Optimization and Discrete Geometry
Theory and Practice
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The simplex method for LP and the Length of a Path

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Linear Programming and a Graph of P

- Linear programming problem (LP):

$$\min\ c^T x,\ \text{subject to } x \in P$$

  where $P \subset R^d$ is a polytope (or a polyhedron).
- P is usually represented by a system of linear equalities and inequalities:

$$P = \{x | Ax \leq b, Cx = d\}.$$

- Let $G = (V, E)$ be a graph of P, where V is a set of vertices of P and E is a set of edges of P.
- Let (LP') be the standard form LP equivalent to (LP).

The simplex method

- The simplex method for solving the standard form LP (LP') was developed by G. Dantzig in 1947.
- The simplex method generates a sequence of basic feasible solutions of the feasible region (or vertices of the polytope $\boldsymbol{P}$).
- Although the simplex method is efficient for almost all practical problems, it needs an exponential number $\mathbf{(2^d - 1)}$ of iterations for some special prolems, e.g., Klee-Minty LP.

A path of vertices

- The sequence of distinct vertices generated by the simplex method for (LP') is a (monotone) path in the graph $G = (V, E)$.
- The length of the sequence (or path) is always finite. Let ℓ be the length of the path.
- If the LP is non-degenerate, then ℓ is equal to the number of iterations.
- Let u and v be the initial vertex and the terminal vertex of the sequence, then

The length of the shortest path (between u and v)
$\leq \ell$
$\leq$ The length of the (monotone) longest path

Contents

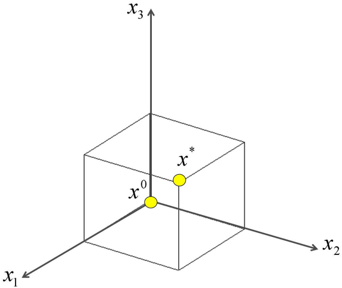
This section

- We discuss the lower and upper bounds of ℓ (the length of the path in the graph $\boldsymbol{G}$ of the feasible region $\boldsymbol{P}$) by using a simple instance.

A simple instance of LP on a cube

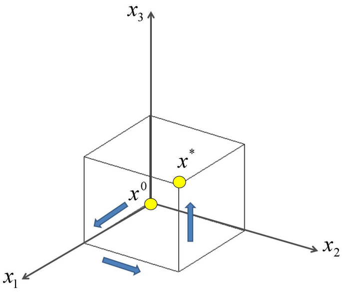
Let $P = \{(x_1, x_2, x_3) | 0 \le x_1, x_2, x_3 \le 1\}$ be the cube in R^3 and consider LP

$$\min \ -(x_1 + x_2 + x_3), \text{subject to } x \in P$$



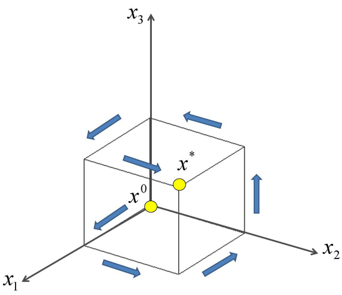
The initial point is $x^0 = (0, 0, 0)^T$ and the optimal solution is $x^* = (1, 1, 1)^T$

The shortest path



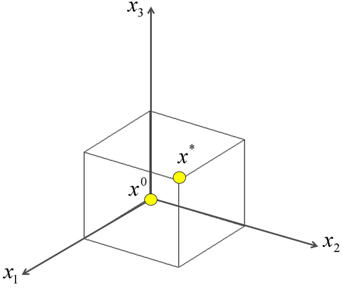
The length (number of edges) of the shortest path from $\boldsymbol{x}^{\mathbf{0}}$ to $\boldsymbol{x}^{*}$ is equal to the dimension $\boldsymbol{d} = \mathbf{3}$.

The longest path



The length of the shortest path is $\boldsymbol{d = 3}$.
The length of the longest path is $\boldsymbol{2^d - 1 = 7}$.

The simplex method on the cube



$\boldsymbol{d} \le \boldsymbol{\ell} \le \mathbf{2^{d} - 1}$, where $\boldsymbol{\ell}$ is the number of distinct vertices (or BFS) generated by the simplex method.

Contents

This section

- We explain the upper bound of ℓ presented by T. Kitahara and S. Mizuno: A bound for the number of different basic solutions generated by the simplex method. *Mathematical Programming*, **137** (2013), 579–586.

