

CS and SWF ENG 4-6TE3: Final

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THIS EXAMINATION PAPER INCLUDES **2** PAGES AND **5** QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

DURATION OF THE EXAM: 3 hours

Special instructions:

You might use a letter-size paper, both sides, with your hand-written notes.

The use of the standard McMaster (Casio FX-991) calculator is allowed.

Questions:

1. Concepts, convexity, duality:

- a.) What is the rate of convergence of the sequence:

$$\alpha^k = \frac{k+1}{k} + \frac{1}{k^k} \quad i = 1, 2, \dots$$

- b.) Show that the function $f(x) : R^2 \rightarrow R$ given by

$$f(x) = \log(\exp(x_1^2 + x_2^2 - 2x_1 - x_2 - 1))$$

is a convex function.

- c.) Consider the nonlinear optimization problem:

$$\begin{aligned} \text{(NLO)} \quad & \min \quad \|(x_1, x_2)\| \\ & \text{s.t.} \quad \exp(x_1^2 - x_2) \leq 1, \\ & \quad \quad 2x_1 + 3x_2 = 1, \\ & \quad \quad x_1, x_2 \geq 0, \end{aligned}$$

Prove that problem (NLO) is a convex optimization problem.

- d.) Give a point $x^0 \in R^2$ that is a Slater point of the problem (NLO).
e.) Give the Lagrange Function and the Wolfe Dual for this problem.
f.) Derive the KKT conditions of the given problem (NLO).

12p

2. Unconstrained Optimization: Consider the function

$$f(x) = (x_1 + x_2 - 2)^4 + \exp(x_1 - 2x_2).$$

- a.) Is $f(x)$ a convex function?
b.) Let $\bar{x} = (2, 1)^T$. Which algorithm would be the best to minimize the function $f(x)$ starting from the given point \bar{x} . Give at least two arguments why would you choose that algorithm
c.) Apply two steps of the Fletcher-Reeves Conjugate Gradient algorithm. (Use an approximate line-search.)
d.) Let $x^0 = (0.5, -0.5)^T$ be an initial point, and $d^1 = (0.5, 0.5)^T$ and $d^2 = (-0.5, 0.5)^T$ be two directions. Apply two steps of the Hooke-Jeeves pattern search method with $\alpha = 1/2$ damping factor.
e.) Give the interpolation polynome $Q(x)$ of the function $f(x)$ through the points $x^1 = (0.5, -0.5)^T$, $x^2 = (0.5, 0.5)^T$, $x^3 = (-0.5, 0.5)^T$ by using the basis functions $\phi_1(x) = 1$, $\phi_2(x) = x_1x_2$ and $\phi_3(x) = x_1^2 + x_2^2$.

10p

SEE THE OTHER SIDE FOR QUESTIONS 3, 4 AND 5!

- 3. Constrained NLO – Modeling:** You have to design a 3-D rectangular block. The sum of the length of the edges is at most 10 m 's. The base of the block has to be at least 10 square m 's, while none of the edges is longer than 4 m 's.
Your task is to design the maximal volume block.

- Model the above problem as a constrained NLO problem.
- Prove that each of the edge lengths is nonzero at the optimal solution.
- Is this a convex optimization problem? Prove if it is convex;
OR try to give an equivalent convex formulation.
- Give the KKT conditions of your NLO formulation.
- Check if the edge lengths $(\sqrt{10}, \sqrt{10}, 10 - 2\sqrt{10})$ give an optimal solution.

10p

- 4. Linearly Constrained NLO :** Consider the following linearly constrained non-linear optimization (NLO) problem:

$$\begin{aligned} \text{(NLO)} \quad & \min \quad x_1 + x_2 + x_1^2 + x_2^2 \\ & \text{s.t.} \quad 2x_1 - x_2 = 4. \end{aligned}$$

- Reduce this problem to an unconstrained optimization problem.
- Make a Newton step for the reduced problem. The current point is $x^1 = (2, 0)^T$.
- Let us assume that the variables x_1 and x_2 are nonnegative. Make one step of the Reduced Gradient algorithm from the point $(2, 0)$ and x_1 is the basis variable.

6p

- 5. Non-linearly Constrained NLO:** Consider the following constrained non-linear optimization (NLO) problem:

$$\begin{aligned} \text{(NLO)} \quad & \min \quad e^{x_1+x_2} \\ & \text{s.t.} \quad \sqrt{x_1^4 + x_2^2} \leq 1. \end{aligned}$$

- Is this a convex optimization problem? Justify your answer.
- Give the Wolfe dual of the (NLO) problem.
- Give the SQP quadratic subproblem for the (NLO) problem at the point $x^1 = (1, 1)^T$ and Lagrange multiplier value $y^1 = 2$.
- Using the logarithmic barrier function reformulate the constrained NLO problem as a series of parameterized unconstrained optimization problem.

12p

THE END