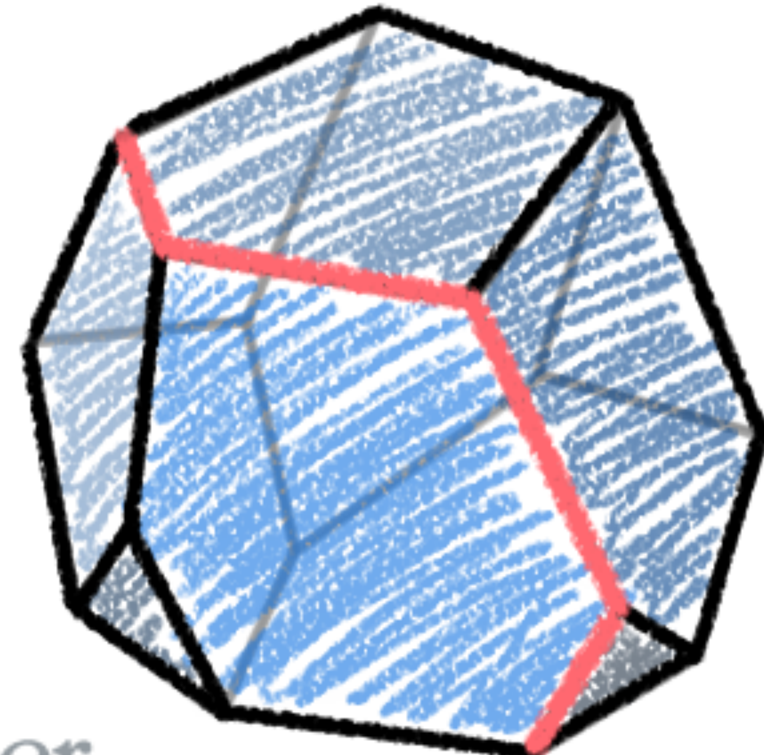
Algorithmic and geometric aspects of combinatorial and continuous optimization



Antoine Deza, McMaster

based on joint works with

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George Manoussakis, Orsay

Shinji Mizuno, Tokyo Tech.

Shmuel Onn, Technion

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linear optimization

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** (full row-rank) matrix ***A*** find, in any, a ***d***-dimensional vector ***x*** such that :

$\boldsymbol{Ax} = \boldsymbol{b}$	$\boldsymbol{Ax} = \boldsymbol{b}$
	$\boldsymbol{x} \geq \boldsymbol{0}$
linear algebra	**linear optimization**

"*Can* ***linear optimization*** *be solved in* ***strongly polynomial*** time?" is listed by Smale (Fields Medal 1966) as one of the top problems for the XXI century

Polynomial : execution time bounded by a ***polynomial*** in ***n***, ***d***, and ***input data length L***

linear optimization

Given an n-dimensional vector $\boldsymbol{b}$ and an n x d (full row-rank) matrix $\boldsymbol{A}$ find, in any, a d-dimensional vector $\boldsymbol{x}$ such that :

$\boldsymbol{Ax} = \boldsymbol{b}$	$\boldsymbol{Ax} = \boldsymbol{b}$ $\boldsymbol{x} \geq \mathbf{0}$
linear algebra	**linear optimization**

*"Can **linear optimization** be solved in **strongly polynomial*** time?" is listed by Smale (Fields Medal 1966) as one of the top problems for the XXI century

Strongly polynomial : ***polynomial*** time; number of arithmetic operations bounded by a polynomial in the ***dimension*** of the problem (***independent*** from the ***input data length*** $\boldsymbol{L}$)

linear optimization algorithms

Given an n-dimensional vector $\boldsymbol{b}$ and an n x d (full row-rank) matrix $\boldsymbol{A}$ and a d-dimensional cost vector $\boldsymbol{c}$, solve : { max $\boldsymbol{c}^T\boldsymbol{x}$: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \mathbf{0}$ }

Simplex methods (Dantzig 1947) pivot-based, combinatorial, ***not proven to be polynomial***, efficient in practice

Ellipsoid methods (Khachiyan 1979)
polynomial ⇒ ***linear optimization is polynomial time solvable***

Interior point methods (Karmarkar 1984)
path-following, ***polynomial***, efficient in practice

Primal-dual interior point (Kojima-Mizuno-Yoshise 1989)

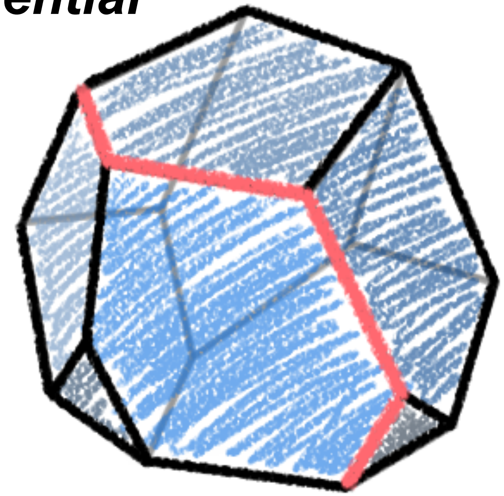
Criss-cross (Terlaky 1983, Wang 1985, Chang 1979)
Volumetric (Vaidya-Atkinson 1993, Anstreicher 1997)
Monotonic build-up simplex (Anstreicher-Terlaky 1994)
…..

linear optimization algorithms
simplex methods

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** (full row-rank) matrix ***A*** and a ***d***-dimensional cost vector ***c***, solve : $\{ \max \boldsymbol{c}^{T}\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \mathbf{0} \}$

Simplex methods (Dantzig 1947): pivot-based, combinatorial, ***not proven to be polynomial***, efficient in practice

- start from a ***feasible basis***
- use a ***pivot rule***
- find an optimal solution after a ***finite number*** of iterations
- most known pivot rules are known to be ***exponential*** (worst case); ***efficient*** implementations exist



Given a ... atrix ***A***

and a *c* ... $x \geq \mathbf{0}$ }

Simple ... ***not***

proven

- start
- use
- find
- mos

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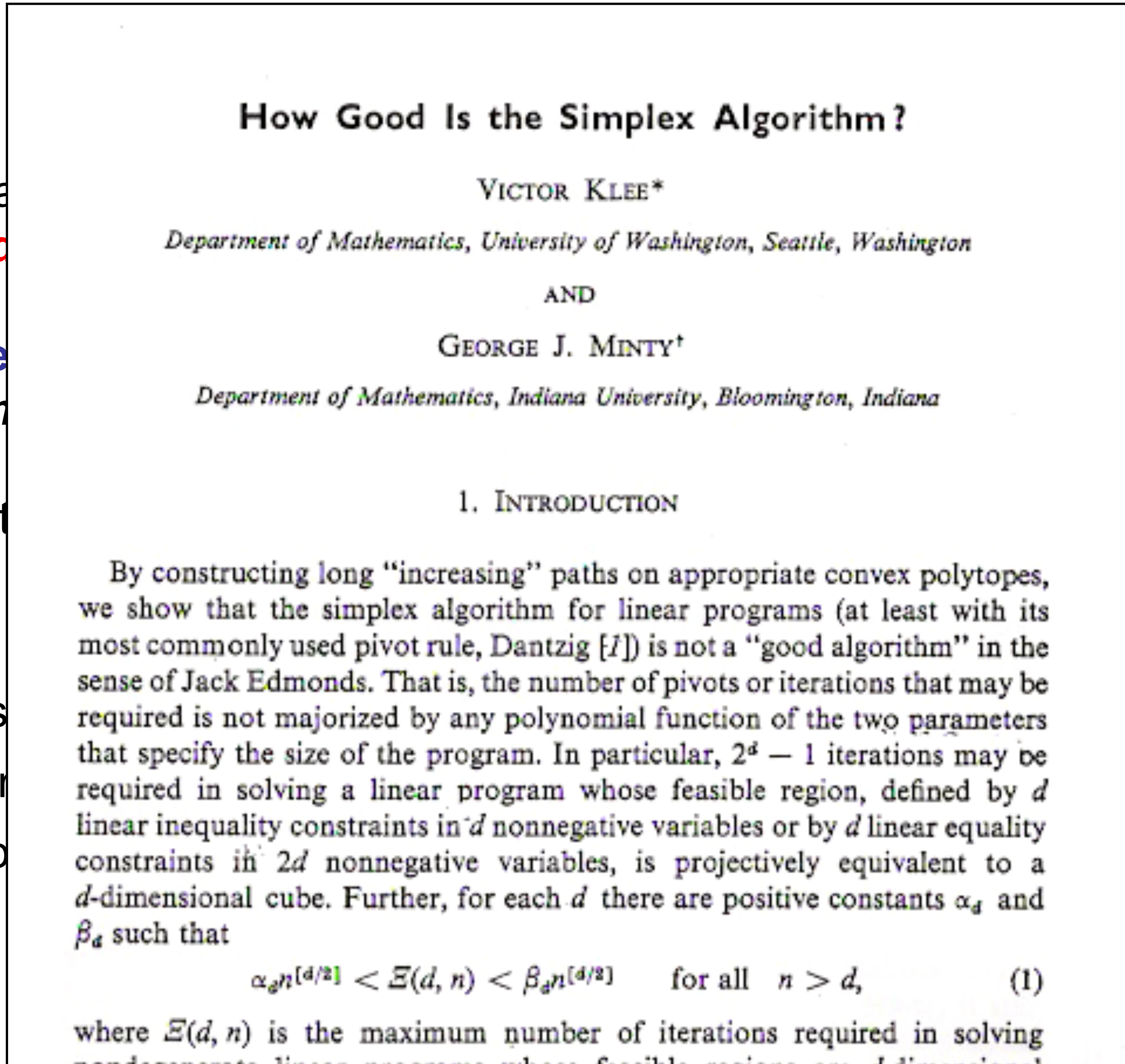

How Good Is the Simplex Algorithm?

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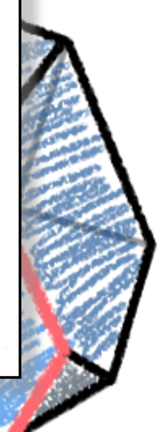
1. INTRODUCTION

By constructing long "increasing" paths on appropriate convex polytopes, we show that the simplex algorithm for linear programs (at least with its most commonly used pivot rule, Dantzig [*1*]) is not a "good algorithm" in the sense of Jack Edmonds. That is, the number of pivots or iterations that may be required is not majorized by any polynomial function of the two parameters that specify the size of the program. In particular, $2^d - 1$ iterations may be required in solving a linear program whose feasible region, defined by d linear inequality constraints in d nonnegative variables or by d linear equality constraints in $2d$ nonnegative variables, is projectively equivalent to a d-dimensional cube. Further, for each d there are positive constants α_d and β_d such that

$$\alpha_d n^{[d/2]} < \Xi(d, n) < \beta_d n^{[d/2]} \quad \text{for all} \quad n > d, \tag{1}$$

where $\Xi(d, n)$ is the maximum number of iterations required in solving nondegenerate linear programs whose feasible regions are d-dimensional



linear optimization algorithms
simplex methods

Klee-Minty 1972: edge-path followed by the simplex method with Dantzig's rule visits the 2^{d} vertices of a ***combinatorial*** cube ($n = 2d$)
$\Rightarrow$ $2^{d} - 1$ pivots required to reach the optimum

Zadeh 1973 : bad network problems

Zadeh 1980 : *deformed products* and *least entered* rule

Amenta-Ziegler 1999 : *deformed products*

Friedmann 2011 : *least entered* rule *is superpolynomial*

Surveys : Terlaky-Zhang 1993, Ziegler 2004, Meunier 2013

… Avis-Friedmann 2016…

Dear Victor,

Please post this offer of $1000 to the first person who can find a counterexample to the least entered rule or prove it to be polynomial. The least entered rule enters the improving variable which has been entered least often.

Sincerely,

Norman Zadeh

Zadeh’s offer (Ziegler 2004)
(Avis’ postface to Zadeh 1980 report, 2009 reprint)

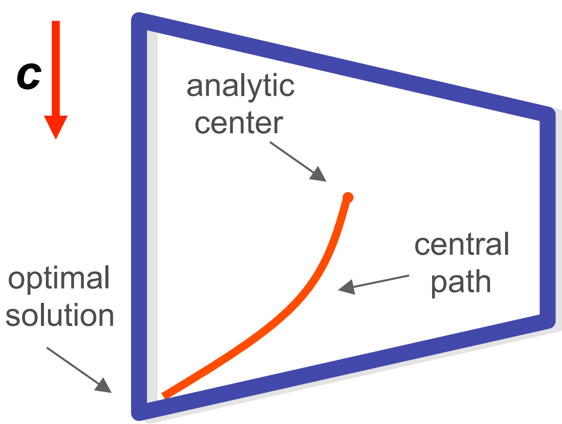
David Avis, Norman Zadeh, Oliver Friedmann, Russ Caflish (IPAM 2011)

linear optimization algorithms
(central path following) interior point methods

Given an n-dimensional vector $\boldsymbol{b}$ and an n x d (full row-rank) matrix $\boldsymbol{A}$ and a d-dimensional cost vector $\boldsymbol{c}$, solve : $\{ \max \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} : \boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0} \}$

Interior Point Methods :
path-following, ***polynomial***, efficient in practice

- start from the ***analytic center***
- follow the ***central path***
- converge to an optimal solution in $O(\sqrt{n}L)$ iterations
  (L: input data length)



$$\min \quad \boldsymbol{c}^{\mathrm{T}}x - \mu \sum_i \ln(b - Ax)_i$$

μ : central path parameter
$x \in \boldsymbol{P} : Ax \leq b$

linear optimization
(some) combinatorial and geometric parameters

Tardos 1985: algorithm polynomial in $\boldsymbol{n}$, $\boldsymbol{d}$, and $L_{\boldsymbol{A}}$ (size of $\boldsymbol{A}$)
⇒ strongly polynomial for minimum cost flow, bipartite matching etc.
… Orlin 1986, Kitahara-Mizuno 2011, Mizuno 2014, Mizuno-Sukegawa-Deza 2015...

Ye 2011 : strongly polynomial simplex for Markov Decision Problem

Vavasis-Ye 1996 : $O(\boldsymbol{d}^{3.5} \log(\boldsymbol{d}\,\chi_{\boldsymbol{A}}))$ primal-dual interior point method
…Megiddo-Mizuno-Tsuchiya 1998, Monteiro-Tsuchiya 2003…

Bonifas-Summa-Eisenbrand-Hähnle-Niemeier 2014: $O(\boldsymbol{d}^{4}\Delta_{\boldsymbol{A}}^{2} \log(\boldsymbol{d}\,\Delta_{\boldsymbol{A}}))$ diameter (${\Delta}_{\boldsymbol{A}}$ largest sub-determinant norm; Dyer-Frieze 1994)

Dadush-Hähnle 2015: $O(\boldsymbol{d}^{3}/\delta_{\boldsymbol{A}} \log(\boldsymbol{d}/\delta_{\boldsymbol{A}}))$ expected (shadow vertex) simplex pivots ($\delta_{\boldsymbol{A}}$ *curvature* ; $1/\delta_{\boldsymbol{A}} \leq \boldsymbol{d}\,\Delta_{\boldsymbol{A}}^{2}$)

….

