McMaster University

Advanced Optimization Laboratory

Title:

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AdvOL-Report No. 2026/2

March 2026, Hamilton, Ontario, Canada

Flat simplices and kissing polytopes

By

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Abstract

We consider how flat a lattice simplex contained in the hypercube $[0,k]^d$ can be. This question is related to the notion of kissing polytopes: two lattice polytopes contained in the hypercube $[0,k]^d$ are kissing when they are disjoint but their distance is as small as possible. We show that the smallest possible distance of a lattice point P contained in the cube $[0,k]^3$ to a lattice triangle in the same cube that does not contain P is

$$\frac{1}{\sqrt{3k^4-4k^3+4k^2-2k+1}}$$

when k is at least 2. We also improve the known lower bounds on the distance between kissing polytopes when d is at least 4 and k is at least 2.

§ 1. Introduction

Consider a full dimensional simplex S contained in $\mathbb{R}^d$. We say that two proper faces P and Q of S are *opposite* when the vertex set of P and the vertex set of Q form a partition of the vertex set of S. We are interested in measuring how close P and Q can be, or equivalently how flat S can be, under the constraint that S is a lattice (d,k)-simplex—here and in the sequel, a lattice (d,k)-polytope refers to a polytope contained in $[0,k]^d$ and whose vertices are lattice points. More precisely, we will study the smallest possible distance $\varepsilon_i(d,k)$ between an i-dimensional face of a lattice (d,k)-simplex and the opposite $(d-i-1)$-dimensional face. Similar measures of how close two polytopes can be appear in optimization [2, 7, 8, 9, 10, 11] and combinatorics [1, 6]. This quantity is also related to *kissing polytopes*: let us denote by $\varepsilon(d,k)$ the smallest possible distance between two disjoint lattice (d,k)-polytopes and say that two

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d	k			
	1	2	3	$k \geq 4$
2	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{13}$	$\sqrt{(k-1)^2+k^2}$
3	$\sqrt{6}$	$5\sqrt{2}$	$\sqrt{299}$	$\sqrt{2(2k^2-4k+5)(2k^2-2k+1)}$
4	$3\sqrt{2}$	$2\sqrt{113}$	$11\sqrt{71}$	
5	$\sqrt{58}$			
6	$\sqrt{202}$			

Table 1. The known values of $1/\varepsilon(d,k)$.

lattice (d,k)-polytopes whose distance is $\varepsilon(d,k)$ are *kissing* following the notation and terminology from [3, 4, 5]. In fact, $\varepsilon(d,k)$ and $\varepsilon_i(d,k)$ are related as follows.

$$\varepsilon(d,k) = \min\left\{\varepsilon_i(d,k) : 0 \leq i \leq \frac{d-1}{2}\right\}. \tag{1.1}$$

General upper and lower bounds on $\varepsilon(d,k)$ have been proven in [5] and expressions for $\varepsilon(2,k)$ and $\varepsilon(3,k)$ have been established in [3, 4]. Moreover a computer assisted strategy allowed to compute additional values of $\varepsilon(d,k)$ when d and k is reasonably small [4]. The values of $\varepsilon(d,k)$ known so far are reported in Table 1. It is shown in [4] that $\varepsilon(2,k)$ is always the distance between a point and a segment. In other words, it is always equal to $\varepsilon_0(2,k)$. Note that, when d is equal to 2, there is no other possible value of $\varepsilon_i(2,k)$ because $\varepsilon_0(2,k)$ and $\varepsilon_1(2,k)$ coincide (and in general $\varepsilon_i(d,k)$ and $\varepsilon_{d-i-1}(d,k)$ are equal). When d is at least 3 however, it is shown in [3, 4] that all the known values of $\varepsilon(d,k)$ are equal to $\varepsilon_1(d,k)$. In these cases, $\varepsilon_0(d,k)$ is unknown, even in the cases where $\varepsilon(d,k)$ itself if known (with the exception of $\varepsilon_0(d,1)$ which is always equal to $1/\sqrt{d}$, see [5, Lemma 2.4]). The first purpose of this article is to provide the value of $\varepsilon_0(d,k)$ corresponding to every known value of $\varepsilon(d,k)$ when d is at least 3 and k at least 2. In particular, we prove the following.

Theorem 1.1. *If k is at least 2, then*

$$\varepsilon_0(3,k) = \frac{1}{\sqrt{3k^4-4k^3+4k^2-2k+1}}$$

and, up to symmetry, this distance is uniquely achieved between the point $(1,1,1)$ and the triangle with vertices $(0,0,1)$, $(k,k-1,0)$, and $(0,k,k)$

When d is at least 4, the values of $\varepsilon_i(d,k)$ corresponding to the known values of $\varepsilon(d,k)$ are obtained using the computational strategy from [4]. Tables 2 and 3 show all the values of $\varepsilon_0(d,k)$ and $\varepsilon_i(d,1)$ known so far where the bolded entries correspond to the

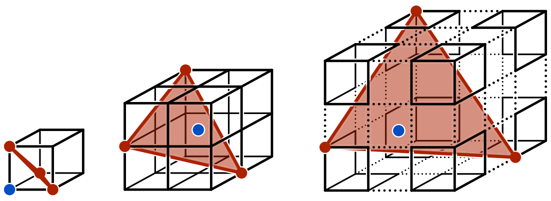

Figure 1. A lattice point and a lattice triangle whose distance achieves $\varepsilon_0(3,k)$ for k equal to 1 (left), 2 (center), and at least 3 (right).

values that we report for the first time. We will adopt a different point of view on how flat a lattice (d,k)-simplex can be by considering the smallest possible distance $\varepsilon_i^u(d,k)$ between the affine hull of an i-dimensional face of a d-dimensional lattice (d,k)-simplex and the affine hull of the opposite $(d-i-1)$-dimensional face, where the exponent u stands for *unbounded*. Ronald Graham and Neil Sloane [6] and Noga Alon and Văn Vũ [1] have considered $\varepsilon_0^u(d,1)$ and shown that

$$\frac{2^{d-1}}{\sqrt{d}^{d+3}} \leq \varepsilon_0^u(d,1) \leq \frac{4^{d+o(d)}}{\sqrt{d}^{d}}.$$

In analogy with $\varepsilon_i(d,k)$, we will denote

$$\varepsilon^u(d,k) = \min\left\{\varepsilon_i^u(d,k) : 0 \leq i \leq \frac{d-1}{2}\right\}. \tag{1.2}$$

Let us first remark that all the known values of $\varepsilon(d,k)$ coincide with $\varepsilon^u(d,k)$: the strategy for the computation of all the values of $\varepsilon(d,k)$ reported in Table 1 was to compute $\varepsilon^u(d,k)$ in order to simplify the argument by only handling affine hulls (instead of convex hulls) and then to observe that an explicit pair P and Q of disjoint lattice (d,k)-polytopes satisfies

$$d(P,Q) = \varepsilon^u(d,k).$$

d	k: 1	2	3	$k \geq 4$
2	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{13}$	$\sqrt{(k-1)^2+k^2}$
3	$\sqrt{3}$	$\boldsymbol{\sqrt{29}}$	$\boldsymbol{\sqrt{166}}$	$\boldsymbol{\sqrt{3k^4-4k^3+4k^2-2k+1}}$
4	2	$\boldsymbol{\sqrt{209}}$	$\boldsymbol{\sqrt{3022}}$	
$\vdots$	$\vdots$			
d	$\sqrt{d}$			

Table 2. The known values of $1/\varepsilon_0(d,k)$.

d	$i=0$	$i=1$	$i=2$
2	$\sqrt{2}$		
3	$\sqrt{3}$	$\sqrt{6}$	
4	2	$3\sqrt{2}$	
5	$\sqrt{5}$	$\sqrt{58}$	$\boldsymbol{\sqrt{55}}$
6	$\sqrt{6}$	$\sqrt{202}$	$\boldsymbol{\sqrt{199}}$

Table 3. The known values of $1/\varepsilon_i(d,1)$.

