

COMP SCI and SFWR ENG 4-6TE3: MidTerm

Tamás Terlaky

March 1, 2005

THIS EXAMINATION PAPER INCLUDES **2** PAGES AND **5** QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

DURATION OF THE EXAM: 2 hours

Special instructions:

You might use a letter-size sheet with your hand-written notes.

The use of the standard McMaster (Casio FX-991) calculator is allowed.

Questions:

1. Consider the functions

$$f(x_1, x_2) = 2x_1 + \frac{1}{4}x_1^4 + e^{(x_1^2)} + e^{-x_1-2x_2}$$

$$h(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + \min\{x_1 + 2x_2 + x_3, (x_1 + 2x_2 + x_3)^2\}$$

- (a) (i) Give the gradient and the Hessian of $f(x_1, x_2)$.
(ii) Give the second-order Taylor series expansion of $f(x_1, x_2)$ at the point $x^0 = (2, 1)^T$.
- (b) Prove that the function $h(x)$ is not convex.
- (c) Make a step with Newton's method from the point $x^0 = (1, 0)^T$ when you minimize the function $f(x)$.
- (d) Examine if the point $x = (0, 0, 0)^T$ is a local/global minimum of $h(x)$ without using its gradient/Hessian information.
- (e) Consider the function $h(x)$. Give the directional derivative of the function $h(x)$ at the point $(1, 2, 3)^T$ in the direction $(1, -2, 1)^T$.
Examine if this direction is a descent or an ascent direction.

12p

2. Convergence:

- (a) Give a sequence that converges to 1 with order 2.5 and prove its order of convergence.
(b) Determine the limit point and the rate of convergence of the following sequence:

$$x^k = \left(\exp(-k), \frac{1}{k!}\right), \quad k = 1, 2, \dots$$

Justify your answer.

6p

3. Consider the function $f(x_1, x_2) = (2x_1 - x_2 - 2)^2$.

- (a) Let $x^0 = (0, -1)^T$. Apply the Fletcher-Reeves conjugate gradient algorithm to minimize the function $f(x)$. Do as many iterations as necessary.
- (b) Let $x^0 = (0, -1)^T$. Apply one cycle of the pattern search algorithm to minimize the function when the search directions are given as $(1, -1)^T$ and $(0.5, 0.5)^T$. Further, let the contraction ratio (damping factor) be $\alpha = 0.5$.
- (c) Make a line-search step by using the Goldstein-Armijo method from the point $x^0 = (0, -1)^T$ in the direction $s^0 = (1, 1)^T$, when $\mu_1 = 0.1$, $\mu_2 = 0.9$ and $\alpha_0 = 0.8$.

8p

SEE THE OTHER SIDE FOR QUESTIONS 4 and 5!

4. Convex functions.

- (a) Prove that if the function $f(x) : R^n \rightarrow R$ is convex, then for any set of points x^1, x^2, x^3 the inequality

$$f\left(\sum_{i=1}^3 \lambda_i x^i\right) \leq \sum_{i=1}^3 \lambda_i f(x^i),$$

where $\sum_{i=1}^3 \lambda_i = 1$ and $0 \leq \lambda_i \leq 1, \forall i$ holds.

- (b) Show that $\exp(-3x + 4) : R \rightarrow R$ is a convex function.

6p

5. Which of the following statements is true/false.

Give a one-sentence justification of your answer.

- (a) A convex two times differentiable function is monotonically increasing.
- (b) The maximum of convex functions is convex.
- (c) The cost of calculating a search direction in the steepest descent algorithm is $O(n)$ arithmetic operations.
- (d) No line-search is used in the trust region algorithm, thus it always uses a full Newton step.
- (e) The Hessian of a convex functions is nonsingular.
- (f) The conjugate gradient algorithm needs a finite number of iterates to find the minimum of a convex quadratic function.
- (g) Newton's method (for zero finding in 1-dimension) converges quadratically to find the root of the function $(x - a)^2$, where $a > 0$.
- (h) The function $\log(x)$ is a convex function.

8p

THE END