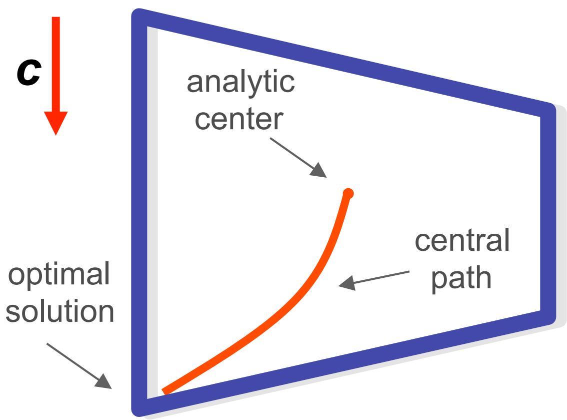
linear optimization diameter and curvature

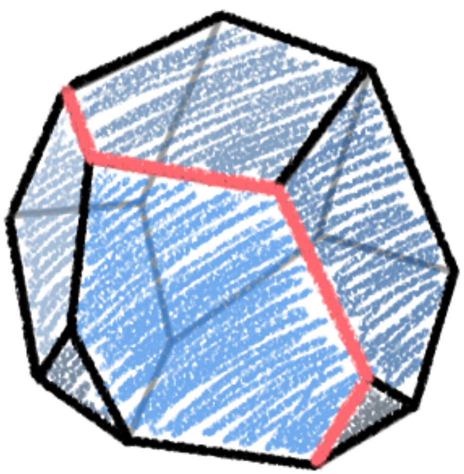
Diameter (*of a polytope*) :

lower bound for the number of iterations for *pivoting* **simplex methods**

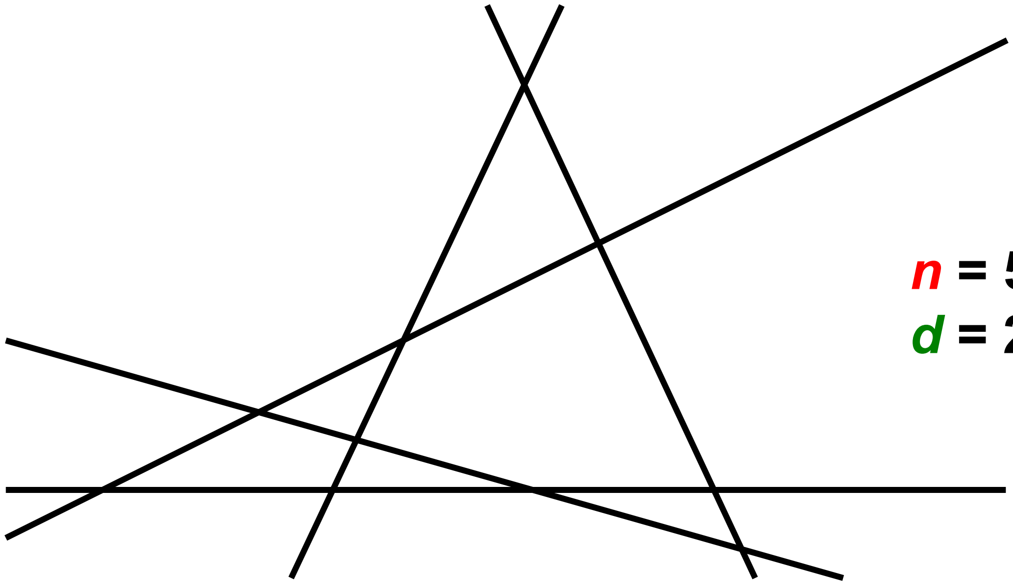
Curvature (*of the central path associated to a polytope*) :

large curvature indicates large number of iterations for *path following* **interior point methods**





linear optimization : diameter and curvature

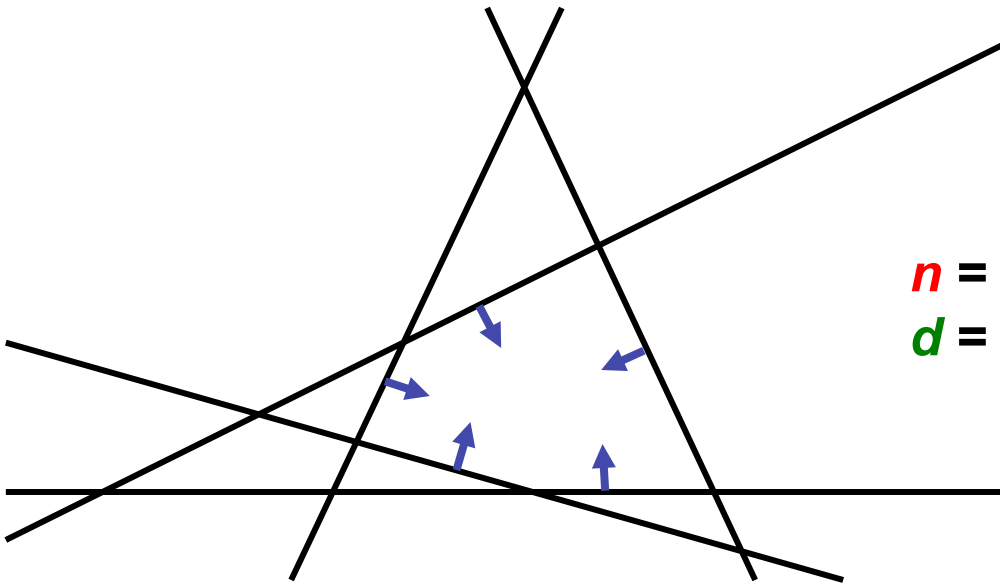


***n* = 5** : *inequalities*
***d* = 2** : *dimension*

Polytope *P* defined by ***n*** **inequalities** in **dimension *d***

❖ polytope : *bounded* polyhedron

linear optimization : diameter and curvature



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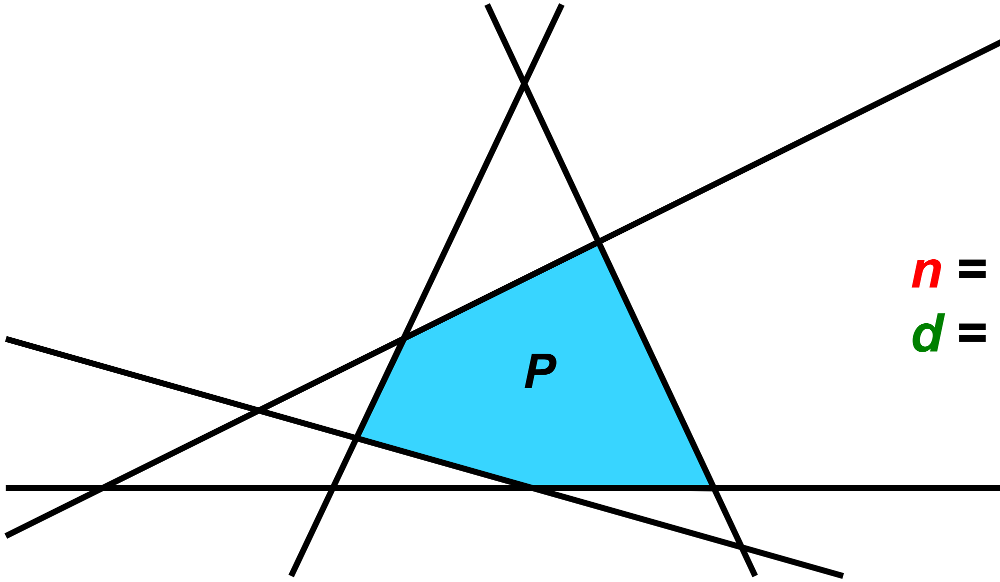
linear optimization : diameter and curvature



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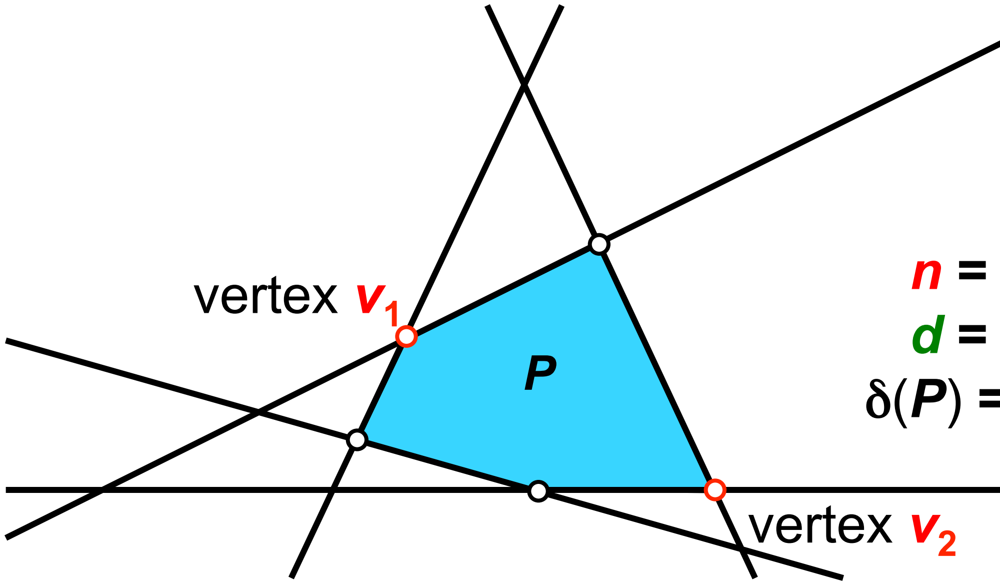
Polytope *P* defined by ***n*** **inequalities** in **dimension *d***

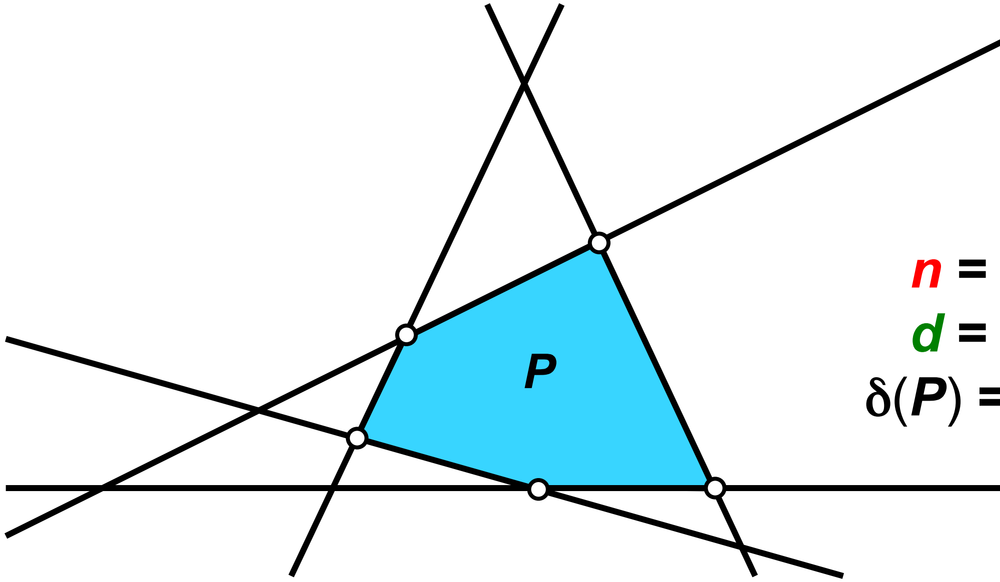
linear optimization : *diameter and curvature*



n **= 5** : *inequalities*
d **= 2** : *dimension*
$\delta(P)$ **= 2** : *diameter*

Diameter $\delta(P)$: smallest number such that **any two vertices** (v_1,v_2) can be connected by a **path with at most** $\delta(P)$ **edges**

linear optimization : diameter and curvature



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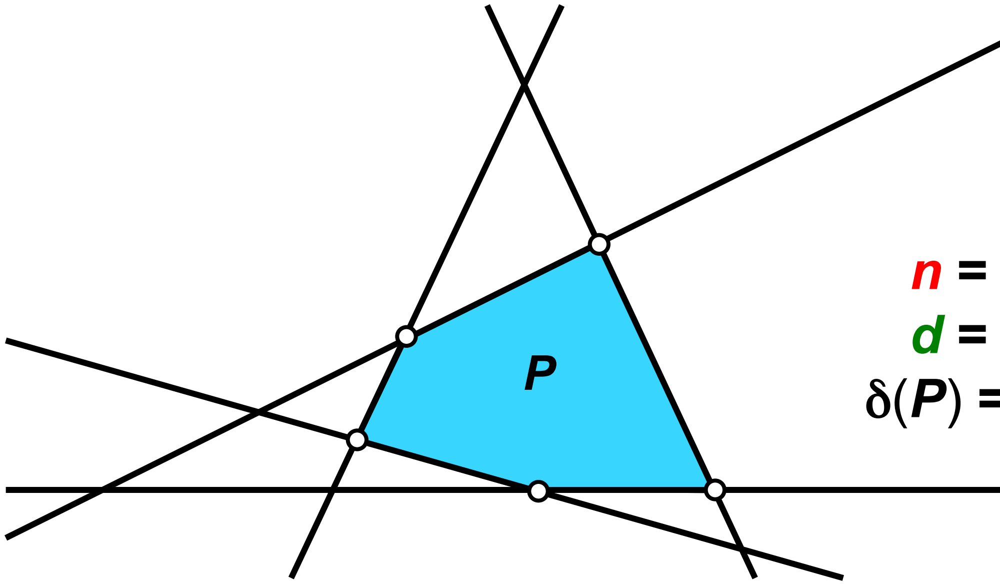
Hirsch Conjecture 1957 : $\delta(P) \leq n - d$

linear optimization : diameter and curvature



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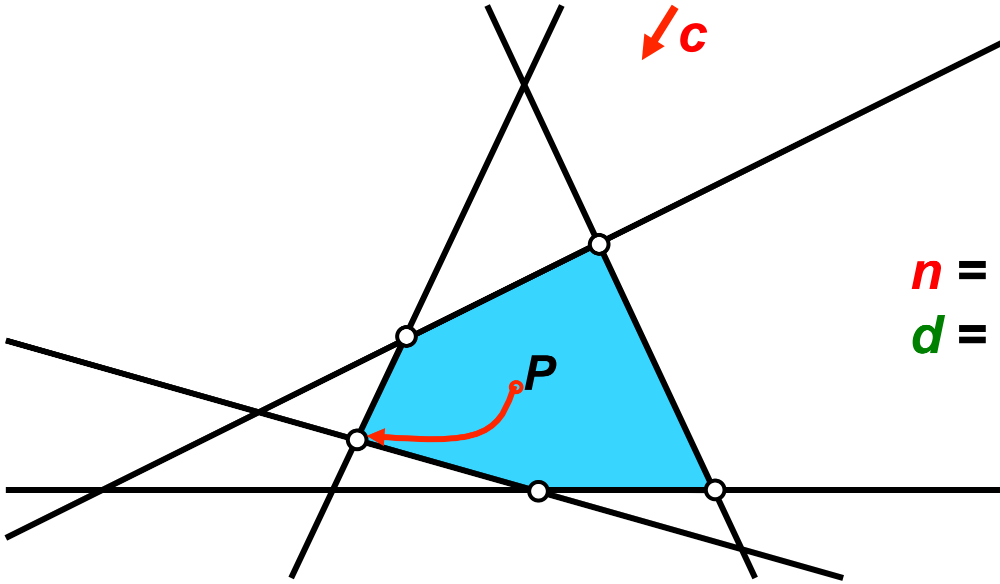
Hirsch Conjecture 1957 : $\delta(\boldsymbol{P}) \leq \boldsymbol{n} - \boldsymbol{d}$

- ***disproved*** by Santos 2012 using construction with ***n*** **= 2*d***

linear optimization : diameter and curvature

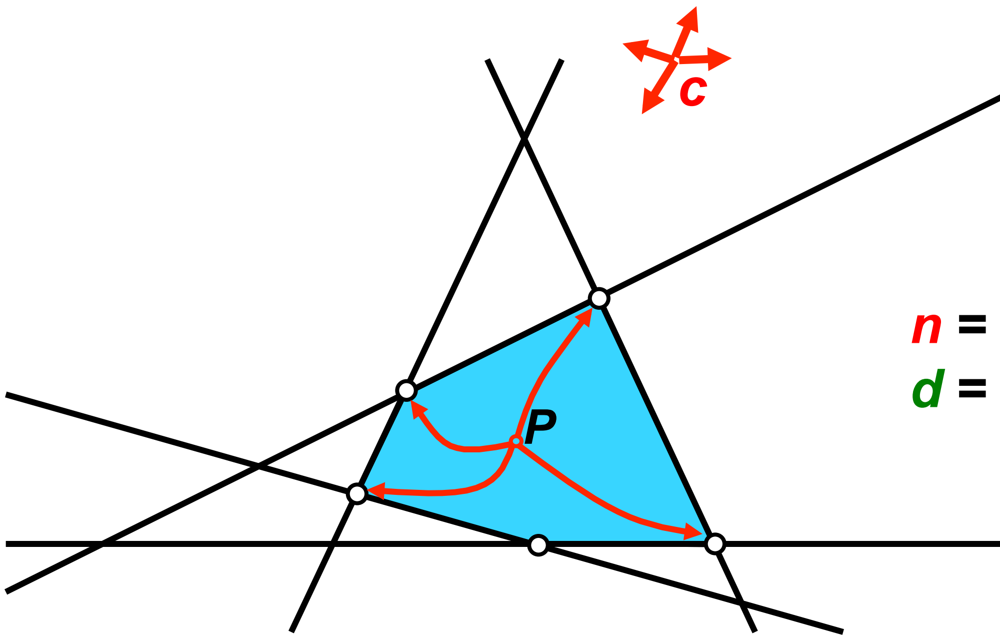


***n* = 5** : *inequalities*
***d* = 2** : *dimension*

$\lambda^{c}(\boldsymbol{P})$: total **curvature** of the primal central path of $\{ \max \boldsymbol{c}^{T}x : x \in \boldsymbol{P} \}$