More precisely, it is shown, in [4] that for all positive k,

$$\varepsilon(2,k)=\varepsilon_0(2,k)=\varepsilon_0^u(2,k)$$

and in [3] that, for all positive k,

$$\varepsilon(3,k)=\varepsilon_1(3,k)=\varepsilon_1^u(3,k).$$

We shall see here when computing $\varepsilon_0(3,k)$ that for all positive k,

$$\varepsilon_0(3,k)=\varepsilon_0^u(3,k).$$

When d is at least 4, computing $\varepsilon_i^u(d,k)$ using the computer-assisted strategy from [4] shows that $\varepsilon_i(d,k)$ and $\varepsilon_i^u(d,k)$ are not always equal. More precisely, all the known values of $\varepsilon_i^u(d,k)$ are equal to $\varepsilon_i(d,k)$ except for $\varepsilon_0^u(d,1)$ when $4\leq d\leq 6$ and $\varepsilon_0^u(4,3)$. The known values of $\varepsilon_0^u(d,k)$ are reported in Table 4 when d is at least 4 (the entries in Table 4 are bolded in order to indicate that all of these values are reported for the first time). Remark that $\varepsilon_0(d,1)$ is greater than $\varepsilon_0^u(d,1)$ when d is at least 4. We provide a lower bound on $\varepsilon^u(d,k)$ for arbitrary d and k, which improves the lower bound on $\varepsilon(d,k)$ stated by Theorems 1.1 and 2.3 from [5]. We also recover and marginally improve the lower bound on $\varepsilon_0^u(d,1)$ by Ronald Graham and Neil Sloane [6].

d	$k=1$	$k=2$	$k=3$
4	$\boldsymbol{\sqrt{7}}$	$\boldsymbol{\sqrt{209}}$	$\boldsymbol{\sqrt{3142}}$
5	$\boldsymbol{\sqrt{19}}$		
6	$\boldsymbol{\sqrt{59}}$		

Table 4. The known values of $1/\varepsilon_0^u(d,k)$.

Theorem 1.2. *For every positive integer d,*

$$\varepsilon_0^u(d,1) \geq \frac{2^{d-1}}{\sqrt{d}^{d+1}}$$

and for any positive integers d and k,

$$\varepsilon^u(d,k) \geq \frac{1}{k^{d-1}\sqrt{d}^d}.$$

The article is organised as follows. We extend to $\varepsilon_i(d,k)$ a number of results that have stated in [3, 4, 5] for $\varepsilon(d,k)$ and prove (1.1) in Section 2. We then establish Theorem 1.2 in Section 3 and Theorem 1.1 in Sections 4 and 5.

§ 2. Some properties of $\varepsilon_i(d,k)$ and $\varepsilon_i^u(d,k)$

By [5, Theorem 5.1], $\varepsilon(d,k)$ is a decreasing function of d—we mean this in the strict sense—for all fixed k. We first extend that property to $\varepsilon_i(d,k)$.

Theorem 2.1. *For any positive integer k and non-negative integer i, $\varepsilon_i(d,k)$, $\varepsilon_i^u(d,k)$, and $\varepsilon^u(d,k)$ are decreasing functions of d.*

Proof. Consider two opposite faces P and Q of a lattice (d,k)-simplex S and denote by i the dimension of P. Identify $\mathbb{R}^d$ as the hyperplane of $\mathbb{R}^{d+1}$ spanned by the first d coordinates. Let v be a vertex of P and consider the point $\tilde{v}$ that admits v as its orthogonal projection on $\mathbb{R}^d$ and whose last coordinate is equal to 1. The convex hull of v and Q is a $(d-i)$-dimensional lattice $(d+1,k)$-simplex disjoint from P, which we will denote by $\tilde{Q}$. We will compare the distance between P and Q with the distance between P and $\tilde{Q}$. Let p be a point in P and q a point in Q such that

$$d(p,q) = d(P,Q).$$

Pick a real number λ that satisfies $0 \leq \lambda \leq 1$ and consider the two points

$$\begin{cases} x = \lambda v + (1-\lambda)p, \\ y = \lambda\tilde{v} + (1-\lambda)q. \end{cases}$$

By convexity, x belongs to P and y belongs to $\tilde{Q}$. In particular,

$$d(P,\tilde{Q}) \leq d(x,y). \tag{2.1}$$

As the vectors $\tilde{v}-v$ and $q-p$ are orthogonal, Pythagoras's theorem yields

$$d(x,y)^2 = \lambda^2 + (1-\lambda)^2 d(p,q)^2$$

and as a consequence,

$$\frac{\partial d(x,y)^2}{\partial \lambda} = 2\lambda\bigl(1 + d(p,q)^2\bigr) - 2d(p,q)^2.$$

Therefore, the distance of x and y is a decreasing function of λ when

$$0 \leq \lambda < \frac{d(p,q)^2}{1 + d(p,q)^2}. \tag{2.2}$$

As the distance of x and y is at least equal to the distance of p and q when λ is equal to 0, for any positive λ satisfying (2.2), the distance of x and y is less than the distance of P and Q. Combining this with (2.1) yields

$$d(P,\tilde{Q}) < d(P,Q).$$

Repeating the argument with a point p in the affine hull of P and a point q in the affine hull of Q such that $d(p,q)$ is the distance of these affine hulls shows that

$$d\bigl(\mathrm{aff}(P), \mathrm{aff}(\tilde{Q})\bigr) < d\bigl(\mathrm{aff}(P), \mathrm{aff}(Q)\bigr).$$

As P and $\tilde{Q}$ are opposite faces of a lattice $(d+1,k)$-simplex of dimension $d+1$, choosing P and Q in such a way that their distance is $\varepsilon_i(d,k)$, we obtain

$$\varepsilon_i(d+1,k) < \varepsilon_i(d,k)$$

and choosing them so that the distance of their affine hulls is $\varepsilon_i(d,k)$ yields

$$\varepsilon_i^u(d+1,k) < \varepsilon_i^u(d,k).$$

It immediately follows from the latter inequality that $\varepsilon^u(d,k)$ is a decreasing function of d for every fixed i and k, as desired. □

The property stated by Theorem 2.1 has the following consequence.

Remark. Consider two positive integers d and k and an integer i that satisfy $0 \leq i < d$. Recall that $\varepsilon_i(d,k)$ coincides with $\varepsilon_{d-i-1}(d,k)$. Therefore

$$\varepsilon_i(d,k) = \varepsilon_{d-i-1}(d,k) < \varepsilon_{d-i-1}(d+1,k) = \varepsilon_{i+1}(d+1,k)$$

Let us now state a consequence of Theorem 2.1 (and Remark 2) that will be useful later in order to establish an expression for $\varepsilon_0(3,k)$.

Corollary 2.2. *Consider two opposite faces P and Q of a d-dimensional lattice (d,k)-simplex such that P has dimension i. If*

$$d(P,Q) = \varepsilon_i(d,k)$$

then

$$d(P,Q) = d\bigl(\mathrm{aff}(P), \mathrm{aff}(Q)\bigr).$$

Proof. Assume that the distance between P and Q is equal to $\varepsilon_i(d,k)$ and, for contradiction that this distance is less than the distance between the affine hulls of P and Q. In this case, the distance between P and Q must be the distance between a face F of P and a face G of Q such that either F is $(i-1)$-dimensional and G is equal to Q or F is equal to P and G is $(d-i-2)$-dimensional. In particular, there exists a hyperplane H of $\mathbb{R}^d$ that contains F and G. Denote by a a non-zero vector orthogonal to H. Since a is non-zero, the coordinates of this vector cannot be all equal to zero and we can assume without loss of generality that a_d is non-zero by, if needed permuting the coordinates of $\mathbb{R}^d$, which does not change the distance between P and Q.

