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Abstract

We show that the simplex method with Dantzig's pivoting rule may require an exponential number of iterations over two highly degenerate instances. The feasible region of the first instance is a full dimensional simplex, and a single point for the second one. In addition, the entries of the constraint matrix, the right-hand-side vector, and the cost vector are $\{0, 1, 2\}$-valued. Those instances, with few vertices and small input data length, illustrate the impact of degeneracy on simplex methods.

Keywords: Linear optimization, simplex methods, small degenerate instances

1 Introduction

hile simplex methods are highly efficient in practice for solving linear optimization, many instances requiring an exponential number of iterations are known. One such instance is the Klee-Minty cube [5] and its variants. In dimension m, the simplex method visits all the 2^m non-degenerate basic feasible solutions corresponding to the vertices of the Klee-Minty cube. Thus, the simplex method requires $2^m - 1$ iterations.

In this note, we essentially perturb the right-hand-side of a Klee-Minty cube considered by Kitahara and Mizuno [3, 4] so that the feasible region becomes a full dimensional simplex. Further perturbing the right-hand-side, the feasible region is reduced to a zero-dimensional simplex, i.e. a single point. Let (LO_0) denote the linear optimization instance considered by Kitahara and Mizuno, and (LO_1) and (LO_2) the instances obtained by perturbing the right-hand-side of (LO_0). We observe that the analysis of Kitahara and Mizuno, showing that (LO_0) requires $2^m - 1$ iterations, can be adapted to show that (LO_1) and (LO_2) require, respectively, $2^{m-1} + 1$ and $2^m - 1$ iterations. For both (LO_1) and (LO_2), an exponential number of iterations are performed at a single degenerate vertex. In addition, the entries of the constraint matrix, the right-hand-side vector, and the cost vector are $\{0, 1, 2\}$-valued for both (LO_1) and (LO_2). Those instances, with few vertices and small input data length, illustrate the impact of degeneracy on simplex methods, and could be of instructional interest.

In a 1980 technical report, reprinted as [8] with a postscript by Avis [1], Zadeh introduced and studied instances requiring an exponential number of iterations whose entries are small integers. In addition, Zadeh pointed out that his constructions, and many others requiring an exponential number of iterations, occur in so-called *deformed products of polytopes*. For more details about pivot based algorithms, instances requiring an exponential number of iterations for simplex methods, and related results, we refer to the survey of Meunier [6], Terlaky and Zhang [7], and Ziegler [9], and references therein.

2 Two small degenerate linear optimization instances

The linear optimization instance (LO_0) considered by Kitahrara and Mizuno in [3, 4], with $\boldsymbol{x} \in \mathcal{R}^m$, is:

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{i=1}^{m} x_i & \\
\text{subject to} & x_1 \le 1 & \\
 & \displaystyle 2\sum_{i=1}^{k-1} x_i + x_k \le 2^k - 1 \quad \text{for } k = 2, 3, \ldots, m & \\
 & \boldsymbol{x} \ge \boldsymbol{0} &
\end{array}
\qquad (LO_0)
$$

The feasible region of (LO_0) is a Klee-Minty cube and the simplex method with Dantzig's pivoting rule visits all its vertices. Thus, $2^m - 1$ iterations may be required to solve the standard form of (LO_0) as observed by Kitahara and Mizuno [3, 4].

The first small linear optimization instance (LO_1) is obtained from (LO_0) by multiplying the first inequality of (LO_0) by 2, and setting to 2 the right-hand-side of the next $m-1$ inequalities:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{m} x_i \\ \text{subject to} & 2x_1 \leq 2 \\ & 2\displaystyle\sum_{i=1}^{k-1} x_i + x_k \leq 2 \quad \text{for } k = 2, 3, \ldots, m \\ & \boldsymbol{x} \geq \mathbf{0} \end{array} \tag{LO_1}$$

