COMP SCI and SFWR ENG 4-6TE3: MidTerm

Tamás Terlaky

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THIS EXAMINATION PAPER INCLUDES **2** PAGES AND **5** QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.
DURATION OF THE EXAM: 2 hours

Special instructions:
You might use a letter-size sheet with your hand-written notes.
The use of the standard McMaster (Casio FX-991) calculator is allowed.

Questions:

1. Consider the functions

$$\begin{aligned} f(x_1, x_2) &= -2x_1 + 4x_1^4 + e^{(x_1^2)} + e^{-x_1 - 2x_2} \\ h(x_1, x_2, x_3) &= \log(1 + |x_1 + x_2 + x_3|) \end{aligned}$$

   (a) (*i*) Give the gradient and the Hessian of $f(x_1, x_2)$.
   (*ii*) Give the second-order Taylor series expansion of $f(x_1, x_2)$ at the point $x^0 = (-1, 1)^T$.

   (b) Prove that $f(x)$ is strictly convex.

   (c) Demonstrate that the function $h(x)$ is not convex.

   (d) Examine if the point $x = (-10, 5, 5)^T$ is a local/global minimum of $h(x)$ without using its gradient/Hessian information.

   (e) Make one step of the Nelder-Mead simplex algorithm when you minimize $f(x)$. Let the initial simplex be given by $(1, 1)^T, (1, 2)^T, (2, 1)^T$. Use the following parameter settings: damping factor $\alpha = 1/2$, contraction factor $\beta = 2/3$, extension factor $\gamma = 2$.

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2. Convergence:

   (a) Give a sequence that converges to 1 with order 4 and prove its order of convergence.

   (b) Determine the limit point and the rate of convergence of the following sequence:
   $x^k = (\frac{1}{k^4}, \frac{4}{k+4})^T, \qquad k = 1, 2, ...$

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3. Consider the function $f(x_1, x_2) = (x_1 + x_2 - 2)^2 - \log(1 + x_1 - x_2)^2$.

   (a) Let $x^0 = (1, 1)^T$. Apply a full Newton step and give x^1.

   (b) Let $x^0 = (1, 1)^T$. Calculate the Trust-Region search direction with the initial value $\alpha = 1$. Let choose $\mu = 0.2$, $\eta = 0.8$, $\gamma_1 = 0.5$, $\gamma_2 = 2.5$. Would you accept this step in the Trust Region Algorithm or α should be changed. If it needs to be changes, should α be increased or decreased?

   (c) Give the directional derivative of the function $f(x)$ at the point $(1, 1)^T$ in the direction $(1, 1)^T$.

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SEE THE OTHER SIDE FOR QUESTIONS 4 and 5!

4. Convex functions.

   (a) Prove that the maximum of convex functions is convex, i.e., if $f_1(x), \ldots, f_k(x) : R^n \to R$ are convex, then $f(x) = \max_{1 \le j \le k} f_j(x)$ is convex as well.

   (b) Specify when Newton's method for minimizing a function is applicable and when not.

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5. Which of the following statements is true/false.
   Give a one-sentence justification of your answer.

   (a) A two times differentiable function is strictly convex if and only if its Hessian is positive definite.

   (b) Minimum of convex functions is convex.

   (c) The steepest descent algorithm converges quadratically.

   (d) An exact line-search is used in trust region algorithms.

   (e) The derivative of convex functions is convex as well.

   (f) The bisection method converges superlinearly.

   (g) A two times differentiable function is convex if and only if its Hessian is symmetric.

   (h) If a sequence converges quadratically, then it converges superlinearly too.

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THE END