Checking if Symmetric Matrix is PD or PSD by Computing its Eigenvalues

Definition Any number \( \lambda \) such that the equation \( Ax = \lambda x \) has a non-zero vector-solution \( x \) is called an eigenvalue (or a characteristic root) of the equation.

A symmetric matrix is PD if its eigenvalues \( \lambda_i > 0 \) for all \( i = 1, 2, \ldots, n \) and PSD if \( \lambda_i \geq 0 \).

How to calculate eigenvalues: \( Ax - \lambda x = 0 \Rightarrow (A - \lambda I)x = 0 \). Since \( x \) is non-zero, the determinant of \( A - \lambda I \) should vanish. Therefore all eigenvalues can be calculated as roots of the equation (which is often called the characteristic equation of \( A \)):

\[
\det(A - \lambda I) = 0.
\]

Example

Consider the Hessian matrix

\[
\nabla^2 f(x) = \begin{pmatrix}
3 & -1 & 0 \\
-1 & 3 & 0 \\
0 & 0 & 5
\end{pmatrix}
\]

Computing eigenvalues

\[
\det(\nabla^2 f(x) - \lambda I) = \begin{pmatrix}
3 - \lambda & -1 & 0 \\
-1 & 3 - \lambda & 0 \\
0 & 0 & 5 - \lambda
\end{pmatrix} = (5 - \lambda)(\lambda^2 - 6\lambda + 8) = (5 - \lambda)(\lambda - 2)(\lambda - 4) = 0.
\]

Therefore, the eigenvalues are \( \lambda = 2 \), \( \lambda = 4 \) and \( \lambda = 5 \). As all of them are strictly positive, the Hessian is positive definite (PD).

Properties of Convex Functions

- if \( f \) is convex function, its sublevel set \( f(x) \leq \alpha \) is convex;
• positive multiple of convex function is convex:
  \( f \text{ convex}, \alpha \geq 0 \implies \alpha f \text{ convex} \)

• sum of convex functions is convex:
  \( f_1, f_2 \text{ convex} \implies f_1 + f_2 \text{ convex} \)

• pointwise maximum of convex functions is convex:
  \( f_1, f_2 \text{ convex} \implies \max \{f_1(x), f_2(x)\} \text{ convex} \)
  (corresponds to intersections of epigraphs)

• affine transformation of domain:
  \( f \text{ convex} \implies f(Ax + b) \text{ convex} \)

Composition Rules

Composite function

\[ f(x) = h(g(x)) \]

is convex if:

• \( g \text{ convex}; h \text{ convex nondecreasing} \)

• \( g \text{ concave}; h \text{ convex nonincreasing} \)

Proof (differentiable functions, \( x \in \mathbb{R} \)):

\[ f'' = h''(g')^2 + g''h' \]

Examples:

• \( f(x) = e^{g(x)} \text{ is convex if } g \text{ is convex} \)

• \( f(x) = 1/g(x) \text{ is convex if } g \text{ is concave, positive} \)

• \( f(x) = g(x)^p, p \geq 1 \text{ is convex if } g(x) \text{ is convex, positive} \)
Convexity of Optimization Problems

Show that the function $e^x + \frac{1}{2}x^2$ is convex and solve $\min e^x + \frac{1}{2}x^2$.

First derivative: A function is increasing if $f' > 0$, decreasing if $f' < 0$ and neither if $f' = 0$.

Second derivative: A function is convex if $f'' > 0$ and concave if $f'' < 0$.

Answer: $f'(x) = e^x + x$ and $f''(x) = e^x + 1 > 0$. So, $f$ is convex.

Thus, we can find a solution to an optimization problem by solving $f'(x) = 0$, given $f$ is convex.

Standard form of optimization problems:

Convex optimization problem:

$$\begin{align*}
\min & \quad f(x) & \text{convex} \\
\text{s.t.} & \quad h_i(x) = 0, \quad i = 1, 2, \ldots & \text{linear} \\
& \quad g_i(x) \leq 0, \quad i = 1, 2, \ldots & \text{convex} \\
& \quad x \in C & C \text{ convex}
\end{align*}$$

Show that the following problem is a convex optimization problem:

$$\begin{align*}
\min & \quad \sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2 + (x_3 - x_6)^2} \\
\text{s.t.} & \quad x_1^2 + x_2^2 + x_3^2 \leq 5 \\
& \quad (x_4 - 3)^2 + x_5^2 \leq 1 \\
& \quad 4 \leq x_6 \leq 8
\end{align*}$$

Objective function:

$$\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2 + (x_3 - x_6)^2} = \frac{\begin{bmatrix} x_1 - x_4 \\ x_2 - x_5 \\ x_3 - x_6 \end{bmatrix}}{\begin{bmatrix} x_1 - x_4 \\ x_2 - x_5 \\ x_3 - x_6 \end{bmatrix}}$$

Norm is a convex function.

Hessian of the function $g_1(x) = x_1^2 + x_2^2 + x_3^2 - 5$ is

$$\nabla^2 g_1(x) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \succeq 0$$

Hessian is PSD and so the function $g_1(x)$ is convex. Consequently, set $g_1(x) \leq 0$ is convex.
Hessian of the function \( g_2(x) = (x_4 - 3)^2 + x_3^2 - 1 \) is

\[
\nabla^2 g_2(x) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \succeq 0
\]

Hessian is PSD and so the function \( g_2(x) \) is convex. Consequently, set \( g_2(x) \leq 0 \) is convex.

Set \( 4 \leq x_6 \leq 8 \) is the intersection of hyperplanes and so convex.

As the objective function is convex and all sets in the constraints are also convex, the optimization problem is convex.

Find the local/global minimum of the functions if exists:

- \( e^x \)

\( f'(x) = e^x, f''(x) = e^x > 0 \) - strictly convex function. \( f'(x) = e^x = 0 \implies x \to -\infty \)

- \(-\ln x \)

\( f'(x) = -1/x, f''(x) = 1/x^2 > 0 \) - strictly convex function. \( f'(x) = -1/x = 0 \implies x \to \infty \)
• $x \ln x$

$f'(x) = 1 + \ln x$, $f''(x) = 1/x > 0$ on the domain of $\ln x \Rightarrow$ strictly convex function. $f'(x) = 1 + \ln x = 0 \implies x = 0.37$ (global minimum).

![Graph of $x \ln x$](image)

• $-\sqrt{x}$ when $x \geq 0$

$f'(x) = -0.5x^{-1/2}$, $f''(x) = 0.25x^{-3/2} \geq 0$ when $x \geq 0 \Rightarrow$ convex function. $f'(x) = -0.5x^{-1/2} = 0 \implies x \to \infty$.

![Graph of $-\sqrt{x}$](image)

• $(x_1 - 2)^2 + (x_2 + 1)^2 - 2$

\[
\nabla f(x) = \begin{pmatrix} 2(x_1 - 2) \\ 2(x_2 + 1) \end{pmatrix}
\]

\[
\nabla^2 f(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \succ 0
\]

As $\nabla^2 f(x)$ is PD, $f(x)$ is strictly convex function.

\[
\nabla f(x) = 0 \implies x = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \text{ (global minimum).}
\]
\begin{itemize}
\item $(x - 2)^4 - 10(x - 2)^2$
\[ f'(x) = 4(x - 2)^3 - 20(x - 2), \quad f''(x) = 12(x - 2)^2 - 20 \] - non-convex, non-concave function.
\end{itemize}