Now identify $\mathbb{R}^{d-1}$ with the hyperplane spanned by the first $d-1$ coordinates of $\mathbb{R}^d$ and denote by π the orthogonal projection from $\mathbb{R}^d$ on $\mathbb{R}^{d-1}$. Since a_d is non-zero, π induces a bijection from H to $\mathbb{R}^{d-1}$. Moreover, that bijection sends a lattice (d,k)-polytope contained in H to a lattice $(d-1,k)$-polytope. Recall that the distance between P and Q is the distance between F and G. As π is 1-Lipschitz,

$$d\big(\pi(F),\pi(G)\big)\leq\varepsilon_i(d,k). \tag{2.3}$$

By construction, $\pi(F)$ and $\pi(G)$ are two opposite faces of a $(d-1)$-dimensional lattice $(d-1,k)$-simplex such that $\pi(F)$ is either i- or $(i-1)$-dimensional. If $\pi(F)$ is i-dimensional, then it follows from (2.3) that $\varepsilon_i(d-1,k)$ is at most $\varepsilon_i(d,k)$ which contradicts Theorem 2.1 and if $\pi(F)$ is $(i-1)$-dimensional, then (2.3) implies that $\varepsilon_{i-1}(d-1,k)$ is at most $\varepsilon_i(d,k)$ which contradicts Remark 2. □

Remark. For fixed d and k, the computational procedure presented in [4] amounts to generate all the possible pairs P and Q exhaustively and to compute the distance of their affine hulls. Therefore it really computes $\varepsilon_i^u(d,k)$. However, the quantity studied in [3, 4, 5] is $\varepsilon(d,k)$ and it turns out that for all the values of d and k investigated so far, $\varepsilon(d,k)$ coincides with $\varepsilon^u(d,k)$. Indeed, for these values of d and k, there exists always a pair P and Q of lattice (d,k)-polytopes satifying

$$d(P,Q)=d\big(\mathrm{aff}(P),\mathrm{aff}(Q)\big)=\varepsilon^u(d,k)$$

which proves that $\varepsilon(d,k)$ coincides with $\varepsilon^u(d,k)$. As we mentioned in the introduction, that observation does not carry over to $\varepsilon_i(d,k)$ and $\varepsilon_i^u(d,k)$. However, by Corollary 2.2 the enumeration procedure from [4] allows to compute $\varepsilon_i(d,k)$ in this case nonetheless: it suffices to check, during that procedure, whether the distance of a considered pair P and Q coincides with the distance of their affine hulls using [4, Remark 2].

Now let us recall that $\varepsilon(d,k)$ is defined following [5] as the smallest possible distance between two disjoint lattice (d,k)-polytopes. However, in the introduction we have defined $\varepsilon_i(d,k)$ as the smallest possible distance between an i-dimensional face of a

d-dimensional lattice (d,k)-simplex and the opposite $(d-i-1)$-dimensional face. While opposite faces of a simplex are necessarily disjoint, the converse is not true and (1.1) is therefore not immediate. Theorem 5.2 from [5] goes a long way towards proving (1.1). Indeed, it states that $\varepsilon(d,k)$ is achieved as the distance between two lattice (d,k)-simplices P and Q whose dimensions sum to $d-1$. There only remains to prove that P and Q are opposite faces of a simplex or, equivalently, that the affine hull of P and Q is $\mathbb{R}^d$. We show that this property holds more generally for $\varepsilon_i(d,k)$.

Theorem 2.3. *For any positive integers d and k, the smallest possible distance between two disjoint lattice (d,k)-simplices, one of which is i-dimensional and the other $(d-i-1)$-dimensional is equal to $\varepsilon_i(d,k)$.*

Proof. The proof is by induction on d. Observe that the statement is immediate when d is equal to 1 because $\varepsilon_0(1,k)$ is equal to 1, the smallest possible distance between two distinct integers. Assume that d is at least 2 and consider two disjoint lattice (d,k)-simplices P and Q such that P has dimension i and Q dimension $d-i-1$.

Assume for contradiction that

$$d(P,Q) < \varepsilon_i(d,k). \tag{2.4}$$

It follows, by the definition of $\varepsilon_i(d,k)$, that P and Q cannot be opposite faces of a d-dimensional simplex. Equivalently, there exists a hyperplane H of $\mathbb{R}^d$ that contains P and Q. Let a be a non-zero vector orthogonal to H. Since a is non-zero, so is one of the coordinates of a, and we can assume without loss of generality that a_d is non-zero by permuting the coordinates of $\mathbb{R}^d$ if needed (which does not change the distance between P and Q). Now, identify $\mathbb{R}^{d-1}$ with the hyperplane of $\mathbb{R}^d$ spanned by the first $d-1$ coordinates and denote by π the orthogonal projection from $\mathbb{R}^d$ on $\mathbb{R}^{d-1}$. As a_d is non-zero, π induces a bijection between H and $\mathbb{R}^{d-1}$ and since π is 1-Lipschitz,

$$d\big(\pi(P),\pi(Q)\big) \leq d(P,Q). \tag{2.5}$$

Consider a point p in P and a point q in Q such that the distance of $\pi(P)$ and $\pi(Q)$ is the distance of $\pi(p)$ and $\pi(q)$. If p is in a proper face of P, then by induction

$$\varepsilon_{i-1}(d-1,k) \leq d\big(\pi(P),\pi(Q)\big) \tag{2.6}$$

and similarly, if q is in a proper face of Q, then by induction

$$\varepsilon_i(d-1,k) \leq d\big(\pi(P),\pi(Q)\big). \tag{2.7}$$

In the former case, combining (2.4), (2.5), and (2.6) contradicts Remark 2. In the latter case, combining (2.4), (2.5), and (2.7) contradicts Theorem 2.1. It therefore suffices to prove that the distance between $\pi(P)$ and $\pi(Q)$ is necessaily achieved by a point in $\pi(P)$ and a point in $\pi(Q)$ one of which is in a proper face of the corresponding simplex. Let us assume that p and q are each contained in the relative interior of the corresponding simplex (otherwise we are done). In that case, observe that $\pi(P)$ and $\pi(Q)$ are both necessarily orthogonal to $\pi(q-p)$. As a consequence $\pi(P)-\pi(p)$ and $\pi(Q)-\pi(q)$ are two simplices contained in the linear hyperplane of $\mathbb{R}^{d-1}$ orthogonal to $\pi(q-p)$. As the dimensions of these two simplices sum to $d-1$, their intersection must be a polytope of positive dimension. Pick a point x in a proper face of that intersection. By construction, either $x+\pi(p)$ is in a proper face of $\pi(P)$ or $x+\pi(q)$ is in a proper face of $\pi(Q)$. As the distance between $x+\pi(p)$ and $x+\pi(q)$ is equal to the distance between $\pi(p)$ and $\pi(q)$, this completes the proof. □

The remainder of the section is devoted to recalling the algebraic model for $\varepsilon(d,k)$ introduced in [4]. This model, that we will state in the case of $\varepsilon_i^u(d,k)$, will be one of the main ingredients in the proofs of Sections 3, 4, and 5. Consider two opposite faces P and Q of a lattice (d,k)-simplex S and denote by i the dimension of P. Further denote by p^0 to p^i the vertices of P and by q^0 to q^{d-i-1} the vertices of Q and consider the $d\times(d-1)$ matrix A whose column j is the vector p^j-p^0 if j is at most i and the vector $q^{j-i}-q^0$ otherwise. Since P and Q are opposite faces of S these vectors collectively span a hyperplane of $\mathbb{R}^d$ and therefore A has rank $d-1$. Further denote by b the vector q^0-p^0. According to [4, Lemma 2], if A^tA is non-singular, then

$$d\big(\mathrm{aff}(P),\mathrm{aff}(Q)\big)=\big\|A(A^tA)^{-1}A^tb-b\big\|.$$

In our case, though, A^tA is necessarily non-singular. Indeed, by the Cauchy–Binet formula [12, Example 10.31] the determinant of this matrix is

$$\det\big(A^tA\big)=\sum_{j=1}^{d}\det(A_j)^2 \tag{2.8}$$

where A_j denotes the $(d-1)\times(d-1)$ matrix obtained by removing row j from the matrix A. Since A has rank $d-1$, one of the matrices A_1 to A_d is non-singular and (2.8) implies that A^tA is non-singular as well.

Remark. With these notation, the lattice vector

$$a=\big(\det(A_1),-\det(A_2),\ldots,(-1)^{d+1}\det(A_d)\big)$$

is orthogonal to both P and Q. Indeed, $a\cdot(p^j-p^0)$ is the determinant of the $d\times d$ matrix obtained by prepending p^j-p^0 to A as a first column. That matrix is necessarily singular since two of its columns coincide, proving that $a\cdot(p^j-p^0)$ is equal to 0 and as a consequence that the vector a is orthogonal to P. The same argument shows that $a\cdot(q^j-q^0)$ also vanishes and that a is orthogonal to Q as well.

The entries of A each are a difference between two non-negative numbers at most k and therefore belong to $[-k,k]$. If k is equal to 1, one can always assume, thanks to the symmetries of $[0,1]^d$ that q^0 is the origin of $\mathbb{R}^d$. If in addition, i is equal to 0, then the columns of A are the vectors q^1 to q^{d-1} and therefore, each entry of A is either 0 or 1. We obtain the following statement as a consequence.