❖ $\lambda^{c}(\boldsymbol{P})$: *redundant* inequalities count

linear optimization : diameter and curvature



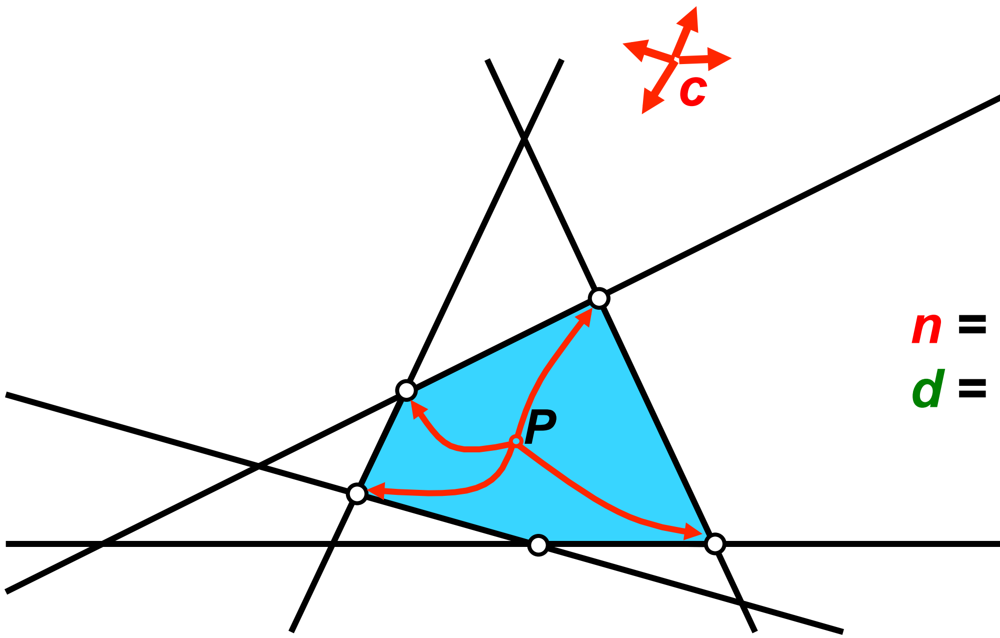
$n = 5$: *inequalities*

$d = 2$: *dimension*

$\lambda^{c}(P)$: total curvature of the primal central path of $\{ \max c^{T}x : x \in P \}$

$\lambda(P)$: *largest* total **curvature** $\lambda^{c}(P)$ *over of all possible* c

linear optimization : diameter and curvature



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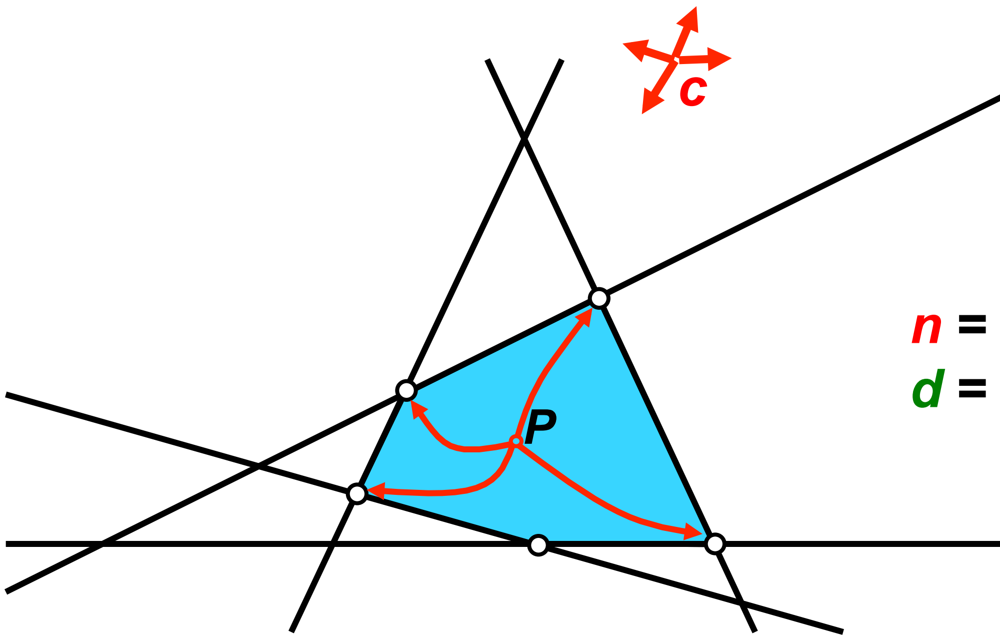
Continuous analogue of Hirsch Conjecture: $\lambda(P) = O(n)$
(Deza-Terlaky-Zinchenko 2008)

❖ Dedieu-Shub 2005 hypothesis : $\lambda(P) = O(d)$

linear optimization : diameter and curvature



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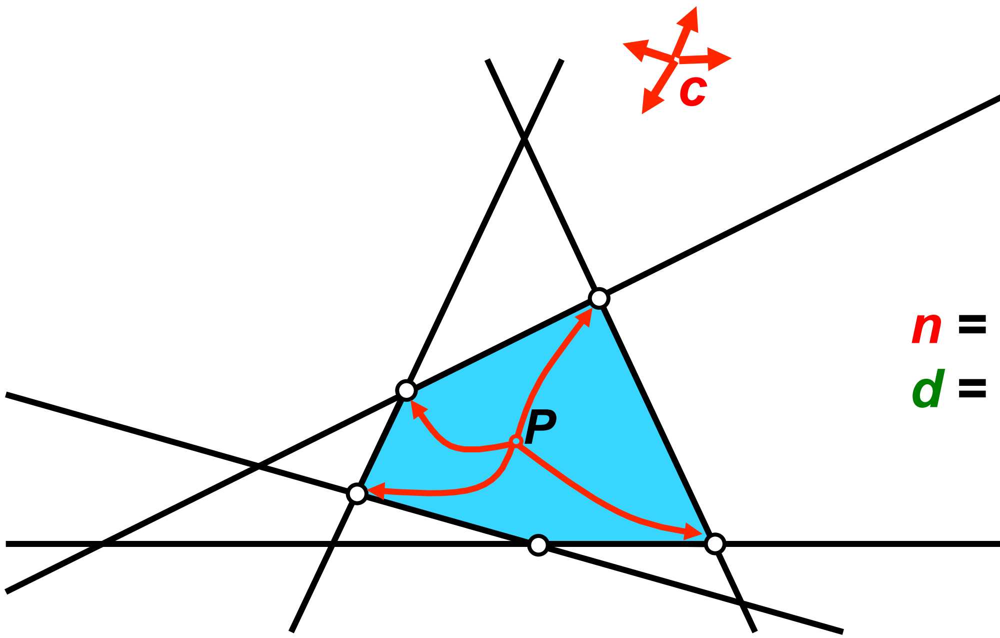
- Dedieu-Shub 2005 hypothesis : $\lambda(P) = O(d)$
- Deza-Terlaky-Zinchenko 2008 : polytope such that: $\lambda(P) = \Omega(2^{d})$

linear optimization : diameter and curvature



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Continuous analogue of Hirsch Conjecture: $\lambda(P) = O(n)$
(Deza-Terlaky-Zinchenko 2008)

- *disproved* by Allamigeon-Benchimol-Gaubert-Joswig 2014

linear optimization : diameter and curvature

Dedieu-Shub 2005 hypothesised $\lambda(\mathbf{P}) = O(d)$
Dedieu-Malajovich-Shub 2005 proved it is true *on average*
(de Loera-Sturmfels-Vinzant 2012)

Deza-Terlaky-Zinchenko 2008: $\boldsymbol{P}$ with exponential $\lambda(\boldsymbol{P})$ and $n = \Omega(2^d)$

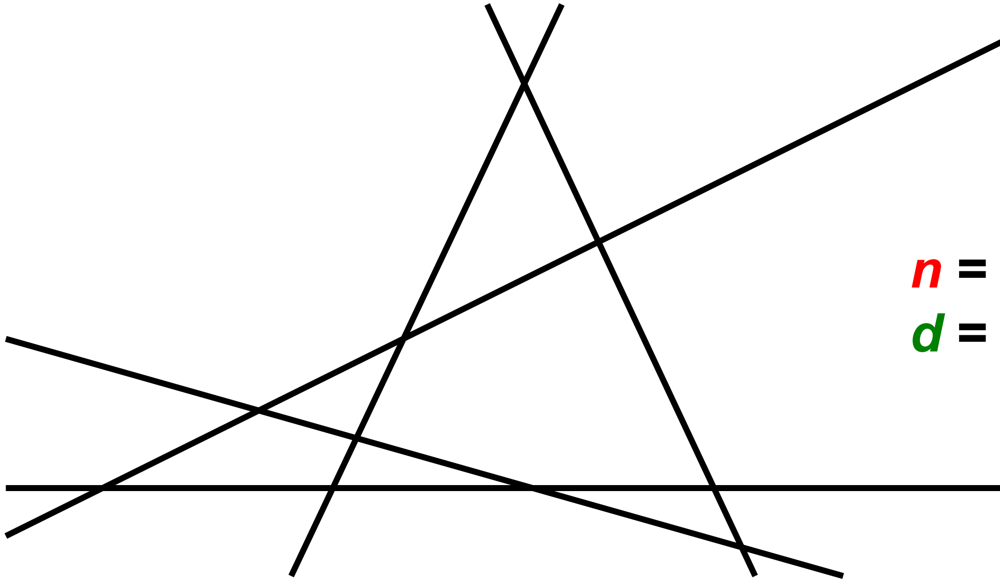
Continuous analogue of Hirsch Conjecture: $\lambda(P) = O(\mathrm{poly}(n,d))$

Allamigeon-Benchimol-Gaubert-Joswig 2014 : linear optimization instance ($2n \approx 3d$) for which central-path following methods require $\Omega(2^{d/2})$ iterations

⇒ ***path-following interior-point methods are not strongly polynomial***

Result obtained using ***tropical geometry***, which reduces the complexity analysis to a ***combinatorial*** problem

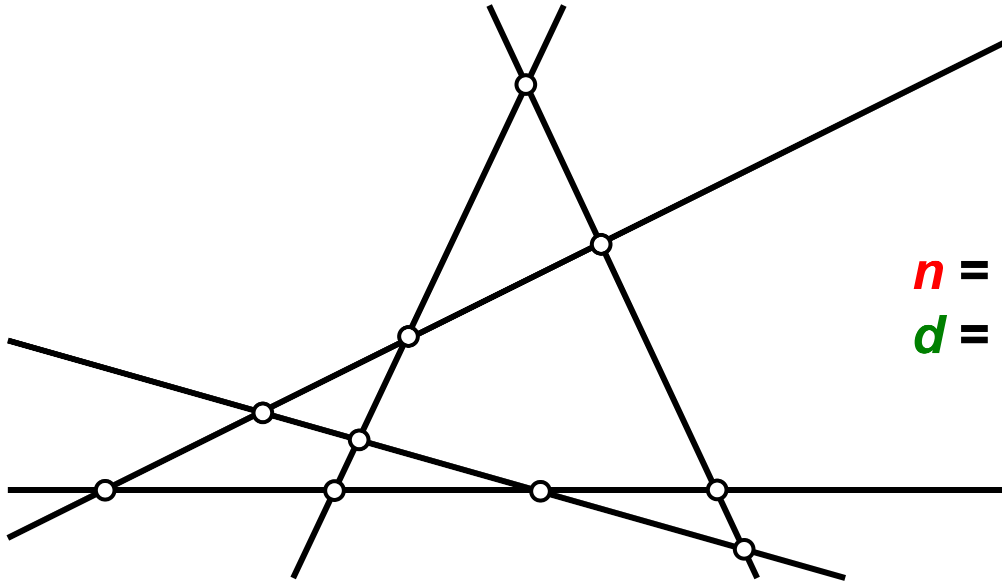
linear optimization : diameter and curvature



***n* = 5** : *hyperplanes*
***d* = 2** : *dimension*

Arrangement *A* defined by ***n*** **hyperplanes** in **dimension *d***

linear optimization : diameter and curvature

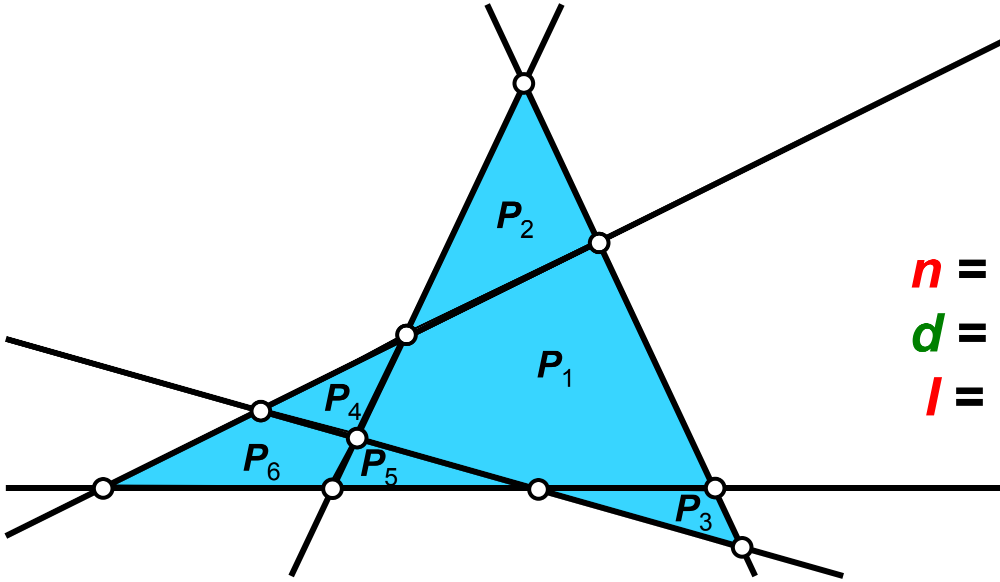


***n* = 5** : *hyperplanes*
***d* = 2** : *dimension*

Simple arrangement:

n > ***d*** and any ***d*** hyperplanes **intersect** at a **unique distinct point**

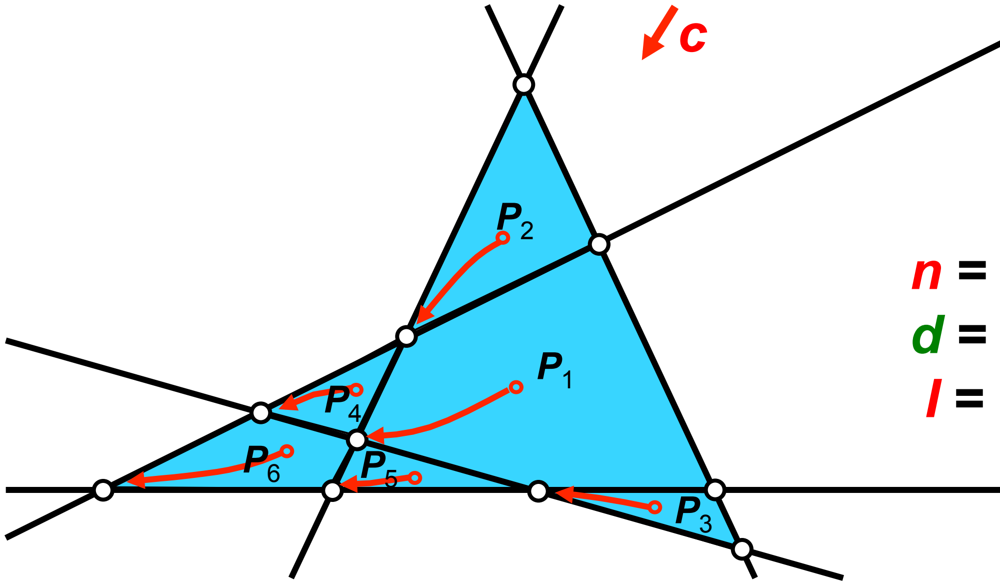
linear optimization : diameter and curvature



n **= 5** : *hyperplanes*
d **= 2** : *dimension*
l **= 6** : *bounded cells*

For a simple arrangement, the number of **bounded cells** l **=** $\binom{n-1}{d}$

linear optimization : diameter and curvature



n = 5 : *hyperplanes*

d = 2 : *dimension*

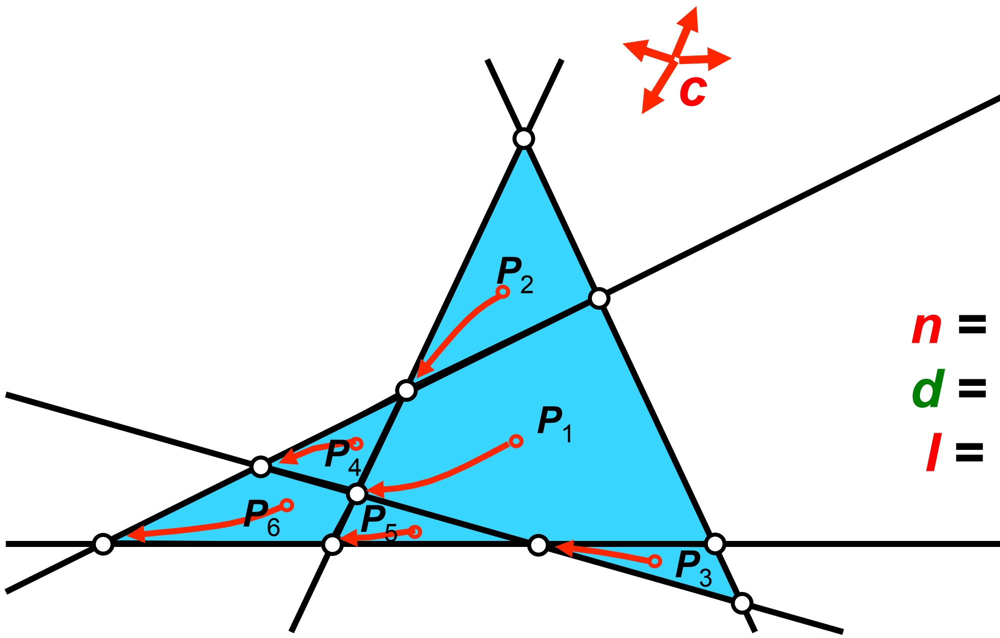
I = 6 : *bounded cells*

$\lambda^c(A)$: average value of $\lambda^c(P_i)$ over the bounded cells P_i of A:

$$\lambda^c(A) = \frac{\sum_{i=1}^{i=I} \lambda^c(P_i)}{I} \qquad \text{with } I = \binom{n-1}{d}$$