One can check that the first $m-1$ inequalities of (LO_1) are redundant, and that the feasible region of (LO_1) is the simplex obtained by intersecting the positive orthant with the half-space defined by $2\sum_{i=1}^{m-1} x_i + x_m \leq 2$. The vertices of this simplex are

$$\{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{m-1}, 2\boldsymbol{e}_m\}$$

where $\boldsymbol{e}_i$ denotes the i-th unit vector of $\mathcal{R}^m$. Note that $\boldsymbol{e}_1$ is a highly degenerate vertex of degree $2m-1$ as it satisfies with equality all the inequalities of (LO_1) except $x_1 \geq 0$. The standard form associated to (LO_1), with slack variable $\boldsymbol{y} \in \mathcal{R}^m$, is:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{m} x_i \\ \text{subject to} & 2x_1 + y_1 = 2 \\ & 2\displaystyle\sum_{i=1}^{k-1} x_i + x_k + y_k = 2 \quad \text{for } k = 2, 3, \ldots, m \\ & \boldsymbol{x} \geq \mathbf{0}, \ \boldsymbol{y} \geq \mathbf{0} \end{array} \tag{LO_1^*}$$

The second small degenerate linear optimization instance (LO_2) is obtained from (LO_0) by setting to 0 the right-hand-side of the first m inequalities:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{i=1}^{m} x_i & \\ \text{subject to} & x_1 \leq 0 & \\ & \displaystyle 2\sum_{i=1}^{k-1} x_i + x_k \leq 0 \quad \text{for } k = 2, 3, \ldots, m & \\ & \boldsymbol{x} \geq \mathbf{0} & \end{array} \qquad (LO_2)$$

One can check that the feasible region of (LO_2) is reduced to the origin $\mathbf{0}$ which forms the unique and highly degenerate optimal point. The standard form associated to (LO_2) is:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{i=1}^{m} x_i & \\ \text{subject to} & x_1 + y_1 = 0 & \\ & \displaystyle 2\sum_{i=1}^{k-1} x_i + x_k + y_k = 0 \quad \text{for } k = 2, 3, \ldots, m & \\ & \boldsymbol{x} \geq \mathbf{0}, \ \boldsymbol{y} \geq \mathbf{0} & \end{array} \qquad (LO_2^*)$$

Proposition 2.1.

(*i*) *For both (LO_1) and (LO_2), the entries of the constraint matrix, the right-hand-side vector, and the cost vector are $\{0, 1, 2\}$-valued.*

(*ii*) *The feasible region of (LO_1) is a full dimensional simplex including a highly degenerate vertex, and that of (LO_2) is reduced to a highly degenerate point.*

(*iii*) *For (LO_1^*), starting from $(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{0}, \mathbf{2})$, the simplex method with Dantzig's pivoting rule visits exactly 3 distinct vertices, and makes $2^{m-1} + 1$ iterations, including $2^{m-1} - 1$ at a highly degenerate vertex.*

(*iv*) *For (LO_2^*), starting from $(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{0}, \mathbf{0})$, the simplex method with Dantzig's pivoting rule visits exactly 1 vertex, and makes $2^m - 1$ iterations at this highly degenerate vertex.*

3 Proof of Proposition 2.1

The first two items of Proposition 2.1 restate the features of (LO_1) and (LO_2). The third item deals with the behaviour of the simplex method with Dantzig's pivoting rule for (LO_1^*). We first outline the simplex pivot sequences for (LO_1^*) with $m = 3$; that is:

$$\begin{array}{lrrrrcl}
\text{maximize} & x_1 & +x_2 & +x_3 & & & \\
\text{subject to} & 2x_1 & & & +y_1 & = & 2 \\
 & 2x_1 & +x_2 & & +y_2 & = & 2 \\
 & 2x_1 & +2x_2 & +x_3 & +y_3 & = & 2 \\
 & \multicolumn{4}{r}{x_1, x_2, x_3, y_1, y_2, y_3} & \geq & 0
\end{array}$$

Setting y_1, y_2, and y_3 as initial basic variables, the first dictionary, or tableau, is:

$$\begin{array}{ccrrrr}
z & = & & x_1 & +x_2 & +x_3 \\
\hline
y_1 & = & 2 & -2x_1 & & \\
y_2 & = & 2 & -2x_1 & -x_2 & \\
y_3 & = & 2 & -2x_1 & -2x_2 & -x_3
\end{array}$$

where nonnegativity conditions $\boldsymbol{x} \geq \mathbf{0}$ and $\boldsymbol{y} \geq \mathbf{0}$ are omitted, and z represents the objective function. The reduced costs, i.e. the coefficients of nonbasic variables x_1, x_2, and x_3 in z, are positive. Thus, dual feasibility is not satisfied and the dictionary is not optimal.

