CS/SE4-6TE3, CES 722/723: Reduced Gradient

Consider the following linearly constrained non-linear optimization (NLO) problem:

$$\text{(NLO)} \qquad \min \quad x_1 + x_2 + x_1 x_2$$
$$\text{s.t.} \quad 3x_1 + 2x_2 = 5.$$

Reduce this problem to an unconstrained optimization problem.

Solution: Substituting $x_2 = (5 - 3x_1)/2$ into the objective function, we get the reduced unconstrained problem

$$\min \quad f_N(x_1) = \frac{5}{2} + 2x_1 - \frac{3}{2}x_1^2.$$

Note that we get the same result choosing x_2 as the basic variable (in that case $B = 2$ and $N = 3$) and eliminating it, that ends up in

$$f_N(x_N) = f_N(x_1) = f(x_N, B^{-1}b - B^{-1}Nx_N) = f(x_1, \frac{5}{2} - \frac{3}{2}x_1) = \frac{5}{2} + 2x_1 - \frac{3}{2}x_1^2.$$

Consider the following non-linear optimization (NLO) problem:

$$\begin{array}{ll} \min & (x_1 - x_2)^2 + (\exp(-x_1 + x_2 - x_3))^2 \\ \text{s.t.} & 10x_1 + 5x_2 + 15x_3 = 60 \\ & 2x_1 + 3x_3 = 10 \end{array} \tag{1}$$

1. Reformulate this linearly constrained NLO problem as a unconstrained optimization problem.

   Solution: Rewrite the constraints into $Ax = b$.

$$\begin{pmatrix} 10 & 5 & 15 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 60 \\ 10 \end{pmatrix} \tag{2}$$

So

$$A = \begin{pmatrix} 10 & 5 & 15 \\ 2 & 0 & 3 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 60 \\ 10 \end{pmatrix}.$$

Given the basis

$$B = \begin{pmatrix} 10 & 5 \\ 2 & 0 \end{pmatrix},$$

thus

$$N = \begin{pmatrix} 15 \\ 3 \end{pmatrix},$$

$$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$x_N = \left(x_3\right).$$

$$\begin{aligned}
Ax &= Bx_B + Nx_N = b \\
\Rightarrow x_B &= B^{-1}b - B^{-1}Nx_N \\
&= \begin{pmatrix}10 & 5\\ 2 & 0\end{pmatrix}^{-1}\begin{pmatrix}60\\10\end{pmatrix} - \begin{pmatrix}10 & 5\\ 2 & 0\end{pmatrix}^{-1}\begin{pmatrix}15\\3\end{pmatrix}(x_3) \\
&= \begin{pmatrix}0 & 0.5\\ 0.2 & -1\end{pmatrix}\begin{pmatrix}60\\10\end{pmatrix} - \begin{pmatrix}0 & 0.5\\ 0.2 & -1\end{pmatrix}\begin{pmatrix}15\\3\end{pmatrix}(x_3) \\
&= \begin{pmatrix}5\\2\end{pmatrix} - \begin{pmatrix}1.5\\0\end{pmatrix}(x_3) \\
&= \begin{pmatrix}5-1.5x_3\\2\end{pmatrix}
\end{aligned}$$

So we can change the objective function as the following:

$$\begin{aligned}
f_N(x_N) &= f(x) = f(5-1.5x_3, 2, x_3) \\
&= (3-1.5x_3)^2 + \exp(0.5x_3 - 3)^2 \\
&= (3-1.5x_N)^2 + \exp(0.5x_N - 3)^2
\end{aligned}$$

Therefore, we reformulate the NLO problem into an unconstrained problem as

$$\min f_N(x_N),$$

where

$$f_N(x_N) = ((3-1.5x_N)^2 + \exp(0.5x_N - 3)^2.$$

2. Give out the null-space of A.

   Solution: The null-space of A is

$$Z = \begin{pmatrix}-B^{-1}N\\ I\end{pmatrix} = \begin{pmatrix}-1.5\\0\\1\end{pmatrix}$$

   Clearly, $AZ = 0$.

3. Let now assume that $x \geq 0$ is also a required constraint. Apply one step of the Reduced Gradient method for this modified problem from the point $x = (5,2,0)^T$ with the initial basis given by x_1 and x_2.

   Solution: From the previous solution, we reformulated the problem as:

$$\begin{aligned}
\min \quad & (3-1.5x_N)^2 + \exp(0.5x_N - 3)^2 \\
s.t. \quad & 5 - 1.5x_N \geq 0 \\
& x_N \geq 0
\end{aligned}$$

And we also get the reduced gradient at the point $(5,2,0)^T$ with the basis given by x_1 and x_2:

$$\begin{aligned}
\nabla f_N(x_N)^T &= (-3(3-1.5x_N)+\exp(0.5x_N-3)^2) = (-8.9975) \\
s_N &= -\nabla f_N(x_N)^T = 8.9975 \\
s_B &= -B^{-1}Ns_N = -\begin{pmatrix}1.5\\0\end{pmatrix} * (8.9975) = \begin{pmatrix}-13.4963\\0\end{pmatrix} \\
(s^1)^T &= (s_B^T, s_N^T) = \begin{pmatrix}-13.4963 & 0 & 8.9975\end{pmatrix}^T \\
\bar{\lambda} &= \min_{1\le i\le n, s_j<0}\left\{\frac{x_i}{-s_i}\right\} = 0.3705 \\
f(x^1+\lambda s^1) &= f((5,2,0)^T + \lambda(-13.4963, 0, 8.9975)^T)) \\
&= f(5-13.4963\lambda, 2, 8.9975\lambda) \\
&= (3-13.4963\lambda)^2 + \exp(4.4988\lambda-3)^2
\end{aligned}$$

So we can make a line search with the Golden section:

$$\begin{aligned}
x^2 &= \arg\min_{0\le\lambda\le\bar{\lambda}} f(x^1+\lambda s^1) \\
&= \arg\min_{0\le\lambda\le 0.3705}(3-13.4963\lambda)^2+\exp(4.4988\lambda-3)^2
\end{aligned}$$

We will get $\lambda = 0.2218$, then

$$x^2 = x^1 + \lambda s^1 = (2.0065, 2, 1.9957)^T,$$

and

$$f(x^2) = 0.0183 < f(x^1) = 9.0025.$$

4. Compute a projected gradient of $f(x)$ at the point $x = (5,2,0)^T$.

Solution:

$$\begin{aligned}
\nabla f(x) &= \begin{pmatrix} 2(x_1-x_2) - 2\exp(-2x_1+2x_2-2x_3) \\ -2(x_1-x_2)+2\exp(-2x_1+2x_2-2x_3) \\ -2\exp(-2x_1+2x_2-2x_3)\end{pmatrix} \\
\nabla f(5,2,0) &= \begin{pmatrix} 6-2e^{-6} \\ -6+2e^{-6} \\ -2e^{-6}\end{pmatrix} \\
Z^T\nabla f(5,2,0) &= \begin{pmatrix}-1.5 & 0 & 1\end{pmatrix}\begin{pmatrix} 6-2e^{-6} \\ -6+2e^{-6} \\ -2e^{-6}\end{pmatrix} = (-9+e^{-6}) = -8.9975
\end{aligned}$$