❖ $\lambda^c(P_i)$: *redundant* inequalities count

linear optimization : diameter and curvature



n **= 5** : *hyperplanes*
d **= 2** : *dimension*
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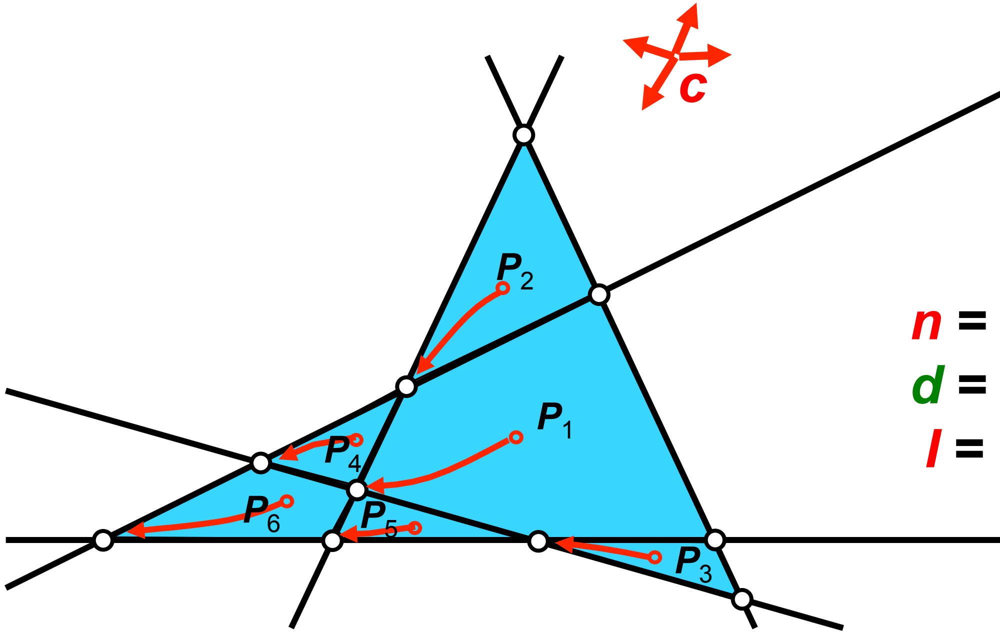
$\lambda^{c}(\boldsymbol{A})$: average value of $\lambda^{c}(\boldsymbol{P_i})$ over the bounded cells $\boldsymbol{P_i}$ of $\boldsymbol{A}$:

$\lambda(\boldsymbol{A})$: *largest* value of $\lambda^{c}(\boldsymbol{A})$ *over all possible* $\boldsymbol{c}$

linear optimization : diameter and curvature



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$\lambda^c(\boldsymbol{A})$: average value of $\lambda^c(\boldsymbol{P}_i)$ over the bounded cells $\boldsymbol{P}_i$ of $\boldsymbol{A}$:

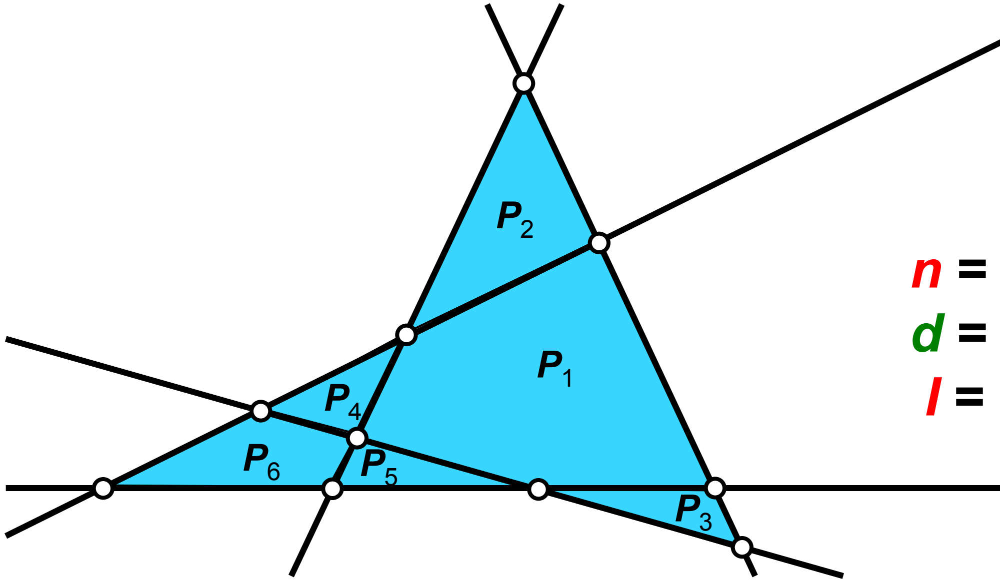
$\lambda(\boldsymbol{A})$: largest value of $\lambda^c(\boldsymbol{A})$ *over all possible* $\boldsymbol{c}$

Dedieu-Malajovich-Shub 2005: $\lambda(\boldsymbol{A}) \leq 2\pi d$

(de Loera-Sturmfels-Vinzant 2012)

❖ $\boldsymbol{A}$: *simple* arrangement

linear optimization : diameter and curvature

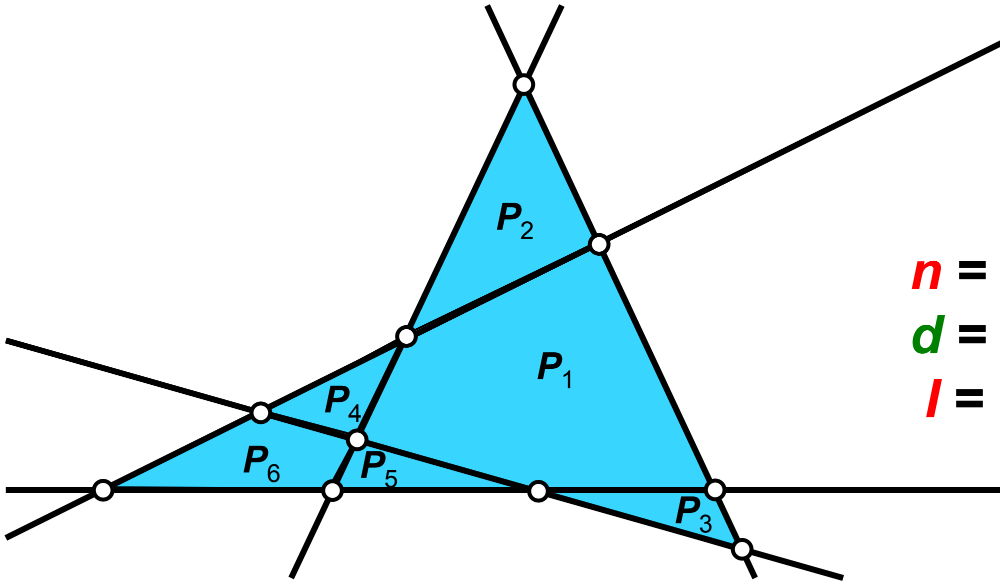


***n* = 5** : *hyperplanes*
***d* = 2** : *dimension*
***I* = 6** : *bounded cells*

$\delta(\boldsymbol{A})$: average diameter of a bounded cell of ***A***:

❖ ***A*** : *simple* arrangement

linear optimization : *diameter and curvature*



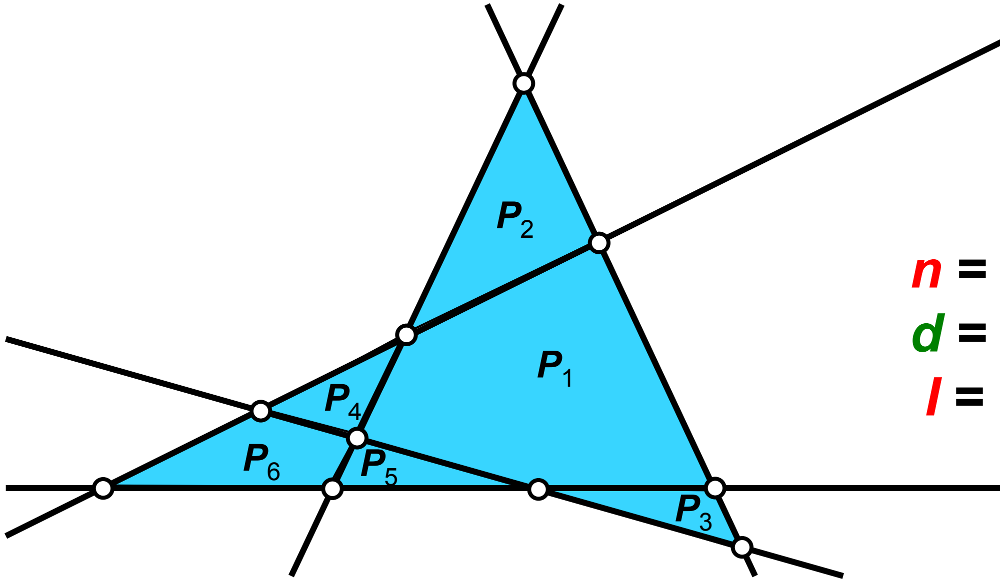
n **= 5** : *hyperplanes*
d **= 2** : *dimension*
I **= 6** : *bounded cells*

δ(***A***) : average diameter of a bounded cell of ***A***:

$$\delta(\boldsymbol{A}) = \frac{\sum_{i=1}^{i=\boldsymbol{I}} \delta(P_i)}{\boldsymbol{I}} \qquad \text{with } \boldsymbol{I} = \binom{\boldsymbol{n}-1}{\boldsymbol{d}}$$

❖ δ(***A***): average diameter ≠ diameter of ***A***
ex: δ(***A***)= 1.333…

linear optimization : diameter and curvature



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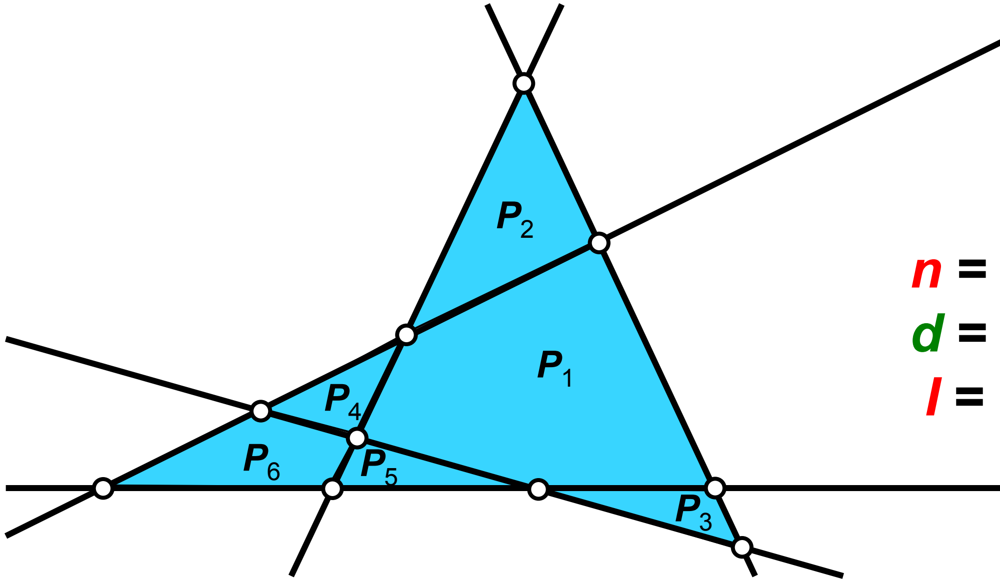
$$\delta(A) = \frac{\sum_{i=1}^{i=I} \delta(P_i)}{I} \quad \text{with } I = \binom{n-1}{d}$$

❖ δ(P_i): only *active* inequalities count

linear optimization : diameter and curvature



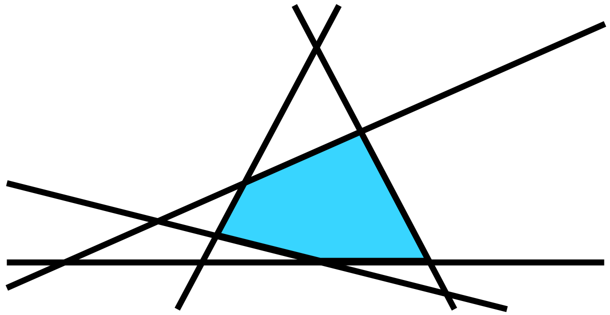
n **= 5** : *hyperplanes*
d **= 2** : *dimension*
I **= 6** : *bounded cells*

$\delta(\boldsymbol{A})$: average diameter of a bounded cell of $\boldsymbol{A}$:

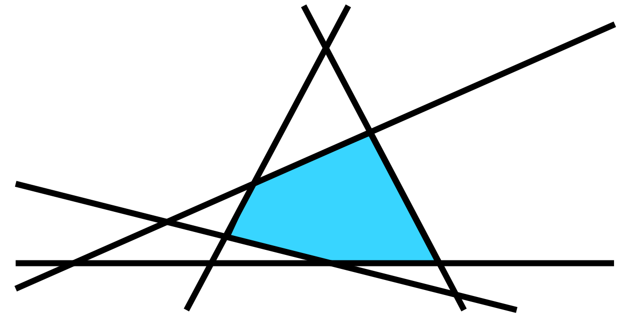
Conjecture : $\delta(\boldsymbol{A}) \leq d$
(Deza-Terlaky-Zinchenko 2008)

(*discrete analogue of Dedieu-Malajovich-Shub result*)

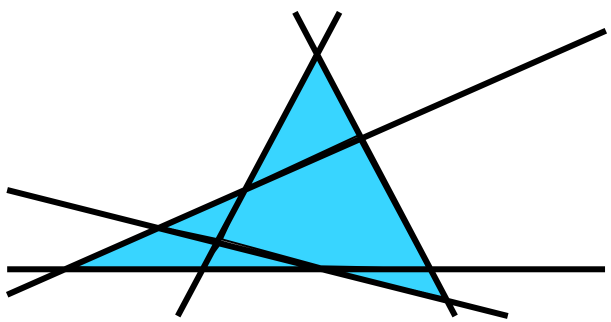
linear optimization : diameter and curvature



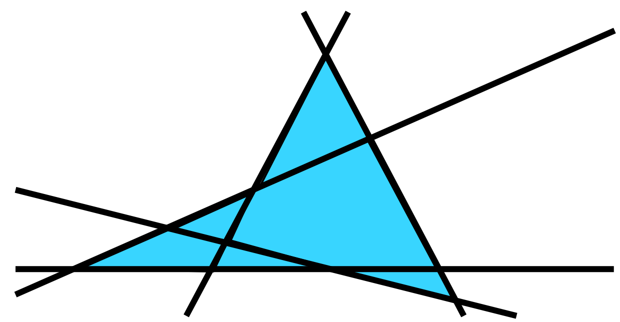
$\cancel{\delta(\boldsymbol{P}) \leq \boldsymbol{n} - \boldsymbol{d}}$? Hirsch conjecture (1957)
Santos 2012