The adopted pivoting rule is Dantzig's rule, and the minimum index rule is used in case of ties as follows:

> The entering variable should be a nonbasic variable with the largest reduced cost. If two or more nonbasic variables have the largest reduced cost, the one with the smallest index is chosen.

> The leaving variable should be basic variable reaching 0 as the entering variable increases. If two or more basic variables reach 0 simultaneously, the one with the smallest index is chosen.

Applying this pivoting rule to the first dictionary, x_1 is the entering variable,

y_1 is the leaving one, and the second dictionary is:

$$\begin{array}{rcrrrr} z & = & 1 & -\frac{y_1}{2} & +x_2 & +x_3 \\ \hline x_1 & = & 1 & -\frac{y_1}{2} & & \\ y_2 & = & & y_1 & -x_2 & \\ y_3 & = & & y_1 & -2x_2 & -x_3 \end{array}$$

x_2 is the next entering variable, y_2 the leaving one, and the third dictionary is:

$$\begin{array}{rcrrrr} z & = & 1 & +\frac{y_1}{2} & -y_2 & +x_3 \\ \hline x_1 & = & 1 & -\frac{y_1}{2} & & \\ x_2 & = & & y_1 & -y_2 & \\ y_3 & = & & -y_1 & +2y_2 & -x_3 \end{array}$$

x_3 is the next entering variable, y_3 the leaving one, and the fourth dictionary is:

$$\begin{array}{rcrrrr} z & = & 1 & -\frac{y_1}{2} & +y_2 & -y_3 \\ \hline x_1 & = & 1 & -\frac{y_1}{2} & & \\ x_2 & = & & y_1 & -y_2 & \\ x_3 & = & & -y_1 & +2y_2 & -y_3 \end{array}$$

y_2 is the next entering variable, x_2 the leaving one, and the fifth dictionary is:

$$\begin{array}{rcrrrr} z & = & 1 & +\frac{y_1}{2} & -x_2 & -y_3 \\ \hline x_1 & = & 1 & -\frac{y_1}{2} & & \\ y_2 & = & & y_1 & -x_2 & \\ x_3 & = & & y_1 & -2x_2 & -y_3 \end{array}$$

y_1 is the next entering variable, x_1 the leaving one, the sixth dictionary is:

$$\begin{array}{rcrrrr} z & = & 2 & -x_1 & -x_2 & -y_3 \\ \hline y_1 & = & 2 & -2x_1 & & \\ y_2 & = & 2 & -2x_1 & -x_2 & \\ x_3 & = & 2 & -2x_1 & -2x_2 & -y_3 \end{array}$$

which is optimal as all reduced costs are nonpositive, and the optimal value is 2.

The observed pivot sequence starts at the initial basic feasible solution $(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{0}, \mathbf{2})$ with an objective value of 0. The highly degenerate second basic feasible solution is $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{e}_1, \boldsymbol{0})$ with an objective value of 1. The following $2^2 - 1$ basic feasible solutions remain at the same vertex with an objective value of 1 until the penultimate iteration. The last iteration reaches the optimal basic feasible solution $(\boldsymbol{x}, \boldsymbol{y}) = (2\boldsymbol{e}_3, \boldsymbol{2} - 2\boldsymbol{e}_3)$ with an objective value of 2. This sequence of 5 simplex pivots is summarized in (S_1^3) where 2 square blocks are highlighted to layout the recursive pattern followed by the sequence (S_1^m) of the 2^{m-1} simplex pivots required for $m \geq 3$. The sequence (S_1^m) is described in Proposition 3.1 which implies the third item of Proposition 2.1

$$\begin{array}{l|c|cc|cc|c}
\text{iteration:} & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
 & y_1 & x_1 & x_1 & x_1 & x_1 & y_1 \\
\text{basic variables:} & y_2 & y_2 & x_2 & x_2 & y_2 & y_2 \\
 & y_3 & y_3 & y_3 & x_3 & x_3 & x_3
\end{array} \qquad (S_1^3)$$

