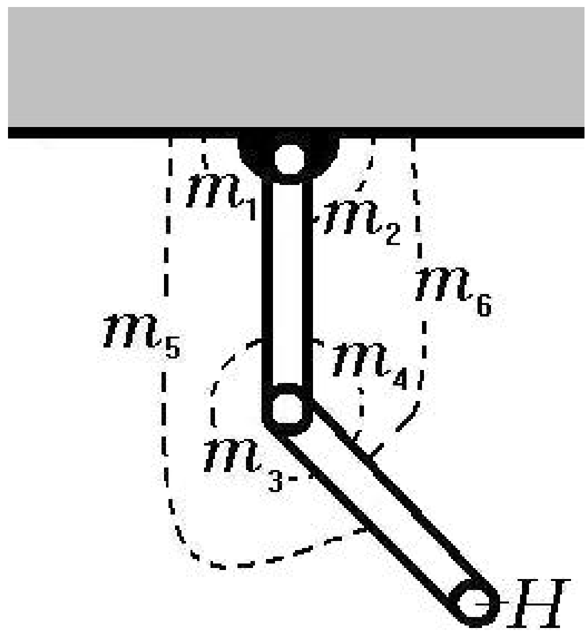
Consider a two joint muscle system:



To exert a force at H we use the 6 muscle groups to exert two forces.
We seek to **minimize** the energy used

$$E(m) = \sum_{i=1}^{6} p_i m_i^\gamma$$

subject to the force equations

$$\begin{aligned} -a_1 m_1 + a_2 m_2 - a_5 m_5 + a_6 m_6 &= F_1, \\ -b_3 m_3 + b_4 m_4 - b_5 m_5 + b_6 m_6 &= F_2, \\ m_i &\geq 0. \end{aligned}$$

Minimize

$$\frac{1}{2} \sum_{i=1}^{6} m_i^2$$

such that

$$\begin{aligned} -m_1 + m_2 - \alpha m_5 + \beta m_6 &= F_1, \\ -m_3 + m_4 - m_5 + m_6 &= F_2, \\ m_i &\geq 0. \end{aligned}$$

Under the assumptions

$F_1 < 0, \quad F_2 = -F_1, \quad \alpha \geq 1$ and $\beta \in (0, 1)$

We could use an algorithm, but we wish to keep α, β and F_i (somewhat) arbitrary.

To write the KKT conditions, note:

$$\begin{aligned} -m_1 + m_2 - \alpha m_5 + \beta m_6 &= F_1, \\ -m_3 + m_4 - m_5 + m_6 &= F_2, \\ m_i &\geq 0 \end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned} -m_1 + m_2 - \alpha m_5 + \beta m_6 &\leq F_1, \\ +m_1 - m_2 + \alpha m_5 - \beta m_6 &\leq -F_1 \\ -m_3 + m_4 - m_5 + m_6 &\leq F_2, \\ +m_3 - m_4 + m_5 - m_6 &\leq -F_2, \\ -m_i &\leq 0. \end{aligned}$$

The KKT conditions are:

$$\begin{aligned} -m_1 + m_2 - \alpha m_5 + \beta m_6 &= F_1 \\ -m_3 + m_4 - m_5 + m_6 &= F_2 \\ m_i &\geq 0 \end{aligned}$$

$$\begin{aligned} -m_1 &= -y_1 - y_7 + y_8 \\ -m_2 &= -y_2 + y_7 - y_8 \\ -m_3 &= -y_3 - y_9 + y_{10} \\ -m_4 &= -y_4 + y_9 - y_{10} \\ -m_5 &= -y_5 - \alpha(y_7 - y_8) - y_9 + y_{10} \\ -m_6 &= -y_6 + \beta(y_7 - y_8) + y_9 - y_{10} \end{aligned}$$

$$\begin{aligned} &-y_1 m_1 - y_2 m_2 - y_3 m_3 \\ &-y_4 m_4 - y_5 m_5 - y_6 m_6 \\ &+(y_7 - y_8)(-m_1 + m_2 - \alpha m_5 + \beta m_6 - F_1) \\ &+(y_9 - y_{10})(-m_3 + m_4 - m_5 + m_6 - F_2) = 0 \end{aligned}$$

$$y_i \geq 0$$

Which simplify to

$$\begin{aligned} F_1 &= -m_1 + m_2 - \alpha m_5 + \beta m_6, \\ F_2 &= -m_3 + m_4 - m_5 + m_6, \end{aligned}$$

$$\begin{aligned} m_1 &= y_1 + \lambda_1 \\ m_2 &= y_2 - \lambda_1 \\ m_3 &= y_3 + \lambda_2 \\ m_4 &= y_4 - \lambda_2 \\ m_5 &= y_5 + \alpha\lambda_1 + \lambda_2 \\ m_6 &= y_6 - \beta\lambda_1 - \lambda_2 \end{aligned}$$

$$y_i m_i = 0,$$

$$y_i \geq 0, \quad m_i \geq 0, \quad \lambda_i \in \mathbf{R}.$$

This can be solved.

Step I: prove $\lambda_1 > 0$ (use $\alpha > \beta$, $F_1 < 0$)

Step II: prove $\lambda_2 < 0$ (use $\lambda_1 > 0$, $F_2 > 0$)

Step III: Show

$$\begin{aligned} F_1 &= -m_1 - \alpha m_5 + \beta m_6 \\ F_2 &= m_4 - m_5 + m_6 \\ 0 &= \alpha m_1 - m_4 - m_5 \\ 0 &= -\beta m_1 + m_4 - m_6 \\ &(m_2 = m_3 = 0). \end{aligned}$$

(use $\lambda_1 = m_1$, $\lambda_2 = -m_4$, and $F_2 = -F_1$)

Step IV: Conclude

$$
\begin{aligned}
m_1 &= F_2 \frac{\alpha+\beta+3}{3+2\alpha^2+2\beta^2-2\alpha\beta} \\
m_2 &= 0 \\
m_3 &= 0 \\
m_4 &= F_2 \frac{\alpha^2+\alpha+\beta+\beta^2+1}{3+2\alpha^2+2\beta^2-2\alpha\beta} \\
m_5 &= F_2 \frac{2\alpha+\alpha\beta-\beta-\beta^2-1}{3+2\alpha^2+2\beta^2-2\alpha\beta} \\
m_6 &= F_2 \frac{-2\beta+1+\alpha^2-\alpha\beta+\alpha}{3+2\alpha^2+2\beta^2-2\alpha\beta}
\end{aligned}
$$

Ex: [Co-contraction]
Let

$$\alpha = 5/4$$

$$\beta = 3/4$$

$$F_1 = -43$$

$$F_2 = 43$$

Then

$$m_1 = 40, \quad m_2 = 0, \quad m_3 = 0$$

$$m_4 = 41, \quad m_5 = 9, \quad m_6 = 11$$

Which shows co-contraction of m_5 and m_6

Talk based on

Herzog, W. and Binding, P.
"Co-contraction of Pairs of Antagonistic Muscles: Analytical Solution for Planar Static Nonlinear Optimization Approaches,"
Mathematical Biosciences,
118, pp. 83–95, (1993).