$\cancel{\lambda(\boldsymbol{P}) \leq 2\pi\, \boldsymbol{n}}$ $\cancel{\text{Poly}(n,d)}$? Deza-Terlaky-Zinchenko 2008
Allamigeon-Benchimol-Gaubert-Joswig 2014



$\delta(\boldsymbol{A}) \leq \boldsymbol{d}$? Deza-Terlaky-Zinchenko 2008



$\lambda(\boldsymbol{A}) \leq 2\pi\, \boldsymbol{d}$ Dedieu-Malajovich-Shub 2005

Terlaky-Mut 2014 : Sonnevend curvature

linear optimization : diameter and curvature

Hirsch bound $\delta(P) \leq n - d$ **implies** $\qquad \delta(A) \leq d\,\frac{n+1}{n-1}$

Hirsch conjecture *holds* for $d = 2$: $\qquad \delta(A) \leq 2\,\frac{n+1}{n-1}$

Hirsch conjecture *holds* for $d = 3$: $\qquad \delta(A) \leq 3\,\frac{n+1}{n-1}$

Larman 1970, Barnette 1974 $\qquad \delta(P) \leq n2^d/12$
(Labbé-Manneville-Santos 2015)

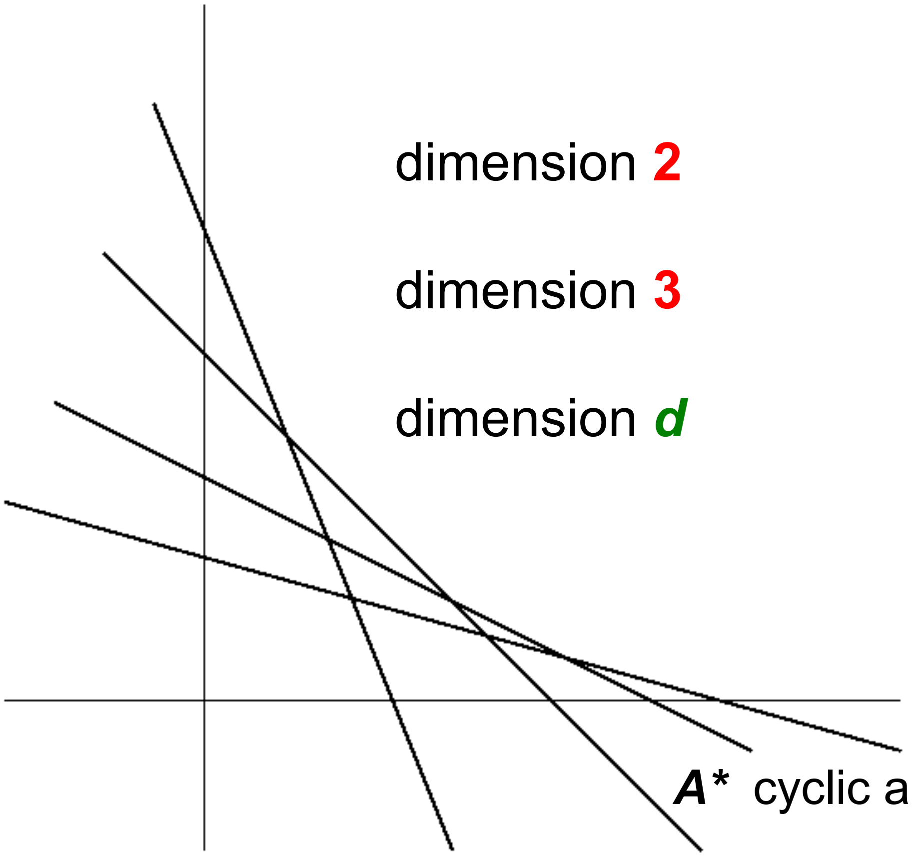
Kalai-Kleitman 1992 $\qquad \delta(P) \leq n^{\log d + 2}$

Todd 2014 $\qquad \delta(P) \leq (n-d)^{\log d}$

Sukegawa-Kitahara 2015 $\qquad \delta(P) \leq (n-d)^{\log(d-1)}$

Sukegawa 2016, Mizuno-Sukegawa 2016
Borgwardt-de Loera-Finhold 2016 (Hirsch holds for transportation polytopes)
.....

linear optimization : diameter and curvature



dimension **2** $\delta(A) = \dfrac{2\lfloor n/2 \rfloor}{(n-1)(n-2)}$

dimension **3** $\delta(A)$ asympotically equal to **3**

dimension ***d*** $d \binom{n-d}{d} / \binom{n-1}{d} \leq \delta(A)$

Deza-Xie 2009

A* cyclic arrangement (mainly consists of cubical cells)

- ❖ Haimovich's probabilistic analysis of shadow-vertex simplex method, Borgwardt 1987
- ❖ Forge-Ramírez Alfonsín 2001: counting *k*-face cells of ***A****

linear optimization : diameter and curvature

Diameter (*of a polytope*) :

lower bound for the number of iterations
for the **simplex method** (*pivoting methods*)

lower bound : (1+ ε) **(*n* – *d*)** **upper bound**: $(n-d)^{\log d}$

Curvature (*of the central path associated to a polytope*) :

large curvature indicates large number of iteration
for *central path following* **interior point methods**

lower bound : $\Omega(2^{d/2})$ with $2n \approx 3d$ **upper bound**: $2\pi d \binom{n-1}{d}$

Allamigeon-Benchimol-Gaubert-Joswig 2014 **exponential lower bound** for λ(***P***) contrasts with the belief that a **polynomial upper bound** for δ(***P***) might exist, e.g. δ(***P***) **≤** ***d*** (***n*** – ***d***)/2

linear optimization : diameter and curvature

$\Delta(d,n)$: largest diameter over all d-dimensional polytopes with n facets

$\Delta(d,n)$		$n-d$				
		4	5	6	7	8
d	4	4	5	5	[6,7]	7+
	5	4	5	6	[7,9]	7+
	6	4	5	[6,7]	[7,9]	8+
	7	4	5	[6,7]	[7,10]	8+

$\Delta(4,10) = \mathbf{5}$, $\Delta(5,11) = \mathbf{6}$ Goodey 1972

linear optimization : diameter and curvature

$\Delta(d,n)$: largest diameter over all d-dimensional polytopes with n facets

$\Delta(d,n)$		$n-d$				
		4	5	6	7	8
d	4	4	5	5	6	7+
	5	4	5	6	[7,8]	7+
	6	4	5	6	[7,9]	8+
	7	4	5	6	[7,10]	8+

$\Delta(4,11) = \Delta(6,12) = 6$ Bremner-Schewe 2011

linear optimization : diameter and curvature

$\Delta(d,n)$: largest diameter over all d-dimensional polytopes with n facets

$\Delta(d,n)$		$n - d$				
		4	5	6	7	8
d	4	4	5	5	6	7
	5	4	5	6	7	[7,9]
	6	4	5	6	[7,8]	[8,11]
	7	4	5	6	[7,9]	[8,12]

$\Delta(4,12) = \Delta(5,12) = 7$ Bremner-Deza-Hua-Schewe 2013

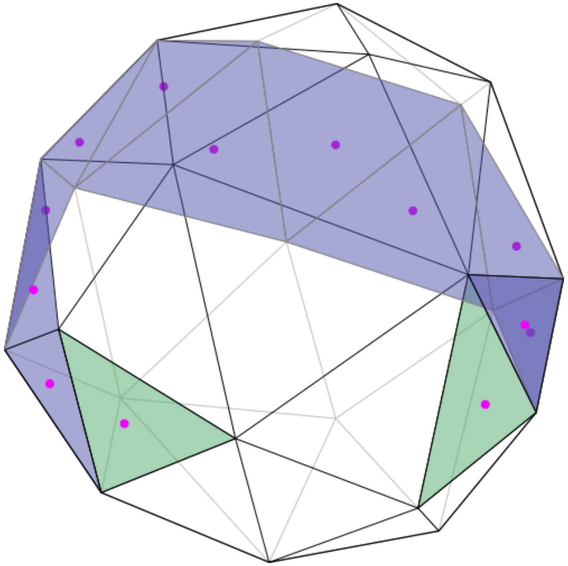
linear optimization : diameter and curvature

$\Delta(d,n)$: largest diameter over all d-dimensional polytopes with n facets

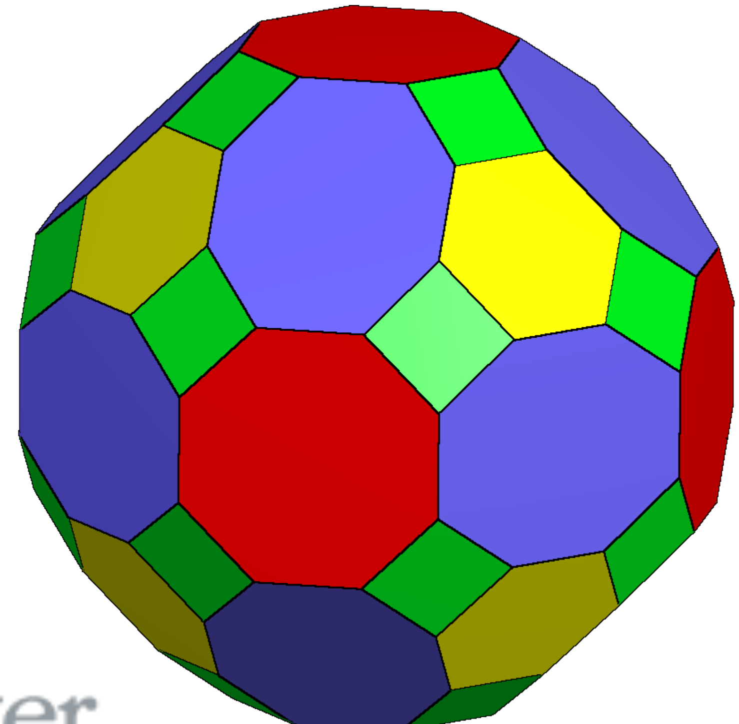
Characterize all combinatorial types of paths of length ℓ

Find necessary conditions for a (chirotope of a) polytope to admit an embedding of a ℓ-path on its boundary (without shortcuts)

If *no* such (chirotope of a) polytope exists: $\Delta(d, n) \neq \ell$



Algorithmic and geometric aspects of combinatorial and continuous optimization



Antoine Deza, McMaster

based on joint works with

David Bremner, New Brunswick

George Manoussakis, Orsay

Shinji Mizuno, Tokyo Tech.

Shmuel Onn, Technion

Lionel Pournin, Paris XIII

Lars Schewe, Erlangen-Nürnberg

Noriyoshi Sukegawa, Chuo

Tamás Terlaky, Lehigh

Feng Xie, Microsoft

Yuriy Zinchenko, Calgary

McMaster University

Advanced Optimization Laboratory

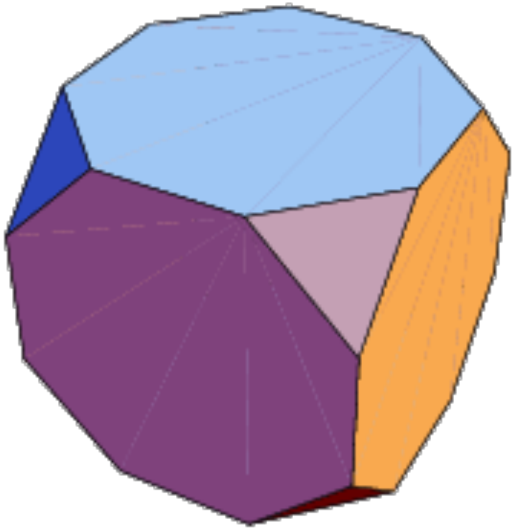
lattice polytopes with large diameter

lattice (d,k)-polytope : convex hull of points drawn from $\{0,1,\ldots,k\}^d$

diameter $\delta(P)$ of polytope P : smallest number such that **any two vertices** of P can be connected by a **path with at most** $\delta(P)$ **edges**

$\delta(d,k)$: largest diameter over all **lattice** (d,k)-polytopes

ex. $\delta(3,3) = 6$ and is achieved by a ***truncated cube***



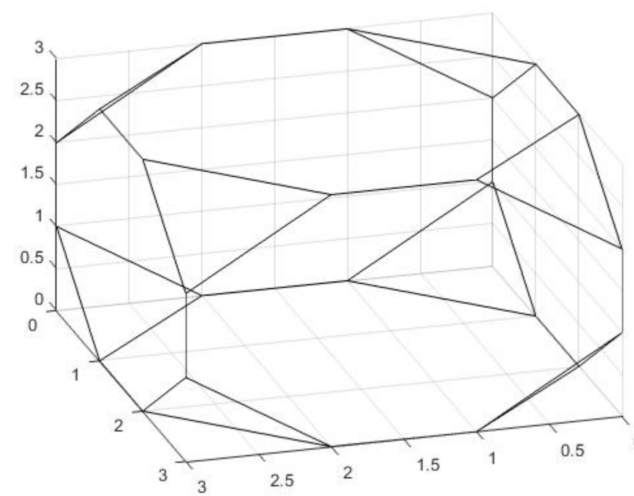
lattice polytopes with large diameter

lattice (d,k)-polytope : convex hull of points drawn from $\{0,1,\ldots,k\}^d$

diameter $\delta(P)$ of polytope P : smallest number such that **any two vertices** of P can be connected by a **path with at most $\delta(P)$ edges**

$\delta(d,k)$: largest diameter over all **lattice** (d,k)-polytopes

ex. $\delta(3,3) = 6$ and is achieved by a ***truncated cube***



lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

upper bounds :

$\delta(d,1) \le d$		[Naddef 1989]
$\delta(2,k) = O(k^{2/3})$		[Balog-Bárány 1991]
$\delta(2,k) = 6(k/2\pi)^{2/3} + O(k^{1/3}\log k)$		[Thiele 1991] [Acketa-Žunić 1995]
$\delta(d,k) \le kd$		[Kleinschmid-Onn 1992]
$\delta(d,k) \le kd - \lceil d/2 \rceil$	for $k \ge 2$	[Del Pia-Michini 2016]
$\delta(d,k) \le kd - \lceil 2d/3 \rceil$	for $k \ge 3$	[Deza-Pournin 2016]
$\delta(d,k) \le kd - \lceil 2d/3 \rceil - (k-2)$	for $k \ge 4$	[Deza-Pournin 2016]

lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

lower bounds :

$\delta(d,1) \geq d$	[Naddef 1989]
$\delta(d,2) \geq \lfloor 3d/2 \rfloor$	[Del Pia-Michini 2016]
$\delta(d,k) = \Omega(k^{2/3}\, d)$	[Del Pia-Michini 2016]
$\delta(d,k) \geq \lfloor (k+1)d\,/2 \rfloor$ for $k < 2d$	[Deza-Manoussakis-Onn 2016]

lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	7+	9+	?	?	?	?
	4	4	6	8	10+	12+	14+	16+	?	?
	5	5	7	10+	12+	15+	17+	20+	22+	25+

$\delta(d,1) = d$ [Naddef 1989]

$\delta(2,k) =$ close form [Thiele 1991] [Acketa-Žunić 1995]

$\delta(d,2) = \lfloor 3d/2 \rfloor$ [Del Pia-Michini 2016]

$\delta(4,3) = 8$ [Deza-Pournin 2016]

lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	7+	9+	?	?	?	?
	4	4	6	8	10+	12+	14+	16+	?	?
	5	5	7	10+	12+	15+	17+	20+	22+	25+