Proposition 3.1. *For $m \geq 3$, the sequence (S_1^m) of the $2^{m-1}+1$ pivots followed by the simplex method with Dantzig's pivoting rule for (LO_1^*) satisfies:*

(i) *the basic variables at iteration 0 are $\{y_1, y_2, \ldots, y_m\}$,*

(ii) *the basic variables at iteration $2^{m-1}+1$ are $\{y_1, y_2, \ldots, y_{m-1}, x_m\}$,*

(iii) *y_m remains a basic variable till iteration 2^{m-2} where it is replaced by x_m which remains a basic variable till iteration $2^{m-1}+1$,*

(iv) *for iterations 1 to 2^{m-2}, the basic variables are obtained by adding y_m to the basic variables of (S_1^{m-1}),*

(v) *for iterations $2^{m-2}+1$ to 2^{m-1}, the basic variables are obtained by adding x_m to the basic variables corresponding to the iterations 2^{m-2} to 1 of (S_1^{m-1}).*

Consequently, starting from the initial basic feasible solution $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{0}, \boldsymbol{2})$ with an objective value of 0, the pivot sequence first reaches the highly degenerate second basic feasible solution $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{e}_1, \boldsymbol{0})$ with an objective value of 1. The following $2^{m-1} - 1$ basic feasible solutions remain at the same vertex with an objective value of 1 until the penultimate iteration. The last iteration reaches the optimal basic feasible solution $(\boldsymbol{x}, \boldsymbol{y}) = (2\boldsymbol{e}_m, \boldsymbol{2} - 2\boldsymbol{e}_m)$ with *an objective value of* 2*. Thus, while visiting exactly 3 vertices, the simplex method with Dantzig's pivoting rule solves* (LO_1^*) *by* $2^{m-1}+1$ *iterations — including* $2^{m-1}-1$ *iterations at a highly degenerate vertex.*

Proof. Since the proof can essentially be adapted from the analysis of Kitahara and Mizuno [4] showing that (LO_0) requires 2^m-1 iterations, we simply illustrate the recursive pattern from (S_1^m) to (S_1^{m+1}) for $m=2$ and 3. One can check that (S_1^2) is as follows.

iteration:	0	1	2	3
basic variables:	y_1	x_1	x_1	y_1
	y_2	y_2	x_2	x_2

(S_1^2)

Note that the first highlighted block of (S_1^3) corresponds to the iterations 1 and 2 of (S_1^2), and that the second highlighted block of (S_1^3) is the mirror image of the first highlighted block. Then, one can check that (S_1^4) is as follows.

iteration:	0	1	2	3	4	5	6	7	8	9
basic variables:	y_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	y_1
	y_2	y_2	x_2	x_2	y_2	y_2	x_2	x_2	y_2	y_2
	y_3	y_3	y_3	x_3	x_3	x_3	x_3	y_3	y_3	y_3
	y_4	y_4	y_4	y_4	y_4	x_4	x_4	x_4	x_4	x_4

(S_1^4)

Note that the first highlighted block of (S_1^4) corresponds to the iterations 1,2,3 and 4 of (S_1^3), and that the second highlighted block of (S_1^4) is the mirror image of the first highlighted block. $\square$

The fourth item of Proposition 2.1 deals with the behaviour of the simplex method with Dantzig's pivoting rule for (LO_2^*). We first outline the simplex pivot sequences for (LO_2^*) with $m=2$; that is:

$$
\begin{array}{lrrrcl}
\text{maximize} & x_1 & +x_2 & & & \\
\text{subject to} & x_1 & & +y_1 & = & 0 \\
 & 2x_1 & +x_2 & +y_2 & = & 0 \\
 & & \multicolumn{2}{r}{x_1, x_2, y_1, y_2} & \geq & 0
\end{array}
$$

Setting y_1 and y_2 as initial basic variables, the first dictionary is:

$$\begin{array}{rcrr} z & = & x_1 & +x_2 \\ \hline y_1 & = & -x_1 & \\ y_2 & = & -2x_1 & -x_2 \end{array}$$

where nonnegativity conditions $\boldsymbol{x} \geq \mathbf{0}$ and $\boldsymbol{y} \geq \mathbf{0}$ are omitted, and z represents the objective function. While $(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{0}, \mathbf{0})$ corresponds to an optimal vertex, the reduced costs, i.e. the coefficients of nonbasic variables x_1 and x_2 in z, are positive. Thus, dual feasibility is not satisfied and the dictionary is not optimal. As for (LO_1^*), the adopted pivoting rule is Dantzig's rule, and the minimum index rule is used in case of ties.

