CS/SE4-6TE3, CES 722/723: Tutorial 5

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- An **eigenvalue** of a matrix is a number λ such that the equation $\mathbf{Ax} = \lambda\mathbf{x}$ has a non-zero vector-solution x.

- **A symmetric matrix is Positive Definite (PD)** if all of eigenvalues are positive.

  A symmetrix matrix is Positive Semi-Definite (PSD) if all of eigenvalues are non-negative.

- **Calculate eigenvalues:** $\mathbf{Ax} - \lambda\mathbf{x} = 0 \Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$, here $\mathbf{I}$ is the identity matrix which has the same size of the matrix $\mathbf{A}$. Because $\mathbf{x}$ is non-zero, the determinant of $(\mathbf{A} - \lambda\mathbf{I})$ should be 0. Therefore all eigenvalues can be calculated as roots of the equation (also called the characteristic equation of the matrix $\mathbf{A}$):

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

- **Example** Consider the Hessian matrix

$$\nabla^2 f(\mathbf{x}) = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

  Computing eigenvalues:

$$\begin{aligned} det(\nabla^2 f(\mathbf{x}) - \lambda\mathbf{I}) &= det\left(\begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= det\begin{pmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{pmatrix} \\ &= (5-\lambda)((3-\lambda)^2 - 1) \\ &= (5-\lambda)(\lambda-2)(\lambda-4) \\ &= 0 \end{aligned}$$

  Therefore, the eigenvalues are $\lambda = 2, 4, 5$. As all of eigenvalues are strictly positive, the Hessian is PD.