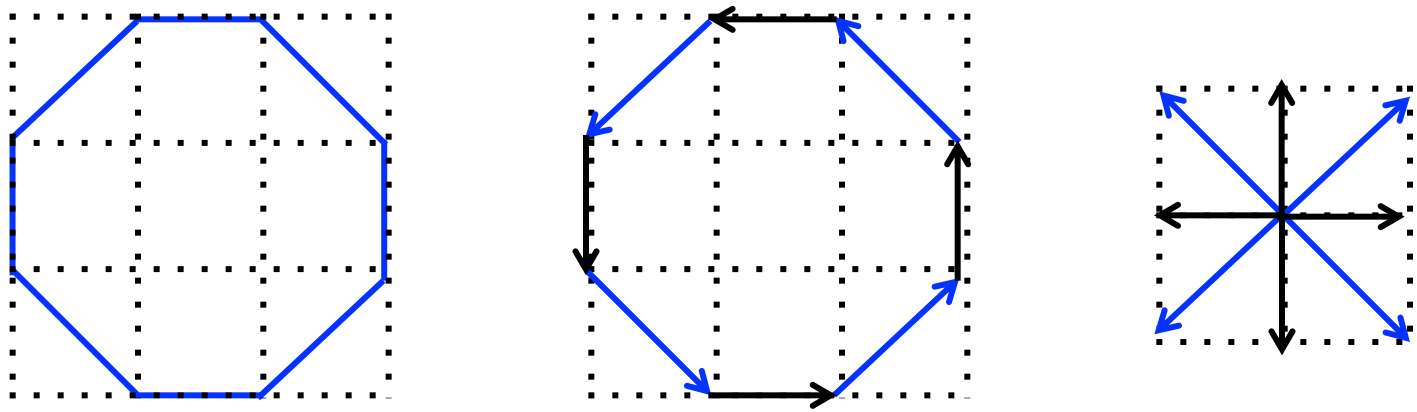
All known entries of $\delta(d,k)$ are achieved, up to translation, by a *Minkowski sum of primitive lattice vectors* (some uniquely)

Conjecture: $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$ [Deza-Manoussakis-Onn 2016]

lattice polygons with many vertices

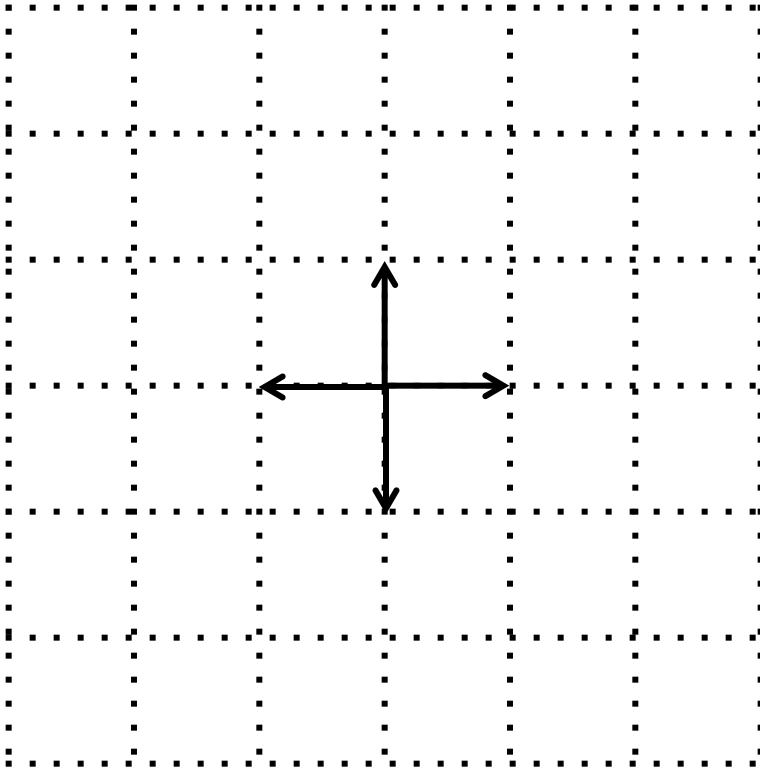
Q. What is δ(2,***k***) : largest diameter of a polygon which vertices are drawn form the ***k*** x ***k*** grid?

A polygon can be associated to a set of vectors (*edges*) *summing up to zero*, and *without a pair of positively multiple vectors*



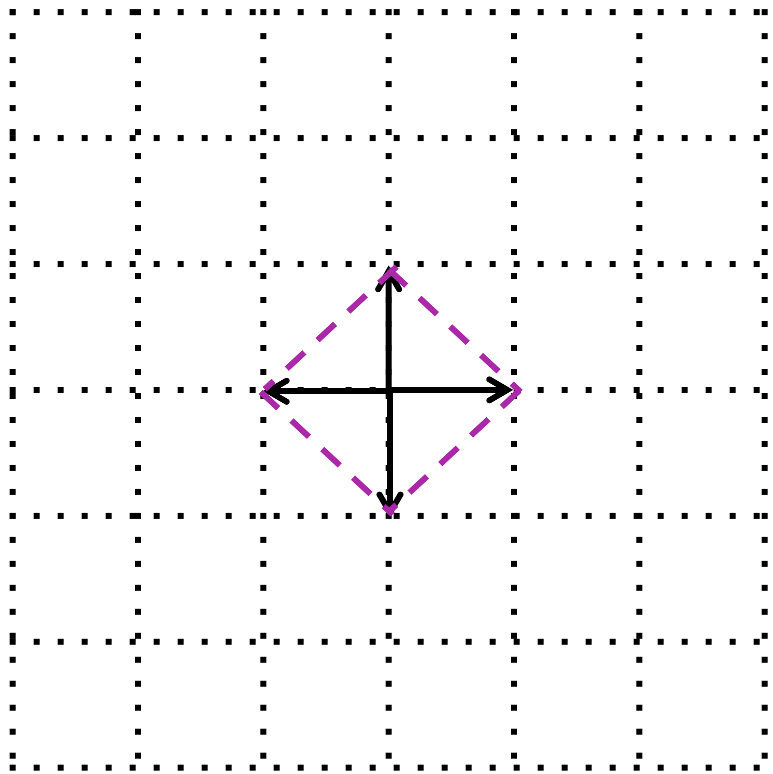
δ(2,3) = 4 is achieved by the 8 vectors : (±1,0), (0,±1), (±1,±1)

lattice polygons with many vertices



$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

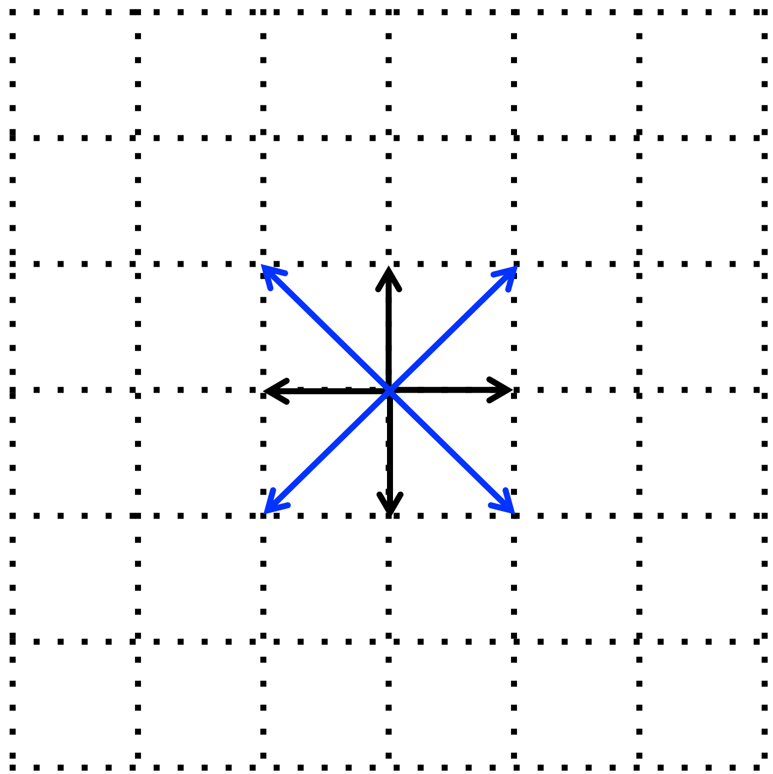
lattice polygons with many vertices



$||x||_1 \leq 1$

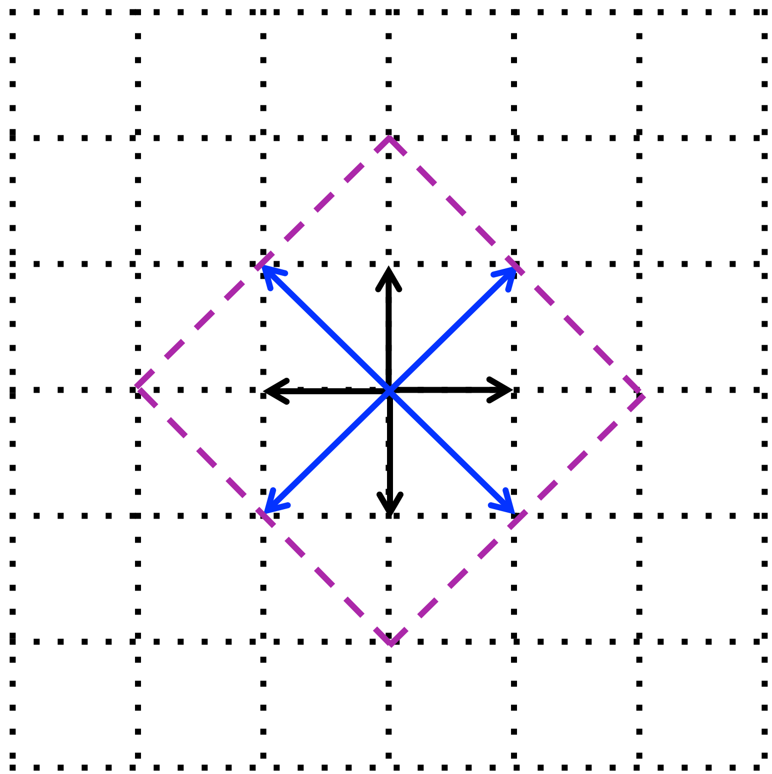
$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

lattice polygons with many vertices



$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$
$\delta(2,3) = 4$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$

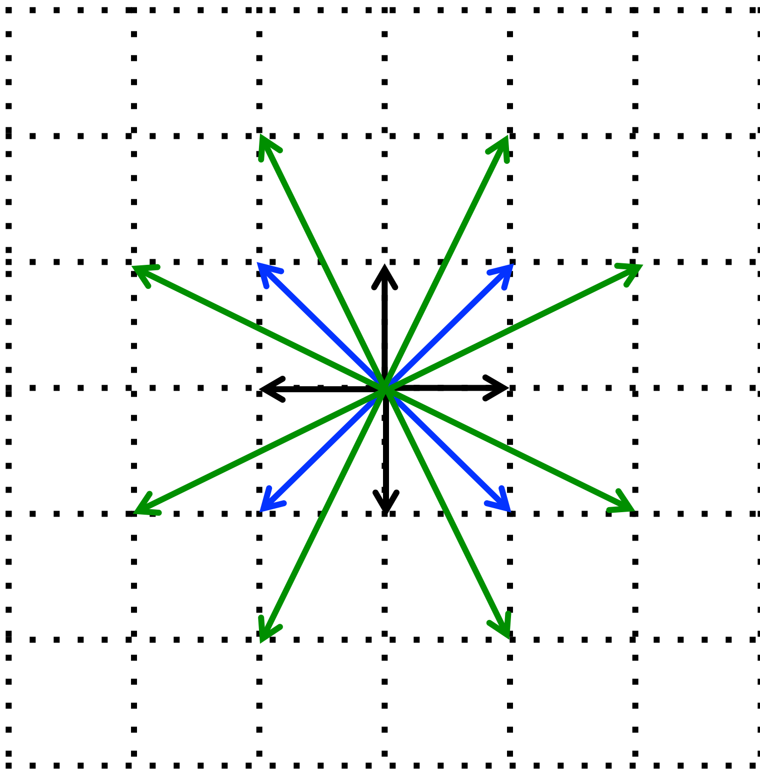
lattice polygons with many vertices



$||x||_1 \leq \mathbf{2}$

$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$
$\delta(2,3) = 4$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$

lattice polygons with many vertices

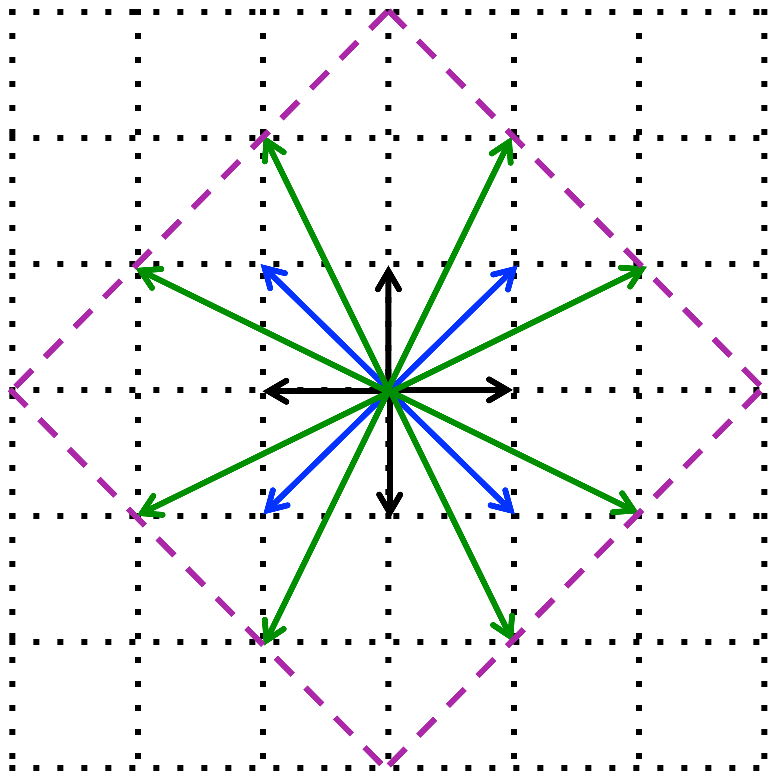


$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

$\delta(2,3) = 4$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$

$\delta(2,9) = 8$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$, $(\pm 1,\pm 2)$, $(\pm 2,\pm 1)$

lattice polygons with many vertices



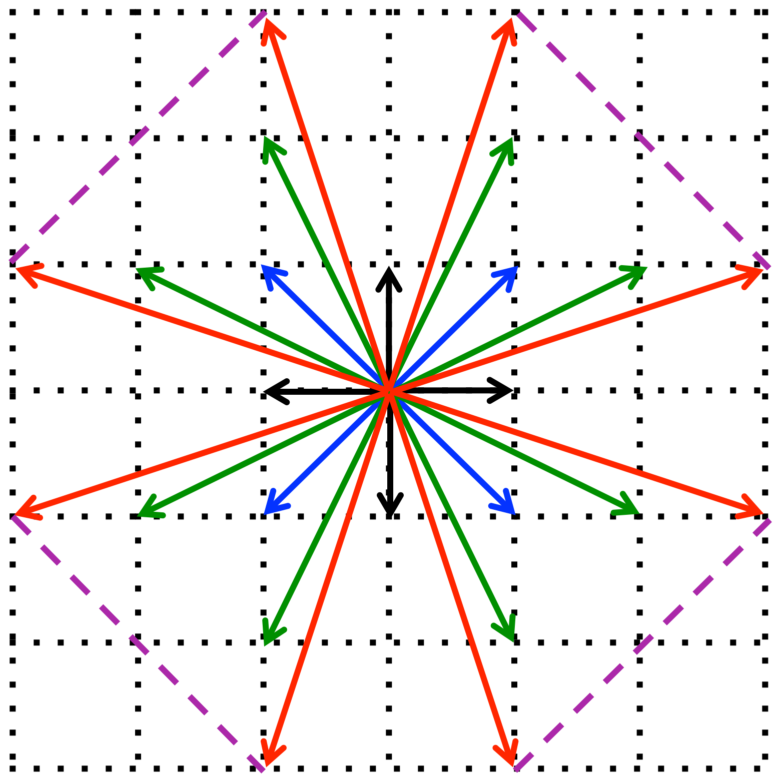
$\|x\|_1 \leq$ **3**

$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

$\delta(2,3) = 4$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$

$\delta(2,9) = 8$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$, $(\pm1,\pm2)$, $(\pm2,\pm1)$

lattice polygons with many vertices



$||x||_1 \leq 4$

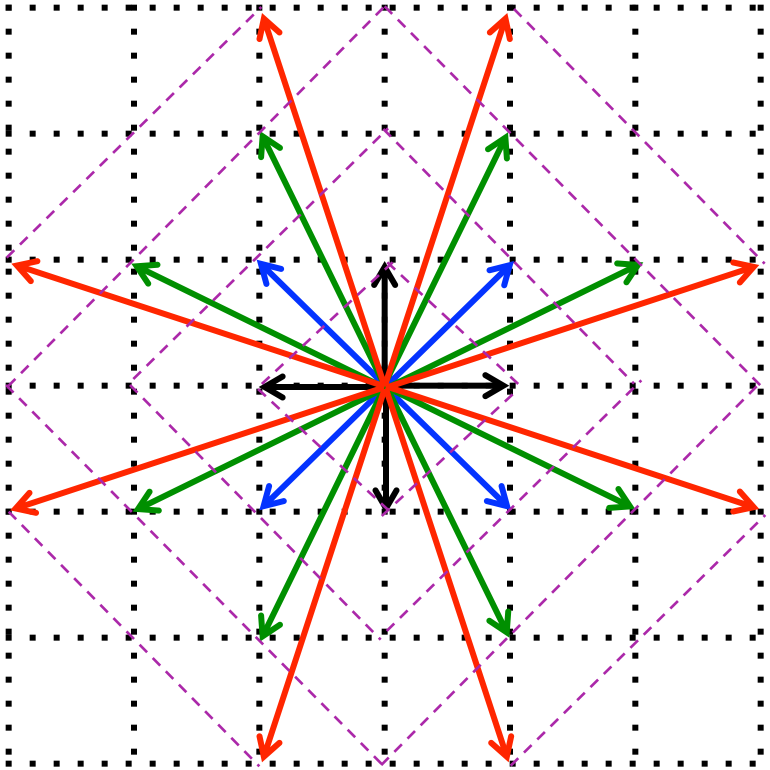
$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