Lemma 2.4. *There exist a $d\times(d-1)$ matrix A and a vector b from $\mathbb{R}^d$ both with integer entries of absolute value at most k such that*

(i) A^tA is non-singular and

(ii) $\varepsilon_i^u(d,k) = \left\|A(A^tA)^{-1}A^tb - b\right\|$.

If in addition, i is equal to 0 and k is equal to 1, then it can be further required that the matrix A has binary entries.

Remark. Even though we will not make use of this in the sequel, Lemma 2.4 still holds when $\varepsilon_i^u(d,1)$ is replaced with $\varepsilon_i(d,1)$, due to Corollary 2.2. In turn, this lemma further holds when $\varepsilon_i^u(d,1)$ is replaced by $\varepsilon(d,1)$ according to Theorem 2.3, and when $\varepsilon_i^u(d,1)$ is replaced by $\varepsilon^u(d,1)$ by (1.2).

§ 3. An improved lower bound on $\varepsilon(d,k)$

This section is devoted to improving the lower bound on $\varepsilon(d,k)$ from [5] and the lower bound on $\varepsilon_0^u(d,1)$ from [6]. The former improvement will in fact be obtained by lower bounding $\varepsilon_i^u(d,k)$ independently from i, therefore providing a lower bound on $\varepsilon^u(d,k)$ and in turn on $\varepsilon(d,k)$. All the proofs are based on a refinement of Lemma 2.4 obtained from arguments on matrix algebra.

Proposition 3.1. *If M is a symmetric $n\times n$ idempotent matrix with real coefficients and b is a vector from $\mathbb{R}^n$, then*

$$\|Mb-b\|^2 = \|b\|^2 - b^tMb.$$

Proof. Denote by I the $n\times n$ identity matrix. If M is symmetric, then

$$\begin{aligned}\|Mb-b\|^2 &= (Mb-b)^t(Mb-b)\\ &= b^t(M-I)^t(M-I)b\\ &= b^t(M-I)^2b.\end{aligned}$$

If in addition M is idempotent, then

$$\begin{aligned}(M-I)^2 &= M^2-2M+I\\ &= I-M\end{aligned}$$

and the proposition follows. □

Recall that Lemma 2.4 provides a formula for $\varepsilon_i^u(d,k)$ as a function of A and b. Thanks to Proposition 3.1 we can refine this formula as follows.

Lemma 3.2. *There exist a $d\times(d-1)$ matrix A and a vector b from $\mathbb{R}^d$ both with integer entries of absolute value at most k such that*

(i) A^tA *is non-singular and*

(ii) $\varepsilon_i^u(d,k)=\sqrt{\|b\|^2-b^tA(A^tA)^{-1}A^tb}$.

If in addition, i is equal to 0 *and k is equal to* 1*, then it can be further required that the matrix A has binary entries.*

Proof. According to Lemma 2.4, there exists a $d\times(d-1)$ matrix A and a vector b from $\mathbb{R}^d$ such that A^tA is non-singular and

$$\varepsilon_i^u(d,k)=\|A(A^tA)^{-1}A^tb-b\|. \tag{3.1}$$

Moreover, both A and b have integer entries of absolute value at most k. Since $A(A^tA)^{-1}A^t$ is symmetric and idempotent, Proposition 3.1 yields

$$\|A(A^tA)^{-1}A^tb-b\|^2=\|b\|^2-b^tA(A^tA)^{-1}A^tb.$$

Combining this with (3.1) completes the proof. □

Remark. According to Remark 2, Lemma 3.2 still holds when, in its statement, $\varepsilon_i^u(d,k)$ is replaced by $\varepsilon_i(d,k)$, $\varepsilon^u(d,k)$, or $\varepsilon(d,k)$.

We shall now lower bound $\varepsilon_i^u(d,k)$ in terms of the determinant of A^tA. Let us point out that by (2.8), this determinant is non-negative.

Theorem 3.3. *There exists a $d\times(d-1)$ matrix A with integer entries of absolute value at most k such that A^tA is non-singular and*

$$\varepsilon_i^u(d,k)\geq\frac{1}{\sqrt{\det(A^tA)}}.$$

If in addition, i is equal to 0 *and k is equal to* 1*, then it can be further required that the matrix A has binary entries.*

Proof. According to Lemma 3.2, there exists a $d\times(d-1)$ matrix A and a vector b from $\mathbb{R}^d$, both of which have integer entries of absolute value at most k, such that A^tA is non-singular and $\varepsilon_i^u(d,k)$ can be expressed as

$$\varepsilon_i^u(d,k)=\sqrt{\|b\|^2-b^tA(A^tA)^{-1}A^tb}. \tag{3.2}$$

Moreover, if i and k are equal to 0 and 1, respectively, then A has binary entries. Denoting by C the cofactor matrix of A^tA,

$$(A^tA)^{-1} = \frac{C^t}{\det(A^tA)}.$$

As a consequence,

$$\|b\|^2 - b^tA(A^tA)^{-1}A^tb = \frac{\|b\|^2\det(A^tA) - b^tAC^tA^tb}{\det(A^tA)}.$$

However, since the entries of A, b, and C are integers, the numerator in the right-hand side is an integer. As in addition the left-hand side is positive and, according to (2.8), the determinant of A^tA is non-negative,

$$\|b\|^2 - b^tA(A^tA)^{-1}A^tb \geq \frac{1}{\det(A^tA)}.$$

Combining this with (3.2) completes the proof □

We conclude the section by proving Theorem 1.2.

Proof of Theorem 1.2. According to Theorem 3.3,

$$\varepsilon_i^u(d,k) \geq \frac{1}{\sqrt{\det(A^tA)}},$$

where A is a $d\times(d-1)$ matrix with integer entries of absolute value at most k such that A^tA is non-singular. Moreover, if i is equal to 0 and k to 1, then A has binary entries. By (2.8), this inequality can rewritten as

$$\varepsilon_i^u(d,k) \geq \frac{1}{\sqrt{\displaystyle\sum_{j=1}^{d}\det(A_j)^2}}. \tag{3.3}$$

where A_j is the $(d-1)\times(d-1)$ matrix obtained from A by removing the jth row. However, by Hadamard's inequality,

$$|\det(A_j)| \leq k^{d-1}\sqrt{d-1}^{\,d-1}$$

and it follows from (3.3) that

$$\varepsilon_i^u(d,k) \geq \frac{1}{\sqrt{d}k^{d-1}\sqrt{d-1}^{\,d-1}}$$

which implies the second announced inequality. Finally, assume that i is equal to 0 and that k is equal to 1. In this case, A and therefore A_j have binary entries. Hadamard's inequality can then be improved into

$$|\det(A_j)| \leq \frac{\sqrt{d}^{\,d}}{2^{d-1}}$$

which, combined with (3.3), proves the first announced inequality. □

§ 4. The distance of a lattice point to a lattice triangle

It is shown in [3] that when k is at least 4, then

$$\varepsilon_1(3,k)=\frac{1}{\sqrt{2(2k^2-4k+5)(2k^2-2k+1)}}. \tag{4.1}$$

It is also shown that $\varepsilon(3,k)$ is always strictly less than $\varepsilon_0(3,k)$ for all positive k (see [3, Theorem 1.2]). It follows in particular that

$$\varepsilon(3,k)=\varepsilon_1(3,k)<\varepsilon_0(3,k).$$

The purpose of this section is to establish an expression similar to (4.1) but for $\varepsilon_0(3,k)$. Let us first remark that the point $P^\star$ of coordinates $(1,1,1)$ and the triangle $Q^\star$ whose vertices are $(0,0,1)$, $(k,k-1,0)$, and $(0,k,k)$ satisfy

$$d(P^\star,Q^\star)=\frac{1}{\sqrt{3k^4-4k^3+4k^2-2k+1}} \tag{4.2}$$

for every k at least 2. In particular, $\varepsilon_0(3,k)$ is at most the right-hand side of (4.2). We shall show that this upper bound on $\varepsilon_0(3,k)$ is always sharp.

