

COMP SCI and SFWR ENG 4-6TE3 Final Exam Solutions (2006)

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1. Concepts, convexity, duality:

a.) What is the order and the rate of convergence of the sequence:

$$\alpha^k = \left\| \left(\frac{1}{2^{(3^k)}}, \frac{1}{3^{(2^k)}} \right) \right\|_{\infty} \quad k = 1, 2, \dots$$

(Hint: $\|x\|_{\infty} = \max\{|x_i|, i = 1, \dots, n\}$.)

b.) Show that the function $f(x) : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f(x) := -\log(x_1 + x_2 - 2x_1 - x_2 - 1)$$

is a convex function.

c.) Consider the nonlinear optimization problem:

$$\begin{aligned} \text{(NLO)} \quad & \min \quad \exp(2x_1) + \exp(-2x_2) \\ & \text{s.t.} \quad x_1^2 + x_2^2 - 3x_1 - 4x_2 \leq 0, \\ & \quad \quad x_1 + x_2 = 1, \\ & \quad \quad x_1, x_2 \geq 0, \end{aligned}$$

Is the problem (NLO) a convex optimization problem or not?

d.) Give a point $x^0 \in \mathbb{R}^2$ that is an Ideal Slater point of the problem (NLO).

e.) Give the Lagrange Function and the Lagrange Dual for this problem.

f.) Derive the KKT conditions of the given problem (NLO).

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Solution

a. Since $3^{(2^k)} < 2^{(3^k)}$, $\frac{1}{3^{(2^k)}}$ is converging slower, we only check the order of $\alpha_k = \frac{1}{3^{(2^k)}}$.

$$\lim_{k \rightarrow \infty} \frac{|\alpha_{k+1}|}{|\alpha_k|^2} = \lim_{k \rightarrow \infty} \frac{1/3^{(2^{k+1})}}{(1/3^{(2^k)})^2} = \lim_{k \rightarrow \infty} \frac{1/3^{(2^{k+1})}}{1/3^{(2^k)2}} = 1$$

$$\lim_{k \rightarrow \infty} \frac{|\alpha_{k+1}|}{|\alpha_k|^{2+\epsilon}} = \lim_{k \rightarrow \infty} \frac{1/3^{(2^{k+1})}}{(1/3^{(2^k)})^{2+\epsilon}} = \lim_{k \rightarrow \infty} \frac{1}{(1/3^{(2^k)})^{\epsilon}} \rightarrow \infty \quad \forall \epsilon > 0$$

so it is of order 2.

b. $x_1 + x_2 - 2x_1 - x_2 - 1$ is linear and, so, concave; $-\log(x)$ is convex and non-increasing function; as a composition of these two functions, the given function is therefore convex.

c. Yes, objective function and all the constraints are convex, and the equality constraint is linear.

d. $(0.5, 0.5)^T$ is Ideal Slater point because none of the inequality constraints are singular.

e. There two versions of lagrange function for this problem:

$$\begin{aligned} L_1(x, \lambda, \mu) &= f(x) + \sum_{j=1}^k \lambda_j g_j(x) + \mu^T (b - Ax) \\ &= e^{2x_1} + e^{-2x_2} + \lambda_1(x_1^2 + x_2^2 - 3x_1 - 4x_2) + \lambda_3(-x_1) + \lambda_4(-x_2) + \mu(x_1 + x_2 - 1) \\ L_2(x, \lambda, \mu) &= e^{2x_1} + e^{-2x_2} + \lambda_1(x_1^2 + x_2^2 - 3x_1 - 4x_2) + \lambda_3(-x_1) + \lambda_4(-x_2) \\ &\quad + \lambda_5(x_1 + x_2 - 1) + \lambda_6(1 - x_1 - x_2) \\ &\quad \lambda \geq 0 \end{aligned}$$

$$\begin{aligned} \nabla L_1(x, \lambda) &= \begin{pmatrix} 2e^{2x_1} + \lambda_1(2x_1 - 3) - \lambda_3 + \mu \\ -2e^{-2x_2} + \lambda_1(2x_2 - 4) - \lambda_4 + \mu \end{pmatrix} \\ \nabla L_2(x, \lambda) &= \begin{pmatrix} 2e^{2x_1} + \lambda_1(2x_1 - 3) - \lambda_3 + \lambda_5 - \lambda_6 \\ -2e^{-2x_2} + \lambda_1(2x_2 - 4) - \lambda_4 + \lambda_5 - \lambda_6 \end{pmatrix} \end{aligned}$$

$$\psi(\lambda, \mu) = \inf_x \{L(x, \lambda, \mu)\}$$

The Lagrange dual of this problem is

$$\begin{aligned} \sup_{x, \lambda, \mu} \quad & \psi(\lambda, \mu) \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