$\delta(2,3) = 4$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$

$\delta(2,9) = 8$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$, $(\pm 1,\pm 2)$, $(\pm 2,\pm 1)$

$\delta(2,17) = 12$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$, $(\pm 1,\pm 2)$, $(\pm 2,\pm 1)$, $(\pm 1,\pm 3)$, $(\pm 3,\pm 1)$

lattice polygons with many vertices



$||x||_1 \leq p$

$$\delta(2,k) = 2\sum_{i=1}^{p} \varphi(i) \text{ for } k = \sum_{i=1}^{p} i\varphi(i)$$

$\varphi(p)$: ***Euler totient function*** counting positive integers less or equal to p relatively prime with p

$\varphi(1) = \varphi(2) = 1$, $\varphi(3) = \varphi(4) = 2, \ldots$

lattice polygons with many vertices

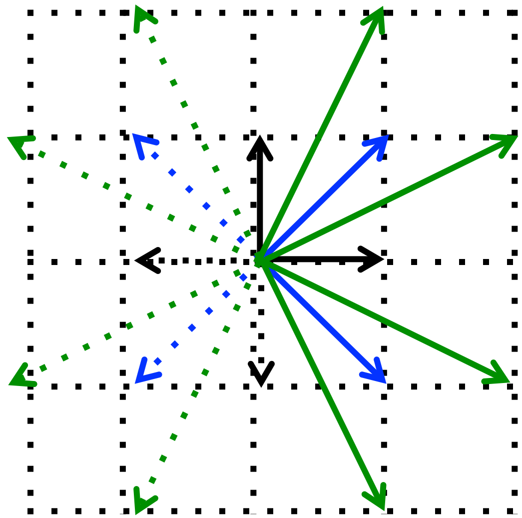
$\delta(2,k)$		k								
		1	2	3	4	5	6	7	8	9
	p	1		2						3
	v	4	6	8	8	10	12	12	14	16
	δ	2	3	4	4	5	6	6	7	8

$\delta(2,k) = 2\sum_{i=1}^{p} \varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

$\varphi(p)$: ***Euler totient function*** counting positive integers less or equal to p relatively prime with p

$\varphi(1) = \varphi(2) = 1$, $\varphi(3) = \varphi(4) = 2, \ldots$

lattice polygons with many vertices



$||x||_1 \leq p$

$H_1(2,p)$: Minkowski sum generated by $\{x \in \mathbb{Z}^2 : ||x||_1 \leq p, \gcd(x)=1, x \succeq 0\}$

$H_1(2,p)$ has diameter $\delta(2,k) = 2\sum_{i=1}^{p}\varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

Ex. $H_1(2,2)$ generated by (1,0), (0,1), (1,1), (1,-1) (fits, *up to translation*, in 3x3 grid)

$x \succeq 0$: first nonzero coordinate of x is nonnegative

primitive lattice polytopes

as generalization of the permutahedron of type B_d

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : ||x||_q \le p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p)$: Zonotope $(x \in \mathbb{Z}^d : ||x||_q \le p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

Given a set G of m vectors (generators)

Minkowski (G) : convex hull of the 2^m sums of the m vectors in G
Zonotope (G) : convex hull of the 2^m ***signed*** sums of the m vectors in G

up to translation $Z(G)$ is the image of $H(G)$ by an homothety of factor 2

❖ ***Primitive lattice polytopes***: Minkowski sum generated by ***short integer*** vectors which are ***pairwise linearly independent***

primitive lattice polytopes

as generalization of the permutahedron of type B_d

$$H_q(d,p) : \text{Minkowski } (x \in \mathbb{Z}^d : \|x\|_q \leq p,\ \gcd(x)=1,\ x \succeq 0)$$

$$Z_q(d,p) : \text{Zonotope } (x \in \mathbb{Z}^d : \|x\|_q \leq p,\ \gcd(x)=1,\ x \succeq 0)$$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $Z_q(d,p)$: invariant under symmetries induced by coordinate permutations and reflections induced by sign flips
- Coordinates of the vertices of $Z_q(d,p)$ are odd, thus the number of vertices of $Z_q(d,p)$ is a multiple of 2^d
- $H_q(d,p)$ is, up to translation, a lattice (d,k)-polytope where k is the sum of the first coordinates of all generators of $Z_q(d,p)$
- diameter of $Z_q(d,p)$ is equal to the number of its generators

primitive lattice polytopes

as generalization of the permutahedron of type B_d

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p)$: Zonotope $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $H_q(d, 1)$: $[0, 1]^d$ cube for finite q

primitive lattice polytopes

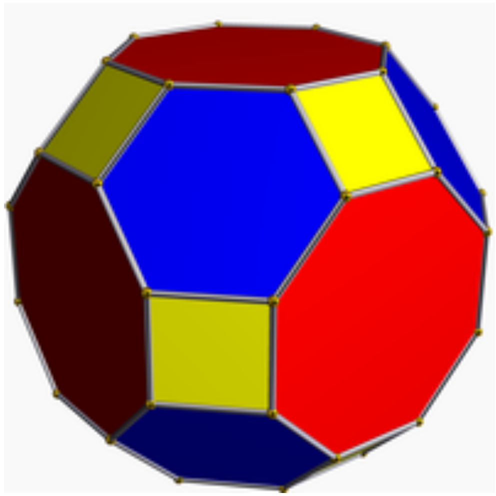
as generalization of the permutahedron of type B_d

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p)$: Zonotope $(x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $H_1(3,2)$: truncated cuboctahedron
  (great rhombicuboctahedron)



primitive lattice polytopes

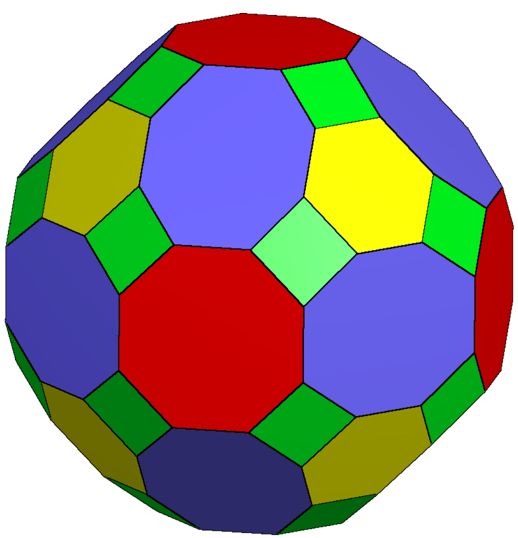
as generalization of the permutahedron of type B_d

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : ||x||_q \leq p, \ \gcd(x)=1, \ x \succeq 0)$

$Z_q(d,p)$: Zonotope $(x \in \mathbb{Z}^d : ||x||_q \leq p, \ \gcd(x)=1, \ x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $H_\infty(3,1)$: truncated small rhombicuboctahedron



primitive lattice polytopes

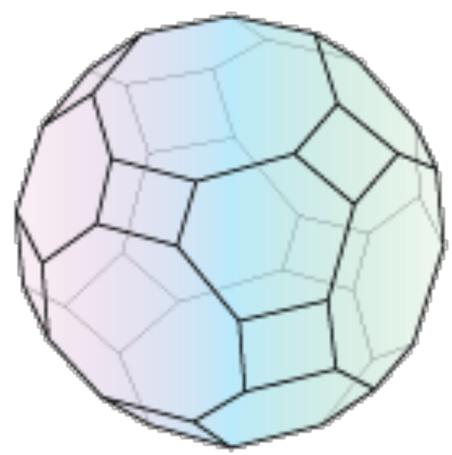
as generalization of the permutahedron of type $B_{\boldsymbol{d}}$

$H_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$: Minkowski ($x \in \mathbb{Z}^{\boldsymbol{d}}$: $||x||_{\boldsymbol{q}} \leq \boldsymbol{p}$, gcd($x$)=1, $x \succeq 0$)

$Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$: Zonotope ($x \in \mathbb{Z}^{\boldsymbol{d}}$: $||x||_{\boldsymbol{q}} \leq \boldsymbol{p}$, gcd($x$)=1, $x \succeq 0$)

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $Z_1(\boldsymbol{d},2)$: permutahedron of type $B_{\boldsymbol{d}}$



primitive lattice polytopes

as generalization of the permutahedron of type B_d

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p)$: Zonotope $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative
H^+ / Z^+: ***positive*** primitive lattice polytope $x \in \mathbb{Z}^d_+$

- $H_1(d,2)^+$: Minkowski sum of the permutahedron with the $\{0,1\}^d$

primitive lattice polytopes

as generalization of the permutahedron of type B_d

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : ||x||_q \le p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p)$: Zonotope $(x \in \mathbb{Z}^d : ||x||_q \le p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

H^+ / Z^+: ***positive*** primitive lattice polytope $x \in \mathbb{Z}^d_+$

- $H_1(d,2)^+$: Minkowski sum of the permutahedron with the $\{0,1\}^d$, i.e., graphical zonotope obtained by the ***d***-clique with a loop at each node

graphical zonotope Z_G: Minkowski sum of segments $[e_i,e_j]$ for all *edges* $\{i,j\}$ of a given graph G

primitive lattice polygons

*as lattice (2,**k**)-polygons with large diameter*

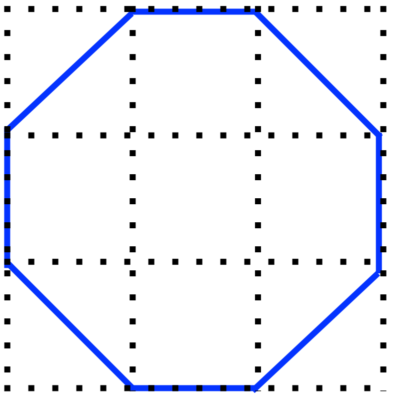
Q. (revisit) What is δ(2,***k***) : largest diameter of a polygon which vertices are drawn form the ***k*** x ***k*** grid?

For any ***k***, there exists ***p*** so that δ(2,***k***) is achieved, up to translation, by the Minkowski sum of a subset of the generators of $H_1(2,p)$.

Moreover, for any ***p***, and for ***k*** = $\sum_{i=1}^{p} i\varphi(i)$, δ(2,***k***) is uniquely achieved, up to translation, by $H_1(2,p)$ (φ : Euler's totient function)

Ex. ***p*** =2

$H_1(2,2)$: lattice (2,3)-polygon with diameter 4



primitive lattice polytopes

as lattice (d,k)-polytopes with large diameter

For $k < 2d$, Minkowski sum of a subset of the generators of $H_1(d,2$ is, up to translation, a lattice (d,k)-polytope with diameter $\lfloor (k+1)d/2 \rfloor$

Proof sketch. Assume d even (odd case similar).
$H_1(d,2)$: lattice $(d,2d\text{-}1)$-polytope with diameter d^2 (permutahedron of type B_d)

removing the $d/2$ generators (0,…,0,1,0,...,0,-1,0,...0) forming one of the d-1 *perfect matchings of the d-clique* [Berge 1983] yields a lattice $(d,k\text{-}1)$-polytope with diameter decreasing by $d/2$. After d removal, one obtains $H_1(d,2)^+$ a lattice (d,d)-polytope with diameter $d(d+1)/2$

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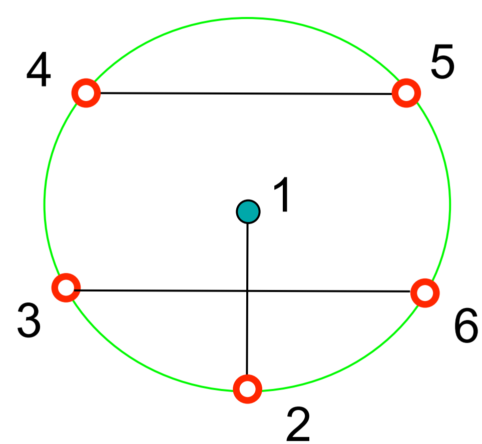
(1,-1,0,0,0,0), (0,0,1,0,0,-1), (0,0,0,1,-1,0)



primitive lattice polytopes

*as lattice (**d**,**k**)-polytopes with large diameter*

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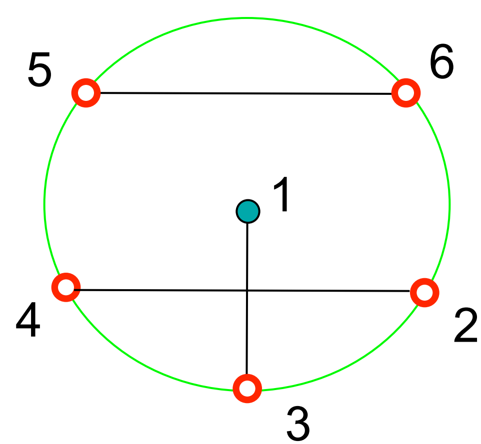
(1,0,-1,0,0,0), (0,1,0,-1,0,0), (0,0,0,0,1,-1)



primitive lattice polytopes

as lattice (***d***,***k***)*-polytopes with large diameter*

For ***k*** < 2***d***, Minkowski sum of a subset of the generators of H_1(***d***,2 is, up to translation, a lattice (***d***,***k***)-polytope with diameter $\lfloor (k+1)d/2 \rfloor$

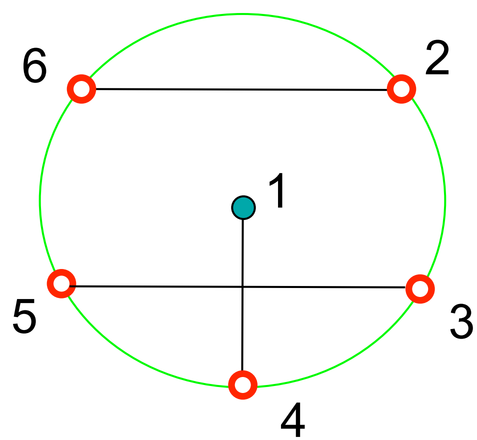
Proof sketch. Assume ***d*** even (odd case similar).
$H_1(d,2)$: lattice (d,2d-1)-polytope with diameter d^2 (permutahedron of type B_d)

removing the ***d***/2 generators (0,…,0,1,0,...,0,-1,0,...0) forming one of the ***d***-1 *perfect matchings of the* ***d****-clique* [Berge 1983] yields a lattice (***d***,***k***-1)-polytope with diameter decreasing by ***d***/2. After ***d*** removal, one obtains $H_1(d,2)^+$ a lattice (***d***,***d***)-polytope with diameter $d(d+1)/2$

(1,-1,0,0,0,0), (0,0,1,0,0,-1), (0,0,0,1,-1,0)

(1,0,-1,0,0,0), (0,1,0,-1,0,0), (0,0,0,0,1,-1)

(1,0,0,-1,0,0), (0,0,1,0,-1,0), (0,1,0,0,0,-1)
….



primitive lattice polytopes

as lattice (d,k)-polytopes with large diameter

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lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

upper bounds :

$\delta(d,1) \le d$ [Naddef 1989]

$\delta(d,k) \le kd$ [Kleinschmid-Onn 1992]

$\delta(d,k) \le kd - \lceil d/2 \rceil$ for $k \ge 2$ [Del Pia-Michini 2016]

$\delta(d,k) \le kd - \lceil 2d/3 \rceil$ for $k \ge 3$ [Deza-Pournin 2016]

lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

Lemma. (Del Pia-Michini 2016) Consider lattice (d,k)-polytope P, u vertex of P, and vector $c \in R^d$ with integer coordinates, then $d(u,F) \leq c\cdot u - \beta$ where $\beta = \min\{ c\cdot x : x \in P \}$ and $F = \{ x \in P : c\cdot x = \beta \}$