When k is reasonably small, $\varepsilon_0(3,k)$ can be obtained computationally by the exhaustive enumeration procedure from [4]. This procedure also allows to recover all the lattice points in $[0,k]^3$ and lattice $(3,k)$-triangles, whose distance is equal to $\varepsilon_0(3,k)$. We obtain the following from that procedure.

Proposition 4.1. *If k is at least 2 and at most 8, then*

$$\varepsilon_0(3,k)=\frac{1}{\sqrt{3k^4-4k^3+4k^2-2k+1}}$$

and, up to symmetry, this distance is uniquely achieved by $P^\star$ and $Q^\star$.

Note that the exhaustive enumeration procedure that provides Proposition 4.1 also shows that $\varepsilon_0(3,k)$ coincides with $\varepsilon_0^u(3,k)$ when $2\leq k\leq 8$.

We will extend the statement of Proposition 4.1 to every integer k greater than 8. Consider a lattice point P contained in the cube $[0,k]^3$ and a lattice $(3,k)$-triangle Q such that P is not contained in the affine hull of Q (so that P and Q are indeed opposite faces of a lattice tetrahedron). Denote by q^0, q^1, and q^2 the vertices of Q. Following the algebraic model from [4] that we recalled at the end of Section 2, we will denote by

A the 3×2 matrix whose first column is $q^1 - q^0$ and whose second column is $q^2 - q^0$ and by b the vector $q^0 - P$. As explained in Section 2, A^tA is non-singular and

$$d\big(\mathrm{aff}(P), \mathrm{aff}(Q)\big) = \big\|A(A^tA)^{-1}A^tb - b\big\|. \tag{4.3}$$

Now consider the lattice point x from the hypercube $[-k,k]^9$ whose coordinates can be obtained from the entries of A and b by identification as

$$A = \begin{bmatrix} x_1\, x_4 \\ x_2\, x_5 \\ x_3\, x_6 \end{bmatrix} \tag{4.4}$$

and

$$b = \begin{bmatrix} x_7 \\ x_8 \\ x_9 \end{bmatrix}. \tag{4.5}$$

The determinant of A^tA is then the function g of x defined by

$$g(x) = (x_1x_5 - x_2x_4)^2 + (x_1x_6 - x_3x_4)^2 + (x_2x_6 - x_3x_5)^2. \tag{4.6}$$

This determinant is non-zero and a sum of squares and therefore $g(x)$ is always positive. With this notation, the right-hand side of (4.3) can be expressed as

$$\|A(A^tA)^{-1}A^tb - b\| = \frac{|f(x)|}{\sqrt{g(x)}} \tag{4.7}$$

where f is the function of x defined by

$$f(x) = (x_1x_5 - x_2x_4)x_9 - (x_1x_6 - x_3x_4)x_8 + (x_2x_6 - x_3x_5)x_7. \tag{4.8}$$

It should be noted that up to six different points x may be obtained from P and Q by permuting q^0, q^1, and q^2. We shall denote by $\mathcal{X}_{P,Q}(k)$ all the lattice points x in the hypercube $[-k,k]^9$ that can be obtained from P and Q as we have just described by possibly permuting q^0, q^1, and q^2 or by applying to both P and Q an isometry of the cube $[0,k]^3$. Note that these isometries send lattice $(3,k)$-polytopes to lattice $(3,k)$-polytopes as they consist in permuting the coordinates of $\mathbb{R}^3$ and performing symmetries with respect to the planes of the form

$$\{x \in \mathbb{R}^3 : x_i = k/2\}$$

where i is equal to either 1, 2, or 3. Our strategy consists in showing that, when k is at least 9 and the distance between the affine hulls of P and Q is $\varepsilon_0^u(3,k)$, some point x in

$\mathcal{X}_{P,Q}(k)$ can be recovered from a finite set $\mathcal{A}$ of lattice points independent from k. More precisely, $\mathcal{A}$ will be the set of the lattice points a contained in $[0,6]^2\times[-4,6]\times[0,6]^3$ such that a_6 is at least $-a_3$. In order to relate $\mathcal{A}$ with $\mathcal{X}_{P,Q}(k)$ we define the affine map $\phi_k : \mathbb{R}^6 \to \mathbb{R}^6$ such that the ith coordinate of $\phi_k(a)$ is given by

$$\big[\phi_k(a)\big]_i = \begin{cases} a_i \text{ if } i \text{ is equal to 3 or 4,} \\ k - a_i \text{ otherwise.} \end{cases}$$

From now on, we identify $\mathbb{R}^6$ with the subspace of $\mathbb{R}^9$ spanned by the first six coordinates. Section 5 will be dedicated to establish the following theorem that provides the announced reduction from $\mathcal{X}_{P,Q}(k)$ to $\mathcal{A}$.

Theorem 4.2. *Consider a lattice point P in $[0,k]^3$ and a lattice $(3,k)$-triangle Q. If k is at least 9 and the distance between P and the affine hull of Q is $\varepsilon_0^u(3,k)$, then the orthogonal projection on $\mathbb{R}^6$ of some point in $\mathcal{X}_{P,Q}(k)$ belongs to $\phi_k(\mathcal{A})$.*

In the remainder of the section, we explain how Theorem 4.2 allows to determine $\varepsilon_0(3,k)$ when k is at least 9. Let us first state two propositions. The first one is established in the proof of Lemma 3.2 from [3].

Proposition 4.3. *Consider a lattice point P in $[0,k]^3$ and a lattice $(3,k)$-triangle Q. If the affine hull of Q does not contain P, then for every point x in $\mathcal{X}_{P,Q}(k)$, the squares $(x_1x_5 - x_2x_4)^2$, $(x_1x_6 - x_3x_4)^2$, and $(x_2x_6 - x_3x_5)^2$ are each at most k^4.*

The second proposition follows from (4.3), (4.7), and Proposition 4.3.

Proposition 4.4. *Consider a lattice point P contained in the cube $[0,k]^3$ and a lattice $(3,k)$-triangle Q. If the distance between P and the affine hull of Q is equal to $\varepsilon_0^u(3,k)$, then for every point x in $\mathcal{X}_{P,Q}(k)$,*

(i) $f(x)$ is equal to 1 or to -1 and

(ii) $g(x)$ is equal to $1/\varepsilon_0^u(3,k)^2$.

Proof. Assume that the distance between P and the affine hull of Q is equal to $\varepsilon_0^u(3,k)$. It follows from (4.3) and (4.7) that

$$\varepsilon_0^u(3,k) = \frac{|f(x)|}{\sqrt{g(x)}}. \tag{4.9}$$

Since $f(x)$ is an integer and $\varepsilon_0^u(3,k)$ is non-zero, the absolute value of $f(x)$ is therefore at least 1. There remains to show that this absolute value is at most 1. Not only will this show that $f(x)$ is equal to 1 or -1 but also that $g(x)$ is the squared inverse of $\varepsilon_0^u(3,k)$ because of (4.9). Assume for contradiction that $|f(x)|$ is greater than 1. In that case, (4.9) and the integrality of $f(x)$ imply that

$$\varepsilon_0^u(3,k) \geq \frac{2}{\sqrt{g(x)}}.$$

However, it follows from (4.6) and Proposition 4.3 that $g(x)$ is at most $3k^4$ and we get that $\varepsilon_0^u(3,k)$, and therefore $\varepsilon_0(d,k)$, is at least $2/(\sqrt{3}k^2)$ which contradicts the fact that $\varepsilon_0(3,k)$ is at most the right-hand side of (4.2). □

Let $\tilde{f}:\mathbb{Z}^9\backslash\{0\}\rightarrow\mathbb{N}$ be the map such that $\tilde{f}(x)$ is the greatest common divisor of $x_1x_5-x_2x_4$, $x_1x_6-x_3x_4$, and $x_2x_6-x_3x_5$. According to (4.8), if the absolute value of $f(x)$ is equal to 1, these three quantities must be relatively prime and, therefore, $\tilde{f}(x)$ is also necessarily equal to 1. In other words, Proposition 4.4 still holds when f is replaced by $\tilde{f}$ in its statement. Now, a crucial observation is that both $\tilde{f}(x)$ and $g(x)$ only depend on the first six coordinates of x. In particular, combining Theorem 4.2 and Proposition 4.4 allows to compute $\varepsilon_0(3,k)$ by only studying the finitely-many functions $k\rightarrow\tilde{f}\circ\phi_k(a)$ and $k\rightarrow g\circ\phi_k(a)$ when a ranges over $\mathcal{A}$.

