Tutorial 2

Fundamentals

CS/SWE 4/6TE3, CES 722/723 September 21, 2010

Checking if Symmetric Matrix is PD or PSD by Computing its Eigenvalues

Definition Any number λ such that the equation $Ax = \lambda x$ has a non-zero vector-solution x is called an eigenvalue (or a characteristic root) of the equation.

A symmetric matrix is PD if its eigenvalues $\lambda_i > 0$ for all $i = 1, 2, \ldots, n$ and PSD if $\lambda_i \geq 0$.

How to calculate eigenvalues: $Ax - \lambda x = 0 \Rightarrow (A - \lambda I)x = 0$. Since x is non-zero, the determinant of $(A - \lambda I)$ should vanish. Therefore all eigenvalues can be calculated as roots of the equation (which is often called the characteristic equation of A):

$$\det(A - \lambda I) = 0.$$

Example

Consider the Hessian matrix

$$\nabla^2 f(x) = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Computing eigenvalues

$$\det(\nabla^2 f(x) - \lambda I) = \begin{pmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{pmatrix} = (5-\lambda)(\lambda^2 - 6\lambda + 8) = (5-\lambda)(\lambda-2)(\lambda-4) = 0.$$

Therefore, the eigenvalues are $\lambda = 2$, $\lambda = 4$ and $\lambda = 5$. As ll of them are strictly positive, the Hessian is positive definite (PD).

Properties of Convex Functions

- if f is convex function, its sublevel set $f(x) \leq \alpha$ is convex;

- positive multiple of convex function is convex:

  f convex, $\alpha \geq 0 \implies \alpha f$ convex

- sum of convex functions is convex:

  f_1, f_2 convex $\implies f_1 + f_2$ convex

- pointwise maximum of convex functions is convex:

  f_1, f_2 convex $\implies \max\{f_1(x), f_2(x)\}$ convex

  (corresponds to intersections of epigraphs)

- affine transformation of domain:

  f convex $\implies f(Ax + b)$ convex

Composition Rules

Composite function

$$f(x) = h(g(x))$$

is convex if:

- g convex; h convex nondecreasing
- g concave; h convex nonincreasing

Proof (differentiable functions, $x \in \Re$):

$$f'' = h''(g')^2 + g''h'$$

Examples:

- $f(x) = e^{g(x)}$ is convex if g is convex
- $f(x) = 1/g(x)$ is convex if g is concave, positive
- $f(x) = g(x)^p,\ p \geq 1$ is convex if $g(x)$ is convex, positive

Convexity of Optimization Problems

Show that the function $e^x + \frac{1}{2}x^2$ is convex and solve $\min\ e^x + \frac{1}{2}x^2$.

First derivative: A function is increasing if $f' > 0$, decreasing if $f' < 0$ and neither if $f' = 0$.

Second derivative: A function is convex if $f'' > 0$ and concave if $f'' < 0$.

Answer: $f'(x) = e^x + x$ and $f''(x) = e^x + 1 \ > \ 0$. So, f is convex.

Thus, we can find a solution to an optimization problem by solving $f'(x) = 0$, given f is convex.

Standard form of optimization problems:

Convex optimization problem:

$$\begin{array}{rcll} \min f(x) & & & \qquad \textit{convex} \\ s.t \quad h_i(x) & = & 0, \quad i = 1, 2, \ldots & \qquad \textit{linear} \\ g_i(x) & \leq & 0, \quad i = 1, 2, \ldots & \qquad \textit{convex} \\ x \in C & & & \qquad C \textit{ convex} \end{array}$$

Show that the following problem is a convex optimization problem:

$$\begin{array}{ll} \min & \sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2 + (x_3 - x_6)^2} \\ \text{s.t.} & x_1^2 + x_2^2 + x_3^2 \quad \leq \quad 5 \\ & (x_4 - 3)^2 + x_5^2 \quad \leq \quad 1 \\ & 4 \leq x_6 \leq 8 \end{array}$$

Objective function:

$$\sqrt{(x_1 - x_4)^2 + (x_2 - x_5)^2 + (x_3 - x_6)^2} = \left\| \begin{array}{c} x_1 - x_4 \\ x_2 - x_5 \\ x_3 - x_6 \end{array} \right\|$$

Norm is a convex function.

Hessian of the function $g_1(x) = x_1^2 + x_2^2 + x_3^2 - 5$ is

$$\nabla^2 g_1(x) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \succeq 0$$

Hessian is PSD and so the function $g_1(x)$ is convex. Consequently, set $g_1(x) \leq 0$ is convex.

Hessian of the function $g_2(x) = (x_4 - 3)^2 + x_5^2 - 1$ is

$$\nabla^2 g_2(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \succeq 0$$

Hessian is PSD and so the function $g_2(x)$ is convex. Consequently, set $g_2(x) \le 0$ is convex.