Applying the pivoting rule to the first dictionary, x_1 is the entering variable, y_1 is the leaving one, and the second dictionary is:

$$\begin{array}{rcrr} z & = & -y_1 & +x_2 \\ \hline x_1 & = & -y_1 & \\ y_2 & = & 2y_1 & -x_2 \end{array}$$

x_2 is the next entering variable, y_2 the leaving one, and the third dictionary is:

$$\begin{array}{rcrr} z & = & y_1 & -y_2 \\ \hline x_1 & = & -y_1 & \\ x_2 & = & 2y_1 & -y_2 \end{array}$$

y_1 is the next entering variable, x_1 the leaving one, and the fourth dictionary is optimal as all the reduced costs are nonpositive, and the optimal value is 0:

$$\begin{array}{rcrr} z & = & -x_1 & -y_2 \\ \hline y_1 & = & -x_1 & \\ x_2 & = & -2x_1 & -y_2 \end{array}$$

The observed pivot sequence starts at the initial basic feasible solution $(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{0}, \mathbf{0})$ with an objective value of 0. The following $2^2 - 1$ basic feasible solutions remain at the same vertex with an objective value of 0 until reaching an optimal basis for the same solution $(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{0}, \mathbf{0})$. Using an approach similar to the one used for the third item of Proposition 2.1, one can derive Proposition 3.2 which implies the fourth item of Proposition 2.1.

Proposition 3.2. *For $m \geq 3$, the sequence (S_2^m) of the $2^m - 1$ pivots followed by the simplex method with Dantzig's pivoting rule for (LO_2^*) satisfies:*

- (i) *the basic variables at iteration* 0 *are* $\{y_1, y_2, \ldots, y_m\}$,
- (ii) *the basic variables at iteration* $2^m - 1$ *are* $\{y_1, y_2, \ldots, y_{m-1}, x_m\}$,
- (iii) y_m *remains a basic variable till iteration* $2^{m-1} - 1$ *where it is replaced by* x_m *which remains a basic variable till iteration* $2^m - 1$,
- (iv) *for iterations* 0 *to* $2^{m-1} - 1$, *the basic variables are obtained by adding* y_m *to the basic variables of* (S_2^{m-1}),
- (v) *for iterations* 2^{m-1} *to* $2^m - 1$, *the basic variables are obtained by adding* x_m *to the basic variables corresponding to the iterations* $2^{m-1} - 1$ *to* 0 *of* (S_2^{m-1}).

Consequently, starting from the initial basic feasible solution $(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{0}, \mathbf{0})$ *with an objective value of* 0, *the following* $2^m - 1$ *basic feasible solutions remain at the same vertex with an objective value of* 0 *until reaching an optimal basis for the same solution* $(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{0}, \mathbf{0})$. *Thus, while visiting exactly one vertex, the simplex method with Dantzig's pivoting rule solves* (LO_2^*) *by* $2^m - 1$ *iterations at a highly degenerate vertex.*

Proof. As for the proof of Proposition 3.1, we simply illustrate the recursive pattern from (S_2^m) to (S_2^{m+1}) for $m = 2$. One can check that (S_2^2) and (S_2^3) are as follows.

iteration:	0	1	2	3
basic variables:	y_1	x_1	x_1	y_1
	y_2	y_2	x_2	x_2

(S_2^2)

iteration:	0	1	2	3	4	5	6	7
basic variables:	y_1	x_1	x_1	y_1	y_1	x_1	x_1	y_1
	y_2	y_2	x_2	x_2	x_2	x_2	y_2	y_2
	y_3	y_3	y_3	y_3	x_3	x_3	x_3	x_3

(S_2^3)

Note that the first highlighted block of (S_2^3) corresponds to the iterations $0, 1, 2$, and 3 of (S_2^2), and that the second highlighted block of (S_2^3) is the mirror image of the first highlighted block. $\square$

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