Lemma. Consider lattice (d,k)-polytope P, $I \subseteq \{1,\ldots,d\}$ such that $l_i \leq x_i \leq h_i$ for $x \in P$ and $i \in I$, then : $\delta(P) \leq \delta(d-|I|,k) + \mathrm{sum}_{i\in I}(h_i - l_i)$

Lemma. Consider lattice (d,k)-polytope P, u, v vertices of P, $I \subseteq \{1,\ldots,d\}$ with $|I| \leq 3$ such that $u_i+v_i \leq k$ when $i \in I$, then

$$d(u,v) \leq \delta(d-|I|,k) + \mathrm{sum}_{i\in I}(u_i+v_i)$$

$|I| = 1$: $\delta(d,k) \leq kd$ [Kleinschmid-Onn 1992]

$|I| = 2$: $\delta(d,k) \leq kd - \lceil d/2 \rceil$ for $k \geq 2$ [Del Pia-Michini 2016]

$|I| = 3$: $\delta(d,k) \leq kd - \lceil 2d/3 \rceil$ for $k \geq 3$ [Deza-Pournin 2016]

lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\dots,k\}^d$

Consider lattice (d,k)-polytope P with $d \geq 3$, $k \geq 3$, u, v vertices of P, then one of the following inequalities holds:

(i) $d(u,v) \leq \delta(d-1,k) + k - 1$

(ii) $d(u,v) \leq \delta(d-2,k) + 2k - 2$

(iii) $d(u,v) \leq \delta(d-3,k) + 3k - 2$

$\Rightarrow \quad \delta(d,k) \leq kd - \lceil 2d/3 \rceil$ for $k \geq 3$

lattice polytopes with large diameter

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$\Rightarrow$ $\delta(d,k) \leq kd - \lceil 2d/3 \rceil$ for $k \geq 3$

$\delta(d,k) \leq kd - \lceil 2d/3 \rceil - (k - 2)$ for $k \geq 4$

primitive lattice polytopes

related questions

[Soprunov-Soprunova 2016] ***Minkowski length*** $L(\boldsymbol{P})$ of a lattice polytope $\boldsymbol{P}$: largest number of lattice segments which Minkowski sum is contained in $\boldsymbol{P}$

denote $L(\{0,1,\ldots,\boldsymbol{k}\}^{\boldsymbol{d}})$ by $L(\boldsymbol{d},\boldsymbol{k})$ (Minkowski length of a box)

$L(2,\boldsymbol{k}) = \delta(2,\boldsymbol{k})$ achieved by a Minkowski sum of a proper subset of generators of $H_1(2,\boldsymbol{p})$ for some $\boldsymbol{p}$

$L(\boldsymbol{d},\boldsymbol{k}) = \lfloor(\boldsymbol{k}+1)\boldsymbol{d}/2\rfloor$ for $\boldsymbol{k} < 2\boldsymbol{d}$ achieved by a Minkowski sum of a proper subset of generators of $H_1(\boldsymbol{d},2)$

Sloane OEI sequences

$H_\infty(\boldsymbol{d},1)^+$ vertices : A034997 = number of generalized retarded functions in quantum Field theory (determined till $\boldsymbol{d}$ =8)

$H_\infty(\boldsymbol{d},1)$ vertices : A009997 = number of regions of hyperplane arrangements with {-1,0,1}-valued normals in dimension $\boldsymbol{d}$ (determined till $\boldsymbol{d}$ =7)

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A034997 Number of Generalized Retarded Functions in Quantum Field Theory. 1

2, 6, 32, 370, 11292, 1066044, 347326352, 419172756930 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

COMMENTS a(d) is the number of parts into which d-dimensional space (x_1,...,x_d) is split by a set of (2^d - 1) hyperplanes c_1 x_1 + c_2 x_2 + ...+ c_d x_d =0 where c_j are 0 or +1 and we exclude the case with all c=0.

Also, a(d) is the number of independent real-time Green functions of Quantum Field Theory produced when analytically continuing from euclidean time/energy (d+1 = number of energy/time variables). These are also known as Generalized Retarded Functions.

The numbers up to d=6 were first produced by T. S. Evans using a Pascal program, strictly as upper bounds only. M. van Eijck wrote a C program using a direct enumeration of hyperplanes which confirmed these and produced the value for d=7. Kamiya et al. showed how to find these numbers and some associated polynomials using more sophisticated methods, giving results up to d=7. T. S. Evans added the last number on Aug 01 2011 using an updated version of van Eijck's program, which took 7 days on a standard desktop computer.

REFERENCES

Björner, Anders. "Positive Sum Systems", in Bruno Benedetti, Emanuele Delucchi, and Luca Moci, editors, Combinatorial Methods in Topology and

Number of Generalized Retarded Functions in Quantum Field Theory.

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T. S. Evans, N-point finite temperature expectation values at real times, Nuclear Physics B 374 (1992) 340-370.

H. Kamiya, A. Takemura and H. Terao, Ranking patterns of unfolding models of codimension one, Advances in Applied Mathematics 47 (2011) 379 - 400.

M. van Eijck, Thermal Field Theory and Finite-Temperature Renormalisation Group, PhD thesis, Univ. Amsterdam, 4th Dec. 1995.

Table of n, a(n) for n=1..8.

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convex matroid optimization

Melamed-Onn 2014:

The optimal solution of max { $\mathbf{f}(\mathbf{W}x) : x \in \mathbf{S}$} is attained at a vertex of the projection integer polytope in $\mathbb{R}^{d}$: conv(**WS**) = **W**conv(**S**)

S : set of feasible point in $\mathbb{Z}^{n}$ (in the talk $\mathbf{S} \in \{0,1\}^{n}$)
W : integer d x n matrix (**W** is mostly {0,1,…, p}-valued)
f : convex function from $\mathbb{R}^{d}$ to $\mathbb{R}$

Q. What is the maximum number **v**(d,n) of vertices of conv(**WS**) when $\mathbf{S} \in \{0,1\}^{n}$ and **W** is a {0,1}-valued d x n matrix ?

Obviously $v(d,n) \leq |\mathbf{WS}| = O(n^{d})$
In particular $v(2,n) = O(n^{2})$, and $v(2,n) = \Omega(n^{0.5})$

convex matroid optimization

Melamed-Onn 2014

Given matroid **S** of order ***n***, {0,1,…,***p***}-valued ***d*** x ***n*** matrix **W**, maximum number **m**(***d***,***p***) of vertices of conv(**WS**) is independent of ***n*** and **S**

convex matroid optimization

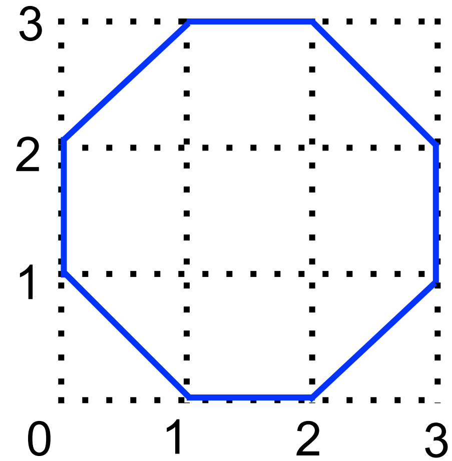
Melamed-Onn 2014

Given matroid **S** of order ***n***, {0,1}-valued ***d*** x ***n*** matrix **W**, maximum number **m**(***d***,1) of vertices of conv(**WS**) is independent of ***n*** and **S**

Ex: maximum number **m**(2,1) of vertices of a planar projection conv(**WS**) of matroid **S** by a binary matrix **W** is attained by the following matrix and uniform matroid of rank 3 and order 8:

$$\mathbf{W} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{S} = U(3,8) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$



conv(**WS**)

convex matroid optimization

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convex matroid optimization

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$$d\,2^{d} \le \mathbf{m}(d,1) \le 2\sum_{i=0}^{d-1}\binom{(3^{d}-3)/2}{i}$$

m(2,1) = 8
24 ≤ **m**(3,1) ≤ 158
64 ≤ **m**(4,1) ≤ 19840

convex matroid optimization

Melamed-Onn 2014

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for ***d*** ≥ *3*

$d\,2^d \leq \mathbf{m}(d,1) \leq 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i}$

m(2,1) = 8
24 ≤ **m**(3,1) ≤ 158
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Deza-Manoussakis-Onn 2016

$2+2d! \leq \mathbf{m}(d,1) \leq 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i} - f(d)$

m(2,1) = 8
48 ≤ **m**(3,1) ≤ **96**
370 ≤ **m**(4,1) ≤ **5376**
11292 ≤ **m**(5,1) ≤ **1 981 440**

primitive lattice polytopes

as lower and upper bound for convex matroid optimization parameter

Given matroid **S** of order ***n***, $\{0,1,\ldots,p\}$-valued ***d*** x ***n*** matrix **W**, maximum number $\mathbf{m}(d,p)$ of vertices of conv(**WS**) is independent of ***n*** and **S**

$$|H_\infty(d,p)^+| \leq \mathbf{m}(d,p) \leq |H_\infty(d,p)|$$

primitive lattice polytopes

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$H_\infty(d,1)^+$ vertices : A034997 = number of generalized retarded functions in quantum Field theory

$H_\infty(d,1)$ vertices : A009997 = number of regions of hyperplane arrangements with {-1,0,1}-valued normals in dimension ***d***

❖ | *P* | : number of vertices of *P*

primitive lattice polytopes

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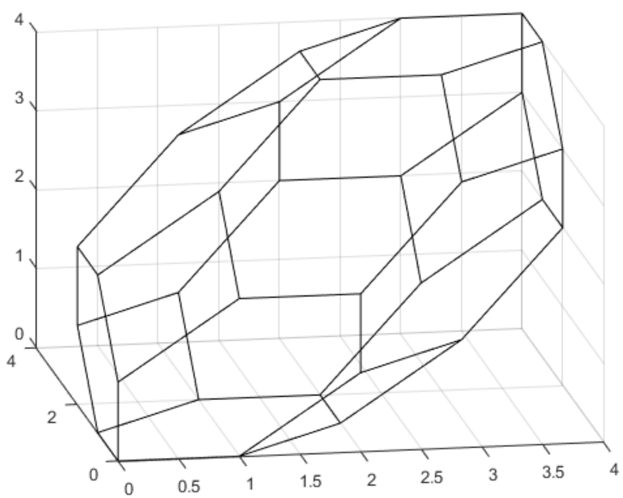
$$| H_\infty(d,1)^+ | \leq \mathbf{m}(d,1) \leq | H_\infty(d,1) |$$

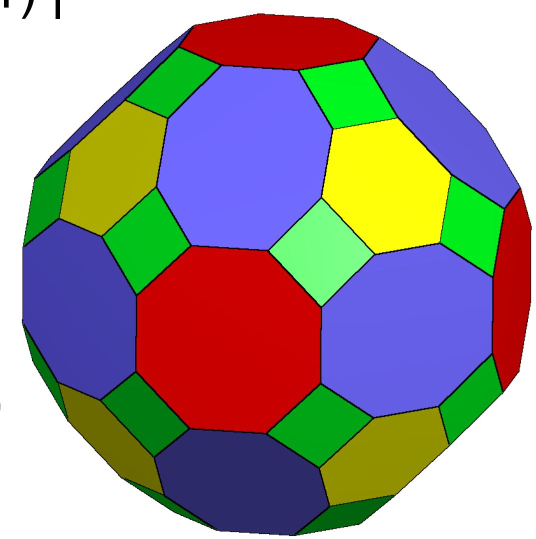
$$32 \leq \mathbf{m}(3,1) \leq 96$$

$$370 \leq \mathbf{m}(4,1) \leq 5\,376$$

$$1\,292 \leq \mathbf{m}(5,1) \leq 1\,981\,440$$



$H_\infty(3,1)^+$



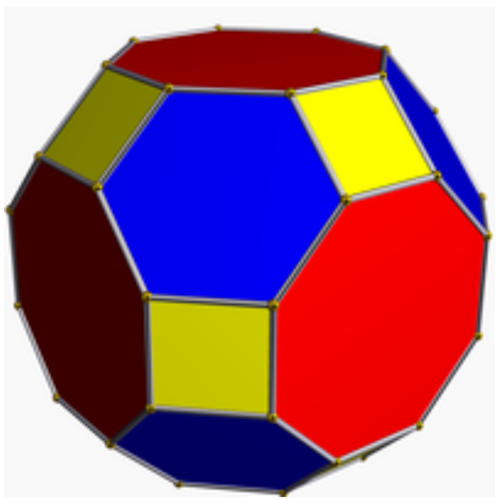
$H_\infty(3,1)$: truncated small rhombicuboctahedron

primitive lattice polytopes

as lower and upper bound for convex matroid optimization parameter

Given matroid **S** of order n, {0,1}-valued d x n matrix **W**, maximum number **m**(d,1) of vertices of conv(**WS**) is independent of n and **S**

$$|\ H_\infty(d,1)^+\ | \leq \mathbf{m}(d,1) \leq |\ H_\infty(d,1)\ |$$



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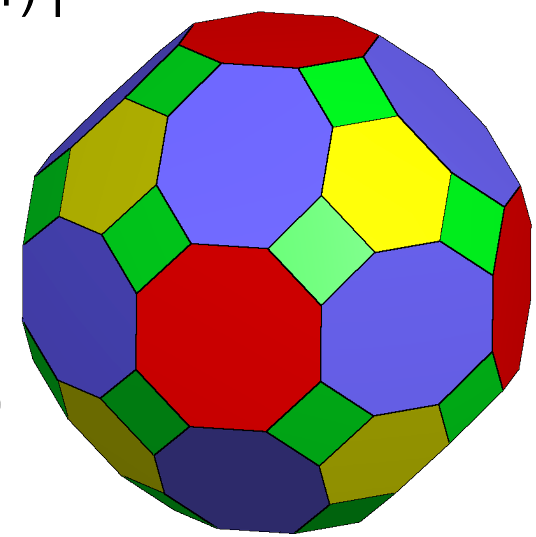

truncated cuboctahedron
(great rhombicuboctahedron)

$H_\infty(3,1)$: truncated small rhombicuboctahedron

❖ lower bound can be further strengthened using computer search for conv(**WS**)

primitive lattice polytopes

complexity questions

For ***fixed*** ***p*** and ***q***, linear optimization over $Z_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$ is polynomial-time solvable, even in ***variable*** dimension ***d*** (polynomial number of generators)

$\Rightarrow$ for ***fixed*** positive ***integers*** ***p*** and ***q***, the following problems are polynomial time solvable:

- ***extremality***: given $x \in \mathbb{Z}^{d}$, decide if x is a vertex of $Z_{q}(d,p)$
- ***adjacency***: given $x_1, x_2 \in \mathbb{Z}^{d}$, decide if $[x_1, x_2]$ is an edge of $Z_{q}(d,p)$
- ***separation***: given rational $y \in \mathbb{R}^{d}$, either assert $y \in Z_{q}(d,p)$, or find $h \in \mathbb{Z}^{d}$ separating y from $Z_{q}(d,p)$ i.e, satisfying $h^{T}y > h^{T}x$ for all $x \in Z_{q}(d,p)$

primitive lattice polytopes

complexity questions

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Q. Existence of a *direct* algorithm for fixed ***p*** and ***q***

Existence of an algorithms for fixed ***p*** and $q = \infty$

Existence of *hole* : $x \in Z_q(d,p)+ \cap \mathbb{Z}^d$ which can not be written as a sum of a subset of generators of $Z_q(d,p)+$

primitive lattice polytopes
diameter and convex matroid optimization bounds

$\delta(d,k)$: largest diameter over all lattice (d,k)-polytopes

- **Conjecture** (holds for all known $\delta(d,k)$: $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$ and $\delta(d,k)$ is achieved, up to translation, by a Minkowski sum of primitive lattice vectors

  $\Rightarrow \delta(d,k) = L(d,k)$ (Minkowski length of cube $\{0,\dots, k\}^d$)

  $\Rightarrow \delta(d,k) = \lfloor (k+1)d/2 \rfloor$ for $k < 2d$

- $| H_\infty(d,1)^+ | \leq \mathbf{m}(d,1) \leq | H_\infty(d,1) |$
  e.g. determination of $\mathbf{m}(3,1)$? ($48 \leq \mathbf{m}(3,1) \leq 96$)

- determination of $\delta(3,k)$ and of $\delta(d,3)$? ($\delta(d,3) = 2d$?)

- Complexity issues, e.g. decide whether a given point is a vertex of $Z_\infty(d,1)$

primitive lattice polytopes
diameter and convex matroid optimization bounds

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- determination of $\delta(3,k)$ and of $\delta(d,3)$? ($\delta(d,3) = 2d$?)

- Complexity issues, e.g. decide whether a given point is a vertex of $Z_\infty(d,1)$

✓ ***thank you***