CS and SWF ENG 4-6TE3: Final

Tamás Terlaky

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THIS EXAMINATION PAPER INCLUDES **2** PAGES AND **5** QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

DURATION OF THE EXAM: 3 hours

Special instructions:
You might use a letter-size paper, both sides, with your hand-written notes.
The use of the standard McMaster (Casio FX-991) calculator is allowed.

Questions:

1. Concepts, convexity, duality:

- **a.** What is the rate of convergence of the following sequence? Justify your answer.
$$x^i = \frac{1}{\log(i+1)} \qquad i = 1, 2, \ldots$$
- **b.** Show that the function $f(x) : R^2 \to R$ given by
$$f(x) := e^{x_1^4 + x_2^2 - 2x_1 - x_2 - 1}$$
is a convex function.
- **c.** Consider the nonlinear optimization problem:
$$\begin{array}{llll} \text{(NLO)} & \min & f(x) & \\ & \text{s.t.} & g_j(x) & \leq 0, \quad j = 1, \ldots, k, \\ & & Ax & = b, \\ & & x & \in R^n, \end{array}$$
where the functions $R^n \to R$ are convex, $A : m \times n$ is a matrix and $b \in R^m$.
Define when a point $x^0 \in R^n$ is a Slater point of this problem.
- **d.** Describe how one can verify if problem (NLO) satisfies the Slater condition. Justify your procedure.
- **e.** Give the Lagrange Function and the Wolfe Dual for this problem.
- **f.** Derive the KKT conditions of the given (NLO) problem.

12p

2. Unconstrained Optimization: Consider the function
$$f(x) = x_1 + x_2 - \frac{1}{|x_1 + x_2|^2 + 1}.$$

- **a.** Is $f(x)$ a convex function?
- **b.** Let $\bar{x}^T = (-1, 1)$. Which algorithm would be the best to minimize the function $f(x)$ starting from the given point $\bar{x}$? Give at least two arguments why would you choose that algorithm.
- **c.** Apply two steps of the Polak-Ribière Conjugate Gradient algorithm starting from $x^0 = (-1, 1)^T$. (Use an approximate line-search.)
- **d.** Let $x^1 = (0.5, -0.5)^T$, $x^2 = (0.5, 0.5)^T$ and $x^3 = (-0.5, 0.5)^T$. Apply one step of the Nelder-Mead Simplex Method with $\alpha = 1/2, \ \beta = 1/3, \ \gamma = 2$.
- **e.** Give the interpolation polynomial $Q(x)$ of the function $f(x)$ through the points $x^1, \ x^2, \ x^3$ by using the basis functions $\phi_1(x) = 1, \ \phi_2(x) = x_1 x_2$ and $\phi_3(x) = 2x_1 + x_2$.

10p

SEE THE OTHER SIDE FOR QUESTIONS 3, 4 AND 5!

3. Linearly Constrained NLO – Modelling: *Tartaglia's (1500-1557) problem:* How to divide the number 8 into two parts such that the result of multiplying the product of the two parts by their difference will be maximal.

a. Model Tartaglia's problem as a constrained NLO problem.

b. Prove that both parts are nonzero at the optimal solution.

c. Either show that your model is convex or give an equivalent convex optimization problem.

d. Prove that the point $x = (4 + 4/\sqrt{3}, 4 - 4/\sqrt{3})$ is the optimal solution for Tartaglia's problem.

8p

4. Linearly Constrained NLO : Consider the following linearly constrained non-linear optimization (NLO) problem:

$$\begin{array}{lll} \text{(NLO)} & \min & x_1 + x_2 + x_1 x_2 \\ & \text{s.t.} & 3x_1 + 2x_2 = 5. \end{array}$$

a. Reduce this problem to an unconstrained optimization problem.

b. Make a Quasi-Newton step with the DFP update for the *reduced* problem from **a**. The current point is $x^1 = (1, 1)^T$ and the current approximate of inverse Hessian is two times the identity matrix.

c. Let us assume that the variables x_1 and x_2 are nonnegative. Make one step of the Reduced Gradient algorithm from the point $(1, 1)$ when x_2 is the basis variable.

8p

5. Non-linearly Constrained NLO: Consider the following constrained non-linear optimization (NLO) problem:

$$\begin{array}{lll} \text{(NLO)} & \min & e^{x_2} \\ & \text{s.t.} & \sqrt{x_1^2 + x_2^2} \le x_1 + 1. \end{array}$$

a. Is this a convex optimization problem? Justify your answer.

b. Give the Wolfe Dual of the (NLO) problem.

c. Give the SQP quadratic subproblem for the (NLO) problem at the point $x^1 = (1, 1)^T$ and Lagrange multiplier value $y^1 = 2$. (Hint: to transform the inequality constraint into equality constraint you can add squared slack variable to the constraint.)

d. Using the logarithmic barrier function reformulate the constrained NLO problem as a series of parameterized unconstrained optimization problems.

12p

THE END