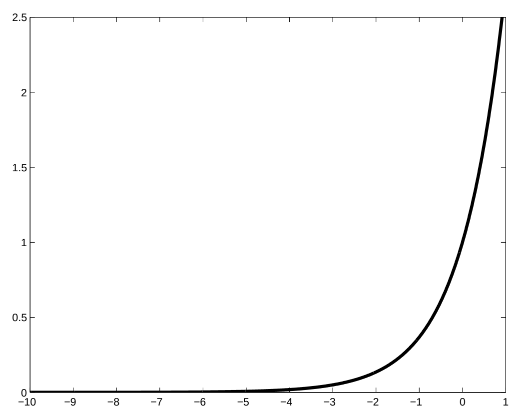
Set $4 \le x_6 \le 8$ is the intersection of hyperplanes and so convex.

As the objective function is convex and all sets in the constraints are also convex, the optimization problem is convex.

Find the local/global minimum of the functions if exists:

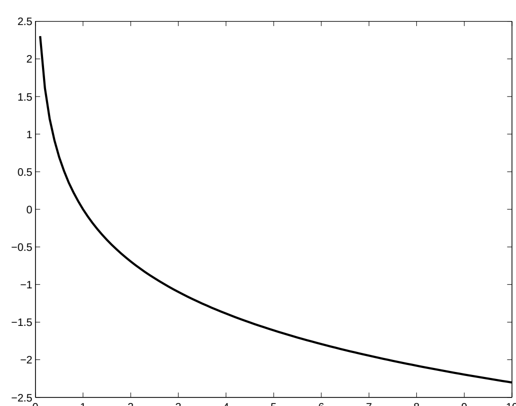
- e^x

  $f'(x) = e^x$, $f''(x) = e^x > 0$ - strictly convex function. $f'(x) = e^x = 0 \implies x \to -\infty$



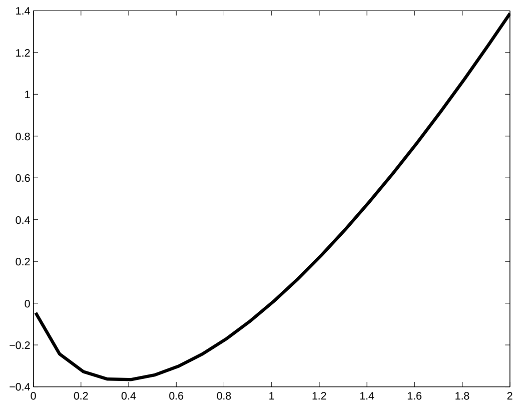
- $-\ln x$

  $f'(x) = -1/x$, $f''(x) = 1/x^2 > 0$ - strictly convex function. $f'(x) = -1/x = 0 \implies x \to \infty$



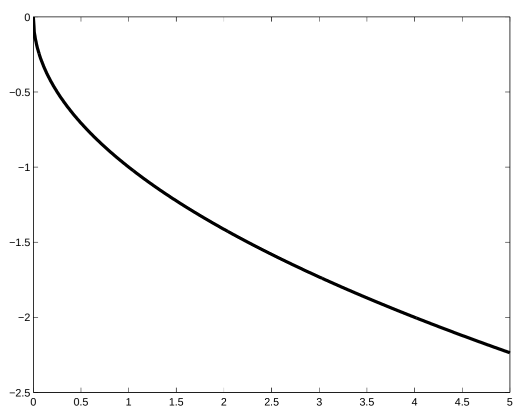
- $x \ln x$

  $f'(x) = 1 + \ln x$, $f''(x) = 1/x > 0$ on the domain of $\ln x \Rightarrow$ strictly convex function. $f'(x) = 1 + \ln x = 0 \implies x = 0.37$ (global minimum).



- $-\sqrt{x}$ when $x \geq 0$

  $f'(x) = -0.5x^{-1/2}$, $f''(x) = 0.25x^{-3/2} \geq 0$ when $x \geq 0 \Rightarrow$ convex function. $f'(x) = -0.5x^{-1/2} = 0 \implies x \to \infty$.



- $(x_1 - 2)^2 + (x_2 + 1)^2 - 2$

$$\nabla f(x) = \begin{pmatrix} 2(x_1 - 2) \\ 2(x_2 + 1) \end{pmatrix}$$

$$\nabla^2 f(x) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \succ 0$$

  As $\nabla^2 f(x)$ is PD, $f(x)$ is strictly convex function.

$$\nabla f(x) = 0 \quad \implies \quad x = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{(global minimum).}$$

- $(x-2)^4 - 10(x-2)^2$

  $f'(x) = 4(x-2)^3 - 20(x-2)$, $f''(x) = 12(x-2)^2 - 20$ - non-convex, non-concave function.