COMP SCI and SFWR ENG 4-6TE3, and CES 722: Final

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THIS EXAMINATION PAPER INCLUDES **2** PAGES AND **5** QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.
DURATION OF THE EXAM: 3 hours

Special instructions:
You might use a letter-size paper, both sides, with your hand-written notes.
The use of the standard McMaster (Casio FX-991) calculator is allowed.

Questions:

1. Concepts, convexity, duality:

a.) What is the order and the rate of convergence of the sequence:

$$\alpha^k = \left\| \left(\frac{1}{2^{(3^k)}}, \frac{1}{3^{(2^k)}} \right) \right\|_\infty \qquad k = 1, 2, \ldots$$

(Hint: $\|x\|_\infty = \max\{|x_i|,\ i = 1, \ldots, n\}$.)

b.) Show that the function $f(x) : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x) := -\log\left(x_1 + x_2 - 2x_1 - x_2 - 1\right)$$

is a convex function.

c.) Consider the nonlinear optimization problem:

$$\begin{array}{llll} \text{(NLO)} & \min & \exp(2x_1) + \exp(-2x_2) & \\ & \text{s.t.} & x_1^2 + x_2^2 - 3x_1 - 4x_2 & \leq 0, \\ & & x_1 + x_2 & = 1, \\ & & x_1, x_2 & \geq 0, \end{array}$$

Is the problem (NLO) a convex optimization problem or not?

d.) Give a point $x^0 \in \mathbb{R}^2$ that is an Ideal Slater point of the problem (NLO).

e.) Give the Lagrange Function and the Lagrange Dual for this problem.

f.) Derive the KKT conditions of the given problem (NLO).

14p

2. Unconstrained Optimization: Consider the function

$$f(x) = (x_1 - x_2)^2 + \frac{1}{1 + x_1^2 + x_2^2}.$$

a.) Is $f(x)$ a convex function? Justify your answer!

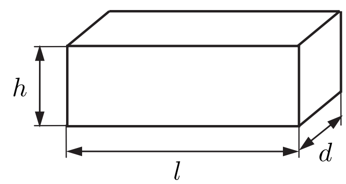
b.) Let $\hat{x}^T = (1, -1)$. Is this point a local/global minimum? Justify your answer!

c.) Let $\bar{x}^T = (1, -1)$. Which algorithm would be the best to minimize the function $f(x)$ starting from the given point $\bar{x}$. Give at least two arguments why would you choose that algorithm.

6p

SEE THE OTHER PAGE FOR QUESTIONS 3, 4 AND 5!

3. Non-linearly Constrained NLO – Modeling: You have to design a 3-D block as a water storage. The volume of the block has to be at least 9 cubic meters. The base area of the block is at most 6 square meters, while the height of the block is at least 1 and at most 2 meters.
Your task is to design the block that satisfies the above conditions and the difference between the base area and the height of the block is maximal.



a. Model the above problem as a constrained NLO problem.

b. Prove that the height is not 1 at an optimal solution.

c. Is this a convex optimization problem? Prove if it is convex;
OR give an evidence that it is not convex.

d. Give the KKT conditions of your NLO formulation.

e. Check if the height=1.5 meters, base-area=6 square meters corresponds to an optimal solution.

12p

4. Linearly Constrained NLO : Consider the following linearly constrained non-linear optimization (NLO) problem:

$$(\text{NLO}) \qquad \begin{array}{ll} \min & x_1^2 + (x_2 - 2)^4 \\ \text{s.t.} & x_1 + x_2 \qquad = 2. \end{array}$$

a. Reduce this problem to an unconstrained optimization problem.

b. Make a full Newton step for the reduced problem. The current point is $x^1 = (2, 0)^T$.

c. Let us assume that the variables x_1 and x_2 are nonnegative. Make one step of the Reduced Gradient algorithm from the point $(0, 2)^T$ using x_2 as the basis variable.

6p

5. Non-linearly Constrained NLO: Consider the following constrained non-linear optimization (NLO) problem:

$$(\text{NLO}) \qquad \begin{array}{ll} \min & 2x_1 + x_2 \\ \text{s.t.} & e^{-x_1} + (x_1 - x_2)^2 \quad \leq 1. \end{array}$$

a. Is this a convex optimization problem? Justify your answer.

b. Give the Wolfe dual of the (NLO) problem.

c. Give the SQP quadratic subproblem for the (NLO) problem at the point $x^1 = (2, 2)^T$ and Lagrange multiplier value $y^1 = 1$. (Hint: to transform the inequality constraint into equality constraint you can add squared slack variable to the constraint.)

d. Using the logarithmic barrier function reformulate the constrained NLO problem as a series of parameterized unconstrained optimization problems.

12p

THE END