The Wolfe dual of this problem is

$$\begin{aligned} \sup_{x, \lambda, \mu} \quad & L(x, \lambda, \mu) \\ \text{s.t.} \quad & \nabla_x L_i(x, \lambda, \mu) = 0, \quad i = 1 \text{ or } 2 \\ & \lambda \geq 0 \end{aligned}$$

f. The KKT condition for this problem is

$$\begin{aligned} g(x) &\leq 0 \\ Ax &= b \\ x &\geq 0 \\ \lambda &\geq 0 \\ \nabla_x L(x, \lambda, \mu) &= 0, \\ \lambda_j g_j(x) &= 0, \quad j = 1, \dots, k. \end{aligned}$$

2. Unconstrained Optimization: Consider the function

$$f(x) = (x_1 - x_2)^2 + \frac{1}{1 + x_1^2 + x_2^2}.$$

- Is $f(x)$ a convex function? Justify your answer!
- Let $\hat{x}^T = (1, -1)$. Is this point a local/global minimum? Justify your answer!
- Let $\bar{x}^T = (1, -1)$. Which algorithm would be the best to minimize the function $f(x)$ starting from the given point \bar{x} . Give at least two arguments why would you choose that algorithm.

6p

Solution

a. Non-convex. We can prove the counterexample: $x^1 = (-1, -1)^T$, $x^2 = (1, 1)^T$ and

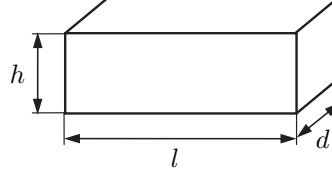
$$f(0.5x^1 + 0.5x^2) = f(0, 0) = 1 > 0.5f(x^1) + 0.5f(x^2) = 1/6 + 1/6 = 1/3.$$

It violates the definition of convexity.

b. Point $\hat{x} = (1, -1)^T$ is not a local/global minimum as $\nabla f(\hat{x}) \neq 0$.

- c. This function is non-convex, so we can use Trust Region algorithm. Trust Region (1) always converges to a local minimum and (2) allows to avoid line search.

- 3. Non-linearly Constrained NLO – Modeling:** You have to design a 3-D block as a water storage. The volume of the block has to be at least 9 cubic meters. The base area of the block is at most 6 square meters, while the height of the block is at least 1 and at most 2 meters. Your task is to design the block that satisfies the above conditions and the difference between the base area and the height of the block is maximal.



- Model the above problem as a constrained NLO problem.
- Prove that the height is not 1 at an optimal solution.
- Is this a convex optimization problem? Prove if it is convex; OR give an evidence that it is not convex.
- Give the KKT conditions of your NLO formulation.
- Check if the height=1.5 meters, base-area=6 square meters corresponds to an optimal solution.

12p

Solution Denote the cylinder height by h and the base-radius by r .

- a. The problem can be formulated as follows

$$\begin{aligned} \max_{l,d,h} \quad & |d \cdot l - h| \\ \text{s.t.} \quad & h \cdot d \cdot l \geq 9 \\ & d \cdot l \leq 6 \\ & h \geq 1 \\ & h \leq 2 \\ & l \geq 0 \\ & d \geq 0 \end{aligned}$$

- If $h = 1$ then the problem is infeasible as $1 \cdot d \cdot l \geq 9$ and $d \cdot l \leq 6$. So, the height h is not 1 in the optimal point as the problem feasible set is non-empty.
- Rewriting the first constraint in the standard form $-h \cdot d \cdot l + 9 \leq 0$ and computing the Hessian of the function $g_1(l, d, h) = -h \cdot d \cdot l + 9$, we get:

$$\nabla^2 g_1(l, d, h) = \begin{pmatrix} 0 & -h & -d \\ -h & 0 & -l \\ -d & -l & 0 \end{pmatrix}.$$

As the determinant of the Hessian is negative for $l, d, h \geq 0$, the function $g_1(l, d, h)$ is not convex. So, the first constraint is not convex and this is not a convex optimization problem.

