CS and SWF ENG 4-6TE3: Final

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THIS EXAMINATION PAPER INCLUDES **2** PAGES AND **5** QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.
DURATION OF THE EXAM: 3 hours

Special instructions:
You might use a letter-size paper, both sides, with your hand-written notes.
The use of the standard McMaster (Casio FX-991) calculator is allowed.

Questions:

1. **Concepts, convexity, duality:**

   a.) What is the order and the rate of convergence of the sequence:
   $$\alpha^k = \|\tfrac{1}{k!}, \tfrac{1}{k^{2^k}}\|_\infty \qquad i = 1, 2, \ldots$$
   (Hint: $\|x\|_\infty = \max |x_i|,\ i = 1, \ldots, n$.)

   b.) Show that the function $f(x) : \mathbb{R}^2 \to \mathbb{R}$ given by
   $$f(x) := \exp\left(x_1^4 + x_2^6 - 2x_1 - x_2 - 1\right)$$
   is a convex function.

   c.) Consider the nonlinear optimization problem:
   $$\begin{array}{lll} \text{(NLO)} & \min & x_1^4 + x_2^4 \\ & \text{s.t.} & \sqrt{x_1^2 + x_2^2} - x_1 - x_2 \leq 0, \\ & & x_1 + x_2 = 1, \\ & & x_1, x_2 \geq 0, \end{array}$$
   Is the problem (NLO) a convex optimization problem or not?

   d.) Give a point $x^0 \in \mathbb{R}^2$ that is a Slater point of the problem (NLO).

   e.) Give the Lagrange Function and the Lagrange Dual for this problem.

   f.) Derive the KKT conditions of the given problem (NLO).

12p

2. **Unconstrained Optimization:** Consider the function
   $$f(x) = (x_1 - x_2)^2 + \exp(x_1 + 2x_2).$$

   a.) Is $f(x)$ a convex function?

   b.) Let $\bar{x}^T = (1, -1)$. Which algorithm would be the best to minimize the function $f(x)$ starting from the given point $\bar{x}$. Give at least two arguments why would you choose that algorithm

   c.) Apply two steps of the Polak-Ribere Conjugate Gradient algorithm starting from the point $x^1 = (1, -1)^T$. (Use the Goldstein-Armijo line-search with $\mu_1 = 0.05$, $\mu_2 = 0.95$, $\rho_1 = 1/2$, $\rho_2 = 1.5$ and $\alpha_0 = 1$.)

   e.) Give the interpolation polynome $Q(x)$ of the function $f(x)$ through the points $x^1 = (0.5, -0.5)^T$, $x^2 = (0.5, 0.5)^T$, $x^3 = (0, 0)^T$ by using the basis functions $\phi_1(x) = 1$, $\phi_2(x) = x_1 + x_2$ and $\phi_3(x) = x_2 + x_2^2$.

10p

SEE THE OTHER SIDE FOR QUESTIONS 3, 4 AND 5!

3. Linearly Constrained NLO – Modeling: You have to design a 3-D cylinder as a water storage. The volume of the cylinder has to be at least 10 cubic meters. The base of the cylinder is at most 6 sq. meters, while the height of the cylinder is at most 2 meters.
Your task is to design the cylinder that satisfies the above conditions and the difference between the radius of the base and the height of the cylinder is minimal.

a. Model the above problem as a constrained NLO problem.

b. Prove that the height is not zero at an optimal solution.

c. Is this a convex optimization problem? Prove if it is convex;
OR give an evidence that it is not convex.

d. Give the KKT conditions of your NLO formulation.

e. Check if the height=2, base-radius=2 is an optimal solution.

10p

4. Linearly Constrained NLO : Consider the following linearly constrained non-linear optimization (NLO) problem:

$$\text{(NLO)} \qquad \min \quad x_1^2 + x_2^4 \qquad \text{s.t.} \quad 3x_1 + 2x_2 = 6.$$

a. Reduce this problem to an unconstrained optimization problem.

b. Make a Newton step for the reduced problem. The current point is $x^1 = (0,3)^T$.

c. Let us assume that he variables x_1 and x_2 are nonnegative. Make one step of the Reduced Gradient algorithm from the point $(0,3)^T$ using x_2 as the basis variable.

6p

5. Non-linearly Constrained NLO: Consider the following constrained non-linear optimization (NLO) problem:

$$\text{(NLO)} \qquad \min \quad (2x_1 + x_2 - 1)^2 \qquad \text{s.t.} \quad e^{x_1+x_2} \leq 1.$$

a. Is this a convex optimization problem? Justify your answer.

b. Give the Wolfe dual of the (NLO) problem.

c. Give the SQP quadratic subproblem for the (NLO) problem at the point $x^1 = (1,2)^T$ and Lagrange multiplier value $y^1 = 3$. (Hint: to transform the inequality constraint into equality constraint you can add squared slack variable to the constraint.)

d. Using the inverse barrier function reformulate the constrained NLO problem as a series of parameterized unconstrained optimization problem.

12p

THE END