Algorithms for Continuous Optimization
(Solving sparse systems with low-rank-update)

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Solving Linear Systems

with Low-rank Updates

Consider the following system of linear equations with the unknown vector \( p \in \mathbb{R}^n \),

\[(Q + RS^T)p = q, \tag{1}\]

where \( Q \) is an \( n \times n \) matrix, \( R \) and \( S \) are \( n \times k \) matrices and \( q \in \mathbb{R}^n \) are given.

Assumptions:

1. The matrix \( Q \) is nonsingular.
2. The rank of both \( R \) and \( S \) are equal to \( k \) (\( \leq n \)).
3. The matrix \( Q + RS^T \) is nonsingular.
4. The matrix \( Q \) is sparse, \( RS^T \) is dense and \( k \leq n \).

Assumptions 1–4 guarantee that the system (1) is solvable efficiently.

As it is intuitively clear, solving a linear system

\[Qp = q, \tag{2}\]

where the sparse matrix \( Q \) is the coefficient matrix is computationally cheap, while directly solving equation system (1) with the dense coefficient matrix is computationally expensive.
The procedure

Goal: To utilize the sparsity of the matrix $Q$.
First step we reformulate (1) as follows

$$Qp = q - RS^T p = q - Rz.$$ 

For a moment neglecting the fact that the vector $z = S^T p$ depends on the unknown $p$ we can decompose the solution process into the following steps.

**Step 1.** Solve the sparse system $Qp_1 = q$.

**Step 2.** Solve the sparse system $QU = R$, where $U$ is an $n \times k$ matrix of unknowns.

**Step 3.** Now we consider how to find an appropriate vector $z$. By the definition of $z$ we have

$$z = S^T p = S^T (p_1 - Uz)$$

which is equivalent to SOLVE

$$(I + S^T U)z = S^T p_1. \quad (3)$$

**Step 4.** Let $U$ be the $n \times k$ matrix with column vectors $u_j$. Then we have the solution

$$p = p_1 - Uz.$$
Step 2 of the above procedure involves the solution of \( k + 1 \) sparse linear systems, all with the same sparse coefficient matrix \( Q \), hence these can be solved easily.

Below we will verify that the \( k \)-dimensional linear equation system (3) has a unique solution and that the above sketched way the linear equation system (1) can indeed be solved efficiently.

Before doing that, let us make a simple estimation of the computational complexity under Assumption 4. By a direct approach the linear system (1) with a dense coefficient matrix can be solved in \( \mathcal{O}(n^3) \) arithmetic operations. Assume that the matrix \( Q \) is sparse and so the equation system \( Qp_1 = q \) can be solved in \( \rho < \mathcal{O}(n^3) \) arithmetic operations. We have to solve \( k + 1 \) such systems and a small dense system with \( k \) unknowns. Thus the total complexity becomes \( (k+1)\rho + \mathcal{O}(k^3) \). Note that, in many applications matrix \( Q \) is either (multi)diagonal, or block-diagonal with small diagonal blocks. As a consequence one has \( \rho = \mathcal{O}(n) \). In many of these cases \( k \leq \mathcal{O}(\sqrt{n}) \) then the total computational complexity becomes \( \mathcal{O}(n\sqrt{n}) \), which is a factor \( n\sqrt{n} \) better than the direct approach.
As it is proposed above the linear system (1) is solved by solving \( k + 1 \) linear equation systems with the same coefficient matrix \( Q \) and a small \( k \times k \) linear system as follows.

Let \( p_1 \in \mathbb{R}^n \) be the solution of the linear system

\[
Qp_1 = q,
\]

(4)

\( u_j \in \mathbb{R}^n \) be the solution of the linear system

\[
Qu_j = r_j
\]

(5)

for \( j = 1, \ldots, k \), where \( r_j \) denotes the \( j \)-th column of matrix \( R \) and finally, let \( z \in \mathbb{R}^k \) be the solution of the linear system

\[
(I + S^T U)z = S^T p_1
\]

(6)

where \( I \) denotes the \( k \)-dimensional identity matrix and \( U = [u_1, \ldots, u_k] \). We prove the following theorem.

**Theorem 1** If Assumptions 1–3 hold then the unique solution of the linear system (1) is given by

\[
p = p_1 - Uz.
\]
Proof

First note that by Assumption 3 the solution $p$ exists and it is unique. By Assumption 1 the equations (4) and (5) have unique solutions. Further, by Assumption 2 the vectors $r_j$, $j = 1, \ldots, k$ are linearly independent and then, by Assumption 1, the solution vectors $u_j$, $j = 1, \ldots, k$ of (5) are linearly independent as well, i.e. $\text{rank}(U) = k$.

Now by proving that the coefficient matrix $I + S^TU$ of equations (6) is nonsingular we verify that equation (6) has a unique solution. Assume to the contrary that there is a nonzero vector $w \in \mathbb{R}^k$ such that

$$(I + S^TU)w = 0,$$

or equivalently $S^TUw = -w$. Then, by multiplying the nonsingular matrix $Q + RST$ by the nonzero vector $Uw$ and using (5) we have

$$(Q + RST)Uw = (QU)w + R(S^TUw) = Rw - Rw$$

yielding a contradiction.

Finally by simple calculations the reader easily verifies that $(Q + RST)(p_1 - Uz) = q$ which completes the proof.  $\square$
The Sherman-Morrison formula

\[(Q + RS^T)^{-1} = Q^{-1} - Q^{-1}R(I + S^TQ^{-1}R)^{-1}S^TQ^{-1} \quad (7)\]

- Having proved the correctness of our procedure the Sherman-Morrison formula (7) can be derived from our procedure. This can be done as follows. By definition – from (4) we have \(p_1 = Q^{-1}q\), from (5) we have \(U = Q^{-1}R\) and from (6) we have \(z = (I + S^TU)^{-1}S^Tp_1\) – the solution \(p\) is given by

\[
\begin{align*}
    p &= p_1 - Uz = Q^{-1}q - Q^{-1}R(I + S^TU)^{-1}S^Tp_1 \\
    &= (Q^{-1} - Q^{-1}R(I + S^TQ^{-1}R)^{-1}S^TQ^{-1})q.
\end{align*}
\]

By Assumption 3 we can write \(p = (Q + RS^T)^{-1}q\). Comparing this with the above expression and observing that Theorem 1 holds for all right-hand-side vector \(q \in \mathbb{R}^n\) the Sherman-Morrison formula (7) follows.

- On the other hand, one can derive our procedure by carefully analyzing the Sherman-Morrison formula (7). Using again that the solution of (1) can be written as \(p = (Q + RS^T)^{-1}q\) we have

\[
    p = (Q^{-1} - Q^{-1}R(I + S^TQ^{-1}R)^{-1}S^TQ^{-1})q.
\]

Here all the “inverse matrix – vector” products have to be replaced by the solution of a linear equation system. Thus we have the expression \(Q^{-1}q\) which is equivalent to the solution of (4) resulting in \(p_1\); the expression \(Q^{-1}R\) which is equivalent to the solution of (5) for each \(j = 1, \ldots k\) resulting in \(U\); and finally, having these done, the expression

\[
(I + S^TQ^{-1}R)^{-1}S^TQ^{-1}q = (I + S^TU)^{-1}S^Tp_1
\]

which is equivalent to the solution of (6) resulting in \(z\).