

1. Concepts, duality:

(a) What is the rate of convergence of the sequence:

$$x^i = \left(\frac{\log(i)}{i^2}, 1 + \frac{i+1}{i}\right) \qquad i = 1, 2, \ldots$$

(b) Show that if the function $f(x): \; R^n \to R$ is concave and the function $g(x): \; R \to R$ is monotone non-increasing convex function, then the function $h(x) = g(f(x))$ is a concave function as well.

(c) Consider the nonlinear optimization problem:

$$\begin{array}{llll} \text{(NLO)} & \min & f(x) & \\ & \text{s.t.} & g_1(x) & \geq 0 \\ & & g_2(x) & = 0 \\ & & g_3(x) & \leq 0 \\ & & x & \in \mathcal{C}, \end{array}$$

where the functions are defined in appropriate spaces and $\mathcal{C}$ is a set.
Define when this problem has an ideal Slater point.

(d) Give the Wolfe dual of the NLO problem presented in (c). Discuss when is the Wolfe dual is a valid dual for this problem.

(e) Let $f(x) = x^2, \; g_1(x) = x - x^2, \; g_2(x) = -x, \; g_3(x) = x^2, \; \mathcal{C} = R_\oplus$.
Does this NLO problem satisfies the Slater condition? Justify your answer.

(f) Derive the KKT conditions of this specific NLO. (Simplify your dual as much as possible.)

14p

2. Unconstrained Optimization: Consider the function $f(x) = -x_1 - x_2 + \exp(x_1 + x_2) - \log(x_2^2)$.

(a) Is this a convex function?

(b) Let $\bar{x}^T = (0.5, -0.5)$. Which algorithm would be the best to minimize the function $f(x)$ starting from the given point $\bar{x}$. Give at least two arguments why would you choose that algorithm

(c) Apply two steps of the Fletcher-Reeves Conjugate Gradient algorithm. (Use an approximate line-search.)

(d) Let $x^1 = (0.5, -0.5)^T$, $x^2 = (0, 0)^T$ and $x^3 = (0.5, 0)^T$. Apply one step of the Nelder-Mead Simplex Method with $\alpha = 1/2, \; \beta = 3/4, \; \gamma = 2$.

(e) Give the interpolation polynom $Q(x)$ of the function $f(x)$ through the points $x^1, \; x^2, \; x^3$ by using the basis functions $\phi_1(x) = 1, \; \phi_2(x) = x_1 x_2$ and $\phi_3(x) = x_1 + x_2$.

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SEE THE OTHER SIDE FOR QUESTIONS 3, AND 4!

3. Reduction, duality, : Consider the following linearly constrained non-linear optimization (NLO) problem:

$$\text{(NLO)} \qquad \begin{array}{ll} \min & x_1 + x_2 + (x_1 - x_2)^4 \\ \text{s.t.} & x_1 + x_2 = 0. \end{array}$$

a. Reduce this problem to an unconstrained optimization problem.

b. Give the Lagrange function of (NLO) and define the saddle point.

c. Give the SQP quadratic subproblem for the (NLO) problem at the point $x^1 = (1,1)^T$ and Lagrange multiplier value $y^1 = 1$.

d. Consider the following NLO problem

$$\text{(NLO)} \qquad \begin{array}{ll} \min & -x_2 \\ \text{s.t.} & \sqrt{x_1^2 + x_2^2} \leq x_1. \end{array}$$

Prove that for this problem the duality gap at optimality is infinity.

10p

4. Linearly Constrained NLO: Consider the following constrained non-linear optimization (NLO) problem:

$$\text{(NLO)} \qquad \begin{array}{lll} \min & 3x_1^2 + x_2^2 - \log(x_1 + x_2 + x_3) & \\ \text{s.t.} & x_1 + x_2^2 + x_3 & = 1 \\ & x_1, x_2, x_3 & \geq 0. \end{array}$$

a. Is this a convex optimization problem? Justify your answer.

b. Apply one step of the Generalized Reduced Gradient method for this problem starting from the point $x = (0, 1, 0)$ with the initial basis given by x_2.

c. Using an inverse barrier function and an exact penalty function reformulate constrained NLO problem as a parameterized optimization problem.

d. What are the optimality conditions for the following primal log-barrier linear optimization problem:

$$\begin{array}{rcl} \min \; c^T x - & \mu \displaystyle\sum_{j=1}^{n} \log x_i & \\ Ax & = & b \\ x & > & 0 \end{array}$$

14p

THE END