- d. The Lagrange function of the optimization problem is:

$$L(l, d, h) = |d \cdot l - h| + y_1(9 - h \cdot d \cdot l) + y_2(d \cdot l - 6) + y_3(1 - h) + y_4(h - 2) - y_5 l - y_6 d, \quad y \geq 0, y \in \mathbb{R}^6.$$

Consequently, the KKT conditions of the NLO formulation are:

$$\begin{aligned} & 9 - h \cdot d \cdot l \leq 0; \quad d \cdot l - 6 \leq 0; \quad 1 - h \leq 0; \quad h - 2 \leq 0; \quad -l \leq 0; \quad -d \leq 0 \\ & \begin{cases} d - y_1(h \cdot d) + y_2 d - y_5 = 0; & l - y_1(h \cdot l) + y_2 l - y_6 = 0; & -1 - y_1(l \cdot d) - y_3 + y_4 = 0 \end{cases} \text{ if } (d \cdot l - h) > 0 \\ & \begin{cases} -d - y_1(h \cdot d) + y_2 d - y_5 = 0; & -l - y_1(h \cdot l) + y_2 l - y_6 = 0; & 1 - y_1(l \cdot d) - y_3 + y_4 = 0 \end{cases} \text{ if } (d \cdot l - h) < 0 \\ & y_1(9 - h \cdot d \cdot l) = 0; \quad y_2(d \cdot l - 6) = 0; \quad y_3(1 - h) = 0; \quad y_4(h - 2) = 0; \quad y_5 l = 0; \quad y_6 d = 0 \\ & y \geq 0 \end{aligned}$$

- e. If $h = 1.5$ and $l \cdot d = 6$, then the problem is feasible as all the constraints are satisfied. For $1 \leq h < 1.5$ the problem gets infeasible. For $1.5 < h \leq 2$ the objective function value is smaller than for $h = 1.5$. So, $h = 1.5$ and $l \cdot d = 6$ is the optimal solution.

4. **Linearly Constrained NLO :** Consider the following linearly constrained non-linear optimization (NLO) problem:

$$\begin{aligned} \text{(NLO)} \quad & \min \quad x_1^2 + (x_2 - 2)^4 \\ & \text{s.t.} \quad x_1 + x_2 = 2. \end{aligned}$$

- a. Reduce this problem to an unconstrained optimization problem.
b. Make a full Newton step for the reduced problem. The current point is $x^1 = (2, 0)^T$.
c. Let us assume that the variables x_1 and x_2 are nonnegative. Make one step of the Reduced Gradient algorithm from the point $(0, 2)^T$ using x_2 as the basis variable.

6p

Solution

- a. Substituting $x_2 = 2 - x_1$ into the objective function, we get the reduced unconstrained problem

$$\min \quad f_N(x_1) = x_1^2 + x_1^4.$$

- b. The gradient of $f_N(x_1)$ is $\nabla_N f(x_1) = 2x_1 + 4x_1^3$.
 $x^1 = 2$, $\nabla_N f(x_1^1) = 36$, $\nabla_N^2 f(x_1^1) = 2 + 12(x_1^1)^2 = 50$;
 $x^2 = x^1 - (\nabla_N^2 f(x_1^1))^{-1} \nabla_N f(x_1^1) = 2 - \frac{36}{50} = 1\frac{7}{25} = 1.28$;
c. $x^1 = (0, 2)^T$, $\nabla f(x^1) = (0, 0)^T$, $B = 1$, $N = 1$.
 $\nabla_N f(x_1^1) = (I, -B^{-1}N) \nabla f(x^1) = (1, -1) \nabla f(x^1) = 0$.
 $s_N^1 = 0$; $s_B^1 = -B^{-1}N s_N^1 = 0$; $s^1 = (0, 0)$;
 $x^2 = x^1 = (0, 2)^T$ as we are already in the optimal point.

