Slater points and Lagrange/Wolfe duals

1 Slater point and Ideal Slater point

- Consider a convex problem with two constraints (1) $x_1 \leq 0$ and (2) $x_1^2 + x_2^2 \leq 1$. Both of them are regular constraints, one linear and the other nonlinear. Point $(0, 0)^T$ is a Slater point of this problem, while it’s not an Ideal Slater point.

- Consider a convex problem with two constraints (1) $x_1 \leq -0.5$ and (2) $x_1^2 + x_2^2 \leq 1$. Both of them are nonsingular constraints, one linear and the other nonlinear. Point $(-0.6, 0)^T$ is an Ideal Slater point of this problem (it is a Slater point as well).

2 Verification of the Slater condition

Check whether the following problem satisfies Slater condition:

\[
\begin{align*}
\min \quad & f(x) = x^2 \\
\text{s.t.} \quad & g_1(x) = x - x^2 \geq 0 \\
& g_2(x) = x^2 \leq 0 \\
& x \geq 0
\end{align*}
\] (1)

Actually, there is only one feasible point $x = 0$, meaning all constraints are singular. However, constraint $g_2(x) \leq 0$ is nonlinear, which means that there is no point which satisfies Slater condition for this problem.

3 Wolfe and Lagrange Dual

Consider the following convex optimization problem:

\[
\begin{align*}
\min \quad & x_1 + e^{x_2} \\
\text{s.t.} \quad & 3x_1 - 2e^{x_2} \geq 10 \\
& x_2 \geq 0 \\
& x \in \mathbb{R}^2.
\end{align*}
\]

Write down the Wolfe dual and Lagrange dual of this problem and show that the optimal value is 5 with $x = (4, 0)^T$. Note that the Slater condition holds for this example.

Solution:

Wolfe dual: The Wolfe dual is given by

\[
\begin{align*}
\sup \quad & f(x) + \sum_{j=1}^{m} y_j g_j(x) \\
\text{s.t.} \quad & \nabla f(x) + \sum_{j=1}^{m} y_j \nabla g_j(x) = 0 \\
& y \geq 0.
\end{align*}
\]
which in our case become

\[
\begin{align*}
&\sup x_1 + e^{x_2} + y_1(10 - 3x_1 + 2e^{x_2}) - y_2x_2 \\
&\text{s.t.} \quad 1 - 3y_1 = 0 \\
&\quad \quad e^{x_2} + 2e^{x_2}y_1 - y_2 = 0 \\
&\quad \quad x \in \mathbb{R}^2, \quad y \geq 0.
\end{align*}
\]

Wolfe dual is a non-convex problem. The first constraint gives \( y_1 = \frac{1}{3} \), and thus the second constraint becomes

\[
\frac{5}{3} e^{x_2} - y_2 = 0.
\]

Now, we can eliminate \( y_1 \) and \( y_2 \) from the objective function. We get the function

\[
f(x_1, x_2) = x_1 - x_1 + \frac{5}{3} e^{x_2} - \frac{5}{3} x_2 e^{x_2} + \frac{10}{3} \implies f(x_2) = \frac{5}{3} e^{x_2} - \frac{5}{3} x_2 e^{x_2} + \frac{10}{3}.
\]

This function has a maximum when

\[
f'(x_2) = -\frac{5}{3} x_2 e^{x_2} = 0,
\]

which is only true when \( x_2 = 0 \) and \( f(0) = 5 \). Hence the optimal value of the Wolfe dual problem is 5 when \((x, y) = (4, 0, \frac{1}{3}, \frac{5}{3})\).

**Lagrange dual**: We can double check this answer by using the Lagrange dual:

\[
\sup \psi(y) \\
\text{s.t.} \quad y \geq 0,
\]

where \( \psi(y) = \inf_x \{ f(x) + \sum_{j=1}^m y_j g_j(x) \} \).

So,

\[
\psi(y) = \inf_{x \in \mathbb{R}^2} \{ x_1 + e^{x_2} + y_1(10 - 3x_1 + 2e^{x_2}) - y_2x_2 \} \\
= \inf_{x_1 \in \mathbb{R}} \{ x_1 - 3y_1x_1 \} + \inf_{x_2 \in \mathbb{R}} \{ (1 + 2y_1)e^{x_2} - y_2x_2 \} + 10y_1.
\]

We have

\[
\inf_{x_1 \in \mathbb{R}} \{ x_1 - 3y_1x_1 \} = \begin{cases} 
0 & \text{for } y_1 = \frac{1}{3} \\
-\infty & \text{otherwise}.
\end{cases}
\]

Now, for fixed \( y_1, y_2 \) let

\[
h(x_2) = (1 + 2y_1)e^{x_2} - y_2x_2.
\]

Then \( h \) has a minimum when

\[
h'(x_2) = (1 + 2y_1)e^{x_2} - y_2,
\]

i.e., when \( x_2 = \log(\frac{y_2}{1 + 2y_1}) \). Further, \( h(x_2) = h(\log(\frac{y_2}{1 + 2y_1})) = y_2 - y_2 \log \left( \frac{y_2}{1 + 2y_1} \right) \). Hence, we have

\[
\inf_{x_2 \in \mathbb{R}} \{ (1 + 2y_1)e^{x_2} - y_2x_2 \} = y_2 - y_2 \log \left( \frac{y_2}{1 + 2y_1} \right)
\]

Thus, the Lagrange dual becomes

\[
\begin{align*}
&\sup \psi(y) = 10y_1 + y_2 - y_2 \log \left( \frac{y_2}{1 + 2y_1} \right) \\
&\text{s.t.} \quad y_1 = \frac{1}{3} \\
&\quad \quad y_2 \geq 0.
\end{align*}
\]
Now we have
\[ \frac{d}{dy_2} \psi \left( \frac{1}{3}, y_2 \right) = \log \left( \frac{3 y_2}{5} \right) = 0, \]
when \( y_2 = \frac{5}{3} \) and \( \psi \left( \frac{1}{3}, \frac{5}{3} \right) = 5. \)