Let us explain how that can be done using symbolic computation. First observe that $\tilde{f}\circ\phi_k(a)$ is the absolute value of the greatest common divisor of

$$\begin{cases} k^2-(a_1+a_4+a_5)k+a_1a_5+a_2a_4,\\ k^2-(a_1+a_6)k+a_1a_6-a_3a_4,\\ k^2-(a_2+a_3+a_6)k+a_2a_6+a_3a_5.\end{cases}$$

For any given point a in $\mathcal{A}$, one can check using symbolic computation that, when the greatest common divisor of these three quadratic polynomials is a linear or quadratic function of k, its absolute value is never equal to 1 when k is at least 9. This just amounts to solve two linear or quadratic equations which is easily done using symbolic computation. Hence, it follows from Theorem 4.2 and Proposition 4.4 that $\varepsilon_0^u(3,k)$ is the inverse of $\sqrt{g\circ\phi_k(a)}$ for some point a in $\mathcal{A}$ such that are such that the greatest common divisor of the above three quadratic polynomials has degree 0. Now consider the set of all the different polynomials $k\rightarrow g\circ\phi_k(a)$ obtained when a ranges over that particular subset of $\mathcal{A}$. Symbolic computation shows that the largest real root of the difference between any two such polynomials is less than 9. One of these polynomials is therefore greater than all the others when k ranges over the interval $[9,+\infty[$. This polynomial turns out to be $3k^4-4k^3+4k^2-2k+1$ and it is obtained for just the four points $(0,0,-1,1,0,1)$, $(0,0,1,0,1,0)$, $(0,1,-1,0,0,1)$, and $(0,1,0,1,0,0)$ from $\mathcal{A}$. Using these computational results, we now prove Theorem 1.1.

Proof of Theorem 1.1. When k is at least 2 and at most 8, the theorem follows from Proposition 4.1 and we will therefore assume that k is at least 9.

First observe that according to (4.2),

$$\varepsilon_0(3,k) \leq \frac{1}{\sqrt{3k^4 - 4k^3 + 4k^2 - 2k + 1}}. \tag{4.10}$$

We shall prove the reverse inequality. Consider a lattice point P in $[0,k]^3$ and a lattice $(3,k)$-triangle Q such that the distance between P and the affine hull of Q is $\varepsilon_0^u(3,k)$. According to Theorem 4.2, there exists a point x in $\mathcal{X}_{P,Q}(k)$ and a point a in $\mathcal{A}$ such that $\phi_k(a)$ is the orthogonal projection of x on $\mathbb{R}^6$. By Proposition 4.4, $|f(x)|$ and therefore $\tilde{f} \circ \phi_k(a)$, is equal to 1. Moreover, $g(x)$, and therefore $g \circ \phi_k(a)$, is the inverse of $\varepsilon_0^u(3,k)^2$. As $\varepsilon_0^u(3,k)$ is at most $\varepsilon_0(3,k)$, (4.10) implies

$$g \circ \phi_k(a) \geq 3k^4 - 4k^3 + 4k^2 - 2k + 1. \tag{4.11}$$

However, as $\tilde{f} \circ \phi_k(a)$ is equal to 1 the above computational results imply that (4.11) is an equality and, as a consequence,

$$\varepsilon_0(3,k) \geq \varepsilon_0^u(3,k) = \frac{1}{\sqrt{3k^4 - 4k^3 + 4k^2 - 2k + 1}}$$

which combined with (4.10) provides the desired expression for $\varepsilon_0(3,k)$ and shows that $\varepsilon_0^u(3,k)$ coincides with $\varepsilon_0(3,k)$. In addition, a is one of the four points $(0,0,-1,1,0,1)$, $(0,0,1,0,1,0)$, $(0,1,-1,0,0,1)$, and $(0,1,0,1,0,0)$. The image of each of these points by ϕ_k provides the first six coordinates of a lattice point in the hypercube $[-k,k]^9$ from which a lattice triangle Q can be reconstructed. All the triangles that can be reconstructed from these points coincide, up to the symmetries of $[0,k]^3$ to the triangle $Q^\star$, which proves that Q coincides with $Q^\star$, up to symmetry.

There remains to show that $P^\star$ is the only lattice point in the cube $[0,k]^3$ whose distance with $Q^\star$ is equal to $\varepsilon_0(3,k)$. Assume for contradiction that P is not equal to $P^\star$ but has the same distance to $Q^\star$ than $P^\star$ and observe that by Corollary 2.2,

$$d(P,Q^\star) = d\big(\mathrm{aff}(P), \mathrm{aff}(Q^\star)\big).$$

As a consequence, the projection of both P and $P^\star$ on the affine hull of $Q^\star$ must belong to $Q^\star$. In turn, this means that one of the points obtained by adding or subtracting $P^\star - P$ to each vertex of $Q^\star$ is a lattice point q contained in $Q^\star$. Observe that q cannot be a vertex of $Q^\star$. Indeed otherwise, the line segment with extremities P and $P^\star$ would be a translate of an edge of $Q^\star$. In that case, the orthogonal projections of P and $P^\star$ on the affine hull of $Q^\star$ are necessarily two vertices of $Q^\star$ because otherwise, one of them would be outside of $Q^\star$. This would imply that $\varepsilon_0(3,k)$ is the distance between two lattice points and therefore at least 1. Hence, q is a lattice point contained in $Q^\star$ and distinct from its three vertices. But in that case, $Q^\star$ contains a strictly smaller lattice $(3,k)$-triangle whose distance to P or $P^\star$ is $\varepsilon_0(3,k)$, which is impossible because any such triangle must coincide with $Q^\star$ up to symmetry. $\square$

§ 5. The proof of Theorem 4.2

In order to prove Theorem 4.2, we first need to single out a representative point in $\mathcal{X}_{P,Q}(k)$ where P is a lattice point in $[0,k]^3$ and Q a lattice $(3,k)$-triangle whose affine hull does not contain P. This can be done as follows.

Lemma 5.1. *For any lattice point P in $[0,k]^3$ and any lattice $(3,k)$-triangle Q such that P and Q are disjoint, there exists a point x in $\mathcal{X}_{P,Q}(k)$ such that*

(i) x_1, x_2, and x_4 to x_6 are non-negative,

(ii) x_1x_5 is at least x_2x_4, and

(iii) x_4 is at most x_2.

Proof. Consider a lattice point P contained in $[0,k]^3$ and a lattice $(3,k)$-triangle Q such that P and Q are disjoint. Let us identify $\mathbb{R}^2$ with the plane spanned by the first two coordinates of $\mathbb{R}^3$ and denote by $\pi:\mathbb{R}^3\rightarrow\mathbb{R}^2$ the orthogonal projection on $\mathbb{R}^2$. Note that $\pi(Q)$ is a triangle or a line segment.

Consider the smallest rectangle of the form $[a,b]\times[c,d]$ that contains $\pi(Q)$. Since $\pi(Q)$ has at most three vertices, one of the edges of the rectangle $[a,b]\times[c,d]$ does not contain a vertex of $\pi(Q)$ in its relative interior. That edge must then admit one of the vertices of $\pi(Q)$ as an extremity because otherwise $[a,b]\times[c,d]$ could be made smaller while still containing $\pi(Q)$. Thanks to the symmetries of the cube $[0,k]^3$, one can assume that (a,c) is a vertex of $\pi(Q)$ while preserving the distance between P and Q. In that case, there is at least one vertex of Q whose orthogonal projection on $\mathbb{R}^2$ is (a,c) and we shall denote one such vertex of Q by q^0. Let q^1 and q^2 be the two vertices of Q distinct from q^0. As $\pi(q^0)$ is equal to (a,c) and both $\pi(q^1)$ and $\pi(q^2)$ are contained in $[a,b]\times[c,d]$, the first two coordinates of both q^1-q^0 and q^2-q^0 are non-negative.