5. **Non-linearly Constrained NLO:** Consider the following constrained non-linear optimization (NLO) problem:

$$\begin{aligned} \text{(NLO)} \quad & \min \quad 2x_1 + x_2 \\ & \text{s.t.} \quad e^{-x_1} + (x_1 - x_2)^2 \leq 1. \end{aligned}$$

- a. Is this a convex optimization problem? Justify your answer.
b. Give the Wolfe dual of the (NLO) problem.
c. Give the SQP quadratic subproblem for the (NLO) problem at the point $x^1 = (2, 2)^T$ and Lagrange multiplier value $y^1 = 1$. (Hint: to transform the inequality constraint into equality constraint you can add squared slack variable to the constraint.)
d. Using the logarithmic barrier function reformulate the constrained NLO problem as a series of parameterized unconstrained optimization problems.

12p

Solution

- a. (NLO) is a convex optimization problem: e^{-x_1} is convex and $(x_1 - x_2)^2$ is convex, hence the constraint is convex as the sum of convex functions; the objective function $2x_1 + x_2$ is linear and, so, convex.

b. The Lagrange function of this problem is

$$L(x, y) = 2x_1 + x_2 + y(e^{-x_1} + (x_1 - x_2)^2 - 1), \quad y \geq 0$$

and

$$\nabla_x L(x, y) = (2 + y(-e^{-x_1} + 2(x_1 - x_2)), 1 - 2y(x_1 - x_2))^T,$$

so its Wolfe dual is

$$\begin{aligned} \sup_{x, y} \quad & 2x_1 + x_2 + y(e^{-x_1} + (x_1 - x_2)^2 - 1) \\ \text{s.t.} \quad & 2 + y(-e^{-x_1} + 2(x_1 - x_2)) = 0 \\ & 1 - 2y(x_1 - x_2) = 0 \\ & y \geq 0 \end{aligned}$$

c. We first add a squared slack variable $x_3 \in \mathbb{R}$ to the constraint as follows

$$e^{-x_1} + (x_1 - x_2)^2 + x_3^2 - 1 = 0.$$

Lets choose $x_3 = 1$ which is actually infeasible for $(2, 2)^T$, in this case we get $(2, 2, 1)^T$ as infeasible starting point of the transformed problem.

The Lagrange function of this transformed problem is

$$L(x, y) = 2x_1 + x_2 + y(e^{-x_1} + (x_1 - x_2)^2 + x_3^2 - 1),$$

$$\nabla_x L(x, y) = (2 + y(-e^{-x_1} + 2(x_1 - x_2)), 1 - 2y(x_1 - x_2), 2yx_3)^T,$$

$$\nabla_{xx} L(x, y) = \begin{pmatrix} 2 + ye^{-x_1} & -2 & 0 \\ -2 & 2y & 0 \\ 0 & 0 & 2y \end{pmatrix}.$$

For the point $(2, 2, 1)^T$ and $y = 1$,

$$\nabla_{xx} L((2, 2, 1), 1) = \begin{pmatrix} 2 + e^{-2} & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\nabla_x L((2, 2, 1), 1) = \begin{pmatrix} 2 - e^{-2} \\ 1 \\ 2 \end{pmatrix}.$$

The Jacobian of constraints at $(2, 2, 1)^T$ is $\nabla H(x) = (-e^{-x_1} + 2(x_1 - x_2), -2(x_1 - x_2), 2x_3) = (-e^{-2}, 0, 2)$, and the function value of the constraint is $H(x) = e^{-2}$. The SQP requested is then

$$\begin{aligned} \min_{\Delta x} \quad & \frac{1}{2} \Delta x^T \begin{pmatrix} 2.135 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Delta x + \begin{pmatrix} 1.865 \\ 1 \\ 2 \end{pmatrix}^T \Delta x \\ \text{s.t.} \quad & (-0.135, 0, 2) \Delta x = 0.135 \end{aligned}$$

d. Using the logarithmic barrier function, this constrained optimization problem can be formulated as a series of the following unconstrained optimization problems

$$\min_x \quad 2x_1 + x_2 - \mu \log(1 - e^{-x_1} - (x_1 - x_2)^2),$$

where $\mu > 0$ is a parameter.

THE END