Now denote by A the 3×2 matrix whose first column is q^1-q^0 and whose second column is q^2-q^0. Further denote by b the vector q^0-P and by x the lattice point in $\mathcal{X}_{P,Q}(k)$ obtained from A and b via (4.4) and (4.5). Since the first two coordinates of q^1-q^0 and q^2-q^0 are non-negative, x_1, x_2, x_4, and x_5 are non-negative. Observe that x_1x_5 may be less than x_2x_4. However, one can recover this property by exchanging the labels of q^1 and q^2 which results in a point x that still belongs to $\mathcal{X}_{P,Q}(k)$ by the definition of this set. Note that exchanging these labels does not affect the sign of x_1, x_2, x_4, or x_5. Similarly, one can make x_4 less than or equal to x_2 by exchanging q^1

with q^2 and by permuting the first two coordinates of $\mathbb{R}^3$. Note that the combination of these two operations does not affect the signs of x_1, x_2, x_4, x_5, or $x_1x_5 - x_2x_4$ and results in a point x that still belongs to $\mathcal{X}_{P,Q}(k)$.

Finally, if the third coordinate of $q^2 - q^0$ (and therefore x_6) is negative, we can make it non-negative without changing the sign of x_1, x_2, x_4, x_5, $x_1x_5 - x_2x_4$, or $x_2 - x_4$ by replacing both P and Q with their symmetric with respect to

$$\{x \in \mathbb{R}^3 : x_3 = k/2\}.$$

This results in a point x in $\mathcal{X}_{P,Q}(k)$ with the desired properties. □

Under the assumption that the distance of P and Q is equal to $\varepsilon_0(3,k)$, we will bound the coordinates of the points x in $\mathcal{X}_{P,Q}(k)$ that satisfy the assertions (i), (ii), and (iii) in the statement of Lemma 5.1. In order to do that, we will use the following three lower bounds on the polynomial expression

$$r(k) = k^4 - 4k^3 + 4k^2 - 2k + 1$$

whose straightforward proofs, that consist in solving polynomial inequalities of degree at most 4 in k or in $\sqrt{2k+1}$, are omitted.

Proposition 5.2. *If k is at least 9, then*

(i) $r(k)$ *is greater than* $k^2(k-3)^2$,

(ii) $r(k)$ *is at least* $\big(k^2 - 2k - 1\big)^2$, *and*

(iii) $r(k)$ *is at least* $\big(k^2 - 4(k - \sqrt{2k+1})\big)^2$,

The announced bounds on the coordinates of x are given depending on the sign of x_3. The following lemma treats the case when x_3 is non-negative.

Lemma 5.3. *Assume that k is at least 9. Consider a lattice point P contained in $[0,k]^3$ and a lattice $(3,k)$-triangle Q such that the distance of P to the affine hull of Q is $\varepsilon_0^u(3,k)$. Further consider a point x in $\mathcal{X}_{P,Q}(k)$ such that*

(i) x_1, x_2, *and* x_4 *to* x_6 *are non-negative,*

(ii) x_1x_5 *is at least* x_2x_4, *and*

(iii) x_4 *is at most* x_2.

If x_3 is non-negative, then x_1, x_5, and x_6 are at least $k-2$ while x_4 is at most 3. Moreover, either x_2 is at least $k-2$ and x_3 is at most 3 or inversely, x_3 is at least $k-2$ and x_2 is at most 3.

Proof. Under the assumption that the distance between P and the affine hull of Q is equal to $\varepsilon_0^u(3,k)$, we obtain from Proposition 4.4 that

$$\varepsilon_0^u(3,k)=\frac{1}{\sqrt{g(x)}}$$

and, in turn, by (4.2) that

$$g(x)\geq 3k^4-4k^3+4k^2-2k+1. \tag{5.1}$$

It then follows from (4.6) and Proposition 4.3 that

$$(x_1x_5-x_2x_4)^2\geq r(k). \tag{5.2}$$

Since all the x_i in the left-hand side of (5.2) are non-negative and x_1x_5 is at least x_2x_4, it follows that $x_1^2x_5^2$ is at least $r(k)$. Now recall that the absolute value of x_1 and x_5 is at most k. As, according to Proposition 5.2, $r(k)$ is greater than $k^2(k-3)^2$, it follows that x_1 and x_5 are both at least $k-2$.

Likewise, it follows from (5.2) that

$$\left(k^2-x_2x_4\right)^2\geq r(k).$$

However, according to Proposition 5.2, $r(k)$ is at least $(k^2-2k-1)^2$ and it follows that x_2x_4 is at most $2k+1$. In summary, we have established that

$$\left\{\begin{array}{l} x_1\geq k-2,\\ x_5\geq k-2,\\ x_2x_4\leq 2k+1. \end{array}\right. \tag{5.3}$$

Since x_4 is at most x_2, it follows from the third inequality that x_4 is at most $\sqrt{2k+1}$. Now, according to (5.1), (4.6), and Proposition 4.3,

$$(x_1x_6-x_3x_4)^2\geq r(k). \tag{5.4}$$

Observe that x_1x_6 must be at least x_3x_4. Indeed, otherwise, as x_3 is at most $\sqrt{2k+1}$, the left-hand side of (5.4) would be less than $k^2(2k+1)$ and therefore less than $k^2(k-3)^2$, which would contradict Proposition 5.2. As x_1x_6 is at least x_3x_4, borrowing the argument we used to prove (5.3) yields

$$\left\{\begin{array}{l} x_6\geq k-2,\\ x_3x_4\leq 2k+1. \end{array}\right. \tag{5.5}$$

In the rest of the proof we consider two cases depending on whether x_2x_6 is at least x_3x_5 or not. First, if x_2x_6 is at least x_3x_5, the argument that we used to establish both (5.3) and (5.5) further yields

$$\left\{\begin{array}{l} x_2\geq k-2,\\ x_3x_5\leq 2k+1. \end{array}\right. \tag{5.6}$$

As x_2 is at least $k-2$, it follows from (5.3) that

$$x_4 \leq \frac{2k+1}{k-2} \tag{5.7}$$

and as x_5 is at least $k-2$, it follows from (5.6) that

$$x_3 \leq \frac{2k+1}{k-2}. \tag{5.8}$$

Since k is at least 9, the right-hand side of (5.7) and (5.8) is at most 3 and we obtain that x_1, x_2, x_5, and x_6 are all at least $k-2$ while x_3 and x_4 are at most 3, as desired. Finally if x_2x_6 is less than x_3x_5, then using the same argument but where x_2 is exchanged with x_3 and x_5 with x_6 shows that x_1, x_3, x_5, and x_6 are all at least $k-2$ while x_2 and x_4 are at most 3, which completes the proof □

In order to prove a statement similar to that of Lemma 5.3 but in the case when x_3 is negative, we will use of the following constraint satisfied by the points in $\mathcal{X}_{P,Q}(k)$ that arises from the embedding of P and Q into $[0,k]^3$.

Proposition 5.4. *Consider a lattice point P in $[0,k]^3$ and a lattice $(3,k)$-triangle Q such that P and Q are disjoint. For every point x in $\mathcal{X}_{P,Q}(k)$,*

$$|x_3 - x_6| \leq k.$$

Proof. Consider a point x in $\mathcal{X}_{P,Q}(k)$. By the definition of $\mathcal{X}_{P,Q}(k)$, there exist a lattice point $\tilde{P}$ contained in $[0,k]^3$ and a lattice $(3,k)$-triangle $\tilde{Q}$ that coincide with P and Q up to some isometry of $[0,k]^3$ and satisfy

$$\begin{cases} x_3 = \tilde{q}^1_3 - \tilde{q}^0_3 \text{ and} \\ x_6 = \tilde{q}^2_3 - \tilde{q}^0_3, \end{cases}$$

where $\tilde{q}^0$, $\tilde{q}^1$, and $\tilde{q}^2$ adequately label the vertices of $\tilde{Q}$. Therefore

$$x_3 - x_6 = \tilde{q}^1_3 - \tilde{q}^2_3$$

and since the points $\tilde{q}^0$, $\tilde{q}^1$, and $\tilde{q}^2$ are contained in the cube $[0,k]^3$, the absolute value of $x_3 - x_6$ is at most k, as desired. □

Thanks to the constraint stated by Proposition 5.4, we can bound the coordinates of the points x in $\mathcal{X}_{P,Q}(k)$ that satisfy the assertions (i), (ii), and (iii) in the statement of Lemma 5.1 and whose third coordinate is negative.

Lemma 5.5. *Assume that k is at least 9. Consider a lattice point P contained in $[0,k]^3$ and a lattice $(3,k)$-triangle Q such that the distance of P to the affine hull of Q is $\varepsilon_0(3,k)$. Further consider a point x in $\mathcal{X}_{P,Q}(k)$ such that* *(i)* x_1, x_2, *and* x_4 *to* x_6 *are non-negative,*

(ii) x_1x_5 *is at least* x_2x_4, *and*

(iii) x_4 *is at most* x_2.

If x_3 *is negative, then* x_1 *and* x_5 *are at least* $k-2$ *while* x_2 *and* x_6 *are at least* $k-6$. *Moreover* x_3 *is at least* -4 *and* x_4 *is at most* 6.

Proof. The proof begins exactly like that of Lemma 5.3. In particular, we get

$$g(x) \geq 3k^4 - 4k^3 + 4k^2 - 2k + 1 \tag{5.9}$$

and

$$\begin{cases} x_1 \geq k-2, \\ x_5 \geq k-2, \\ x_2x_4 \leq 2k+1. \end{cases}$$

As x_4 is at most x_2, it follows that x_4 is at most $\sqrt{2k+1}$. Now, combining (4.6), (5.9), and Proposition 4.3, we obtain

$$(x_1x_6 - x_3x_4)^2 \geq r(k) \tag{5.10}$$

Under the assumption that x_3 is negative, since x_1 is at most k, x_4 at most $\sqrt{2k+1}$, and, by Proposition 5.4, x_6 at most $k+x_3$, this implies that

$$\big(k^2 + \big(k-\sqrt{2k+1}\big)x_3\big)^2 \geq r(k).$$

Developing the left-hand side and expressing $r(k)$ in terms of k yields

$$\big(k-\sqrt{2k+1}\big)^2x_3^2 + 2k^2\big(k-\sqrt{2k+1}\big)x_3 + 4k^3 - 4k^2 + 2k - 1 \geq 0.$$

Treating this as a quadratic inequality in $(k-\sqrt{2k+1})x_3$, the discriminant is $4r(k)$ which is positive because k is at least 9. Hence, either

$$\big(k-\sqrt{2k+1}\big)x_3 \leq -k^2 - \sqrt{r(k)} \tag{5.11}$$

or

$$\big(k-\sqrt{2k+1}\big)x_3 \geq -k^2 + \sqrt{r(k)}. \tag{5.12}$$

However, by Proposition 5.2, $r(k)$ is at least $k^2(k-3)^2$ and (5.11) implies

$$x_3 \leq -k\frac{2k-3}{k-\sqrt{2k+1}}.$$

As the right-hand side of this inequality is less than $-k$ and x_3 is at least $-k$, this inequality cannot hold and this proves that (5.12) does. Hence,

$$x_3 \geq \frac{\sqrt{r(k)} - k^2}{k - \sqrt{2k+1}}. \tag{5.13}$$

By Proposition 5.2, the numerator of the right-hand side is greater than minus four times its denominator and we get that x_3 is at least -4.

Let us now bound x_6. As x_1 and x_4 are both at most k, (5.10) implies that $r(k)$ is at most $k^2(x_6 - x_3)^2$ and therefore at most $k^2(x_6 + 4)^2$. This proves that x_6 is at least $k - 6$ as by Proposition 5.2, $r(k)$ is greater than $k^2(k-3)^2$.

We combine (4.6), (5.9), and Proposition 4.3 again in order to obtain

$$(x_2x_6 - x_3x_5)^2 \geq r(k)$$

which, as x_5 and x_6 are at most k and x_3 is at least -4 implies

$$x_2 \geq \frac{\sqrt{r(k)}}{k} - 4$$

By Proposition 5.2, $r(k)$ is greater than $k^2(k-3)^2$ and therefore x_2 is at least $k-6$. Finally, as by (5.3), x_2x_4 is at most $2k+1$,

$$x_4 \leq \frac{2k+1}{k-6}.$$

Since k is at least 9, the right-hand side of this inequality is less than 7. Therefore x_4 is not greater than 6, as desired. □

Combining Lemmas 5.1, 5.3 and 5.5, we now prove Theorem 4.2.

Proof of Theorem 4.2. Assume that k is at least 9 and that the distance between P and the affine hull of Q is $\varepsilon_0^u(3,k)$. By Lemma 5.1, there exists a point x in $\mathcal{X}_{P,Q}(k)$ whose first six coordinates are non-negative, except maybe the third, while x_1x_5 is at least x_2x_4 and x_2 is at least x_4. We consider two cases depending on the sign of x_3 corresponding to either Lemma 5.3 or Lemma 5.5.

First, if x_3 is non-negative then, it follows from Lemma 5.3 that x_4 is at most 3 while x_1, x_5, and x_6 are at least $k-2$. Moreover, either x_2 is at least $k-2$ and x_3 is at most 3 or x_3 is at least $k-2$ and x_2 is at most 3. Observe that exchanging x_i and x_{i+1} when i is equal to 2,5, and 8 results in a lattice point whose coordinates satisfy the same bounds. Moreover, the resulting point still belongs to $\mathcal{X}_{P,Q}(k)$ because these three exchanges amount to transform P and Q by permuting the second and third coordinates of $\mathbb{R}^3$. Hence, we can assume without loss of generality that x in a point from $\mathcal{X}_{P,Q}(k)$ such that x_1, x_2, x_5, and x_6 are at least $k-2$ while x_3 and x_4 are both at most 3.

If however x_3 is negative. It then follows from Lemma 5.5 that x_1, x_2, x_5, and x_6 are at least $k-6$ while x_3 is at least -4 and x_4 is at most 6.

In particular, in both cases, the lattice point a from $\mathbb{R}^6$ such that a_i is equal to $k-x_i$ when i is equal to 1, 2, 5, or 6 and to x_i when i is equal to 3 or 4 belongs to $[0,6]^2\times[-4,6]\times[0,6]^3$ and by construction, $\phi_k(a)$ is precisely the orthogonal projection of x on $\mathbb{R}^6$. In order to prove that a belongs to $\mathcal{A}$, it therefore suffices to show that a_6 is at least $-a_3$. By Proposition 5.4, x_6-x_3 is at most k which in terms of the coordinates of a implies that a_6 is at least $-a_3$. Hence a belongs to $\mathcal{A}$ as desired. $\square$

References

[1] Noga Alon and Văn H. Vũ, *Anti-Hadamard matrices, coin weighing, threshold gates, and indecomposable hypergraphs*, Journal of Combinatorial Theory, Series A **79** (1997), 133–160.

[2] Amir Beck and Shimrit Shtern, *Linearly convergent away-step conditional gradient for non-strongly convex functions*, Mathematical Programming **164** (2017), 1–27.

[3] Antoine Deza, Zhongyuan Liu and Lionel Pournin, *Kissing polytopes in dimension* 3, Experimental Mathematics, *to appear* (2025).

[4] Antoine Deza, Zhongyuan Liu and Lionel Pournin, *Small kissing polytopes*, Vietnam Journal of Mathematics **53** (2025), no. 4, 901–913.

[5] Antoine Deza, Shmuel Onn, Sebastian Pokutta, and Lionel Pournin, *Kissing polytopes*, SIAM Journal on Discrete Mathematics **38** (2024), no. 4, 2643–2664.

[6] Ronald L. Graham and Neil J. A. Sloane, *Anti-Hadamard matrices*, Linear Algebra and its Applications **62** (1984), 113–137.

[7] David H. Gutman and Javier F. Peña, *The condition number of a function relative to a set*, Mathematical Programming **188** (2021), 255–294.

[8] Simon Lacoste-Julien and Martin Jaggi, *On the global linear convergence of Frank–Wolfe optimization variants*, Proceedings of the 28th International Conference on Neural Information Processing Systems (NIPS), 2015, pp. 496–504.

[9] Javier F. Peña, *Generalized conditional subgradient and generalized mirror descent: duality, convergence, and symmetry*, arXiv:1903.00459 (2019).

[10] Javier F. Peña and Daniel Rodríguez, *Polytope conditioning and linear convergence of the Frank–Wolfe algorithm*, Mathematics of Operations Research **44** (2019), no. 1, 1–18.

[11] Luis Rademacher and Chang Shu, *The smoothed complexity of Frank–Wolfe methods via conditioning of random matrices and polytopes*, Mathematical Statistics and Learning **5** (2022), 273–310.

[12] Igor R. Shafarevich and Alexey O. Remizov, *Linear algebra and geometry*, Springer, 2013.