A Pivoting Rule

- The number ℓ of vertices generated by the simplex method depends on the pivoting rule.
- Kitahara and M obtain an upper bond of ℓ for the simplex method with Dantzig's rule (the most negative pivoting rule).

An upper bound of ℓ by Ki-M

- The upper bound of ℓ by Ki-M:

  $$\ell \leq nm\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta}),$$

  where m is the number of constraints, n is the number of variables, δ and γ are the minimum and the maximum values of all the positive elements of primal BFSs.
- If the primal problem is non-degenerate, it becomes a bound for the number of iterations.

How good is the bound

- Ki-M show that there exits an LP instance for which
$$\ell = \frac{\gamma}{\delta} \text{ and } \ell = 2^m - 1.$$
Hence $nm\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})$ is a good upper bound.
- The LP instance is

$$\begin{array}{ll} \max & \sum_{i=1}^{m} x_i \\ \text{s. t.} & x_1 \leq 1 \\ & 2x_1 + \cdots + 2x_{k-1} + x_k \leq 2^k - 1 \\ & \qquad \text{for } k = 2, 3, \ldots, m \\ & x \geq 0 \end{array}$$

Contents

This section

- We explain that the simplex method using Tardos's basic algorithm is strongly polynomial for linear programming with totally unimodular matrix.

Summary of this section

Problem: Standard form linear programming problem

$$\min c^T x, \text{ subject to } Ax = b, x \geq 0.$$

Algorithm: Dual simplex method with Dantzig's rule

Analysis: Kitahara-Mizuno + Cramer's rule

Result 1: Number of distinct solutions (ℓ) is bounded by a polynomial of n, Δ, and $||c||$, where Δ is maximum subdeterminant of A.

Result 2: By using Tardos' basic algorithm, the number is bounded by a polynomial of n and Δ.

Result 3: If A is totally unimodular and all auxiliary problems are nondegenerate, then the algorithm is strongly polynomial.

A bound by Kitahara-M. (Primal)

- Ki-M shows that the number (ℓ) of distinct solutions generated by the primal simplex method with Dantzig's rule is bounded by

$$mn\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta}), \tag{1}$$

  where δ and γ are the minimum and the maximum values of all the positive elements of basic feasible solutions.
- The bound is independent of $\boldsymbol{c}$.

Ki-M and Cramer's rule

- Define the maximum subdeterminant of $\boldsymbol{A}$:

$$\boldsymbol{\Delta} = \max\{|\det \boldsymbol{D}| \,\big|\, \boldsymbol{D} \text{ is a square submatrix of } \boldsymbol{A}\}.$$

- By Cramer's rule, each element of basic solutions is a rational number $\frac{\boldsymbol{p}}{\boldsymbol{q}}$ where

$$|\boldsymbol{p}| \leq \boldsymbol{m}\boldsymbol{\Delta}||\boldsymbol{b}||_\infty, \ 1 \leq \boldsymbol{q} \leq \boldsymbol{\Delta}.$$

  Hence $\gamma \leq \boldsymbol{m}\boldsymbol{\Delta}||\boldsymbol{b}||_\infty$ and $\delta \geq 1/\boldsymbol{\Delta}$.
- The bound by Ki-M is represented as

$$\boldsymbol{m}^2\boldsymbol{n}\boldsymbol{\Delta}^2||\boldsymbol{b}||_\infty \log(\boldsymbol{m}^2\boldsymbol{\Delta}^2||\boldsymbol{b}||_\infty).$$

- If $\boldsymbol{\Delta}$ and $||\boldsymbol{b}||$ are bounded by a polynomial of $\boldsymbol{n}$, then it is bounded by a polynomial function of $\boldsymbol{n}$.

Dual simplex method

- The number (ℓ) of distinct solutions generated by the dual simplex method with Dantzig's rule is bounded by

$$m^2 n \Delta^2 ||c||_\infty \log(mn\Delta^2 ||c||_\infty).$$

- If Δ and $||c||$ are bounded by a polynomial of n, then it is bounded by a polynomial function of n. (Especially when A is totally unimodular, $\Delta = 1$.)

When $||\boldsymbol{c}||$ is big

- We use a variant of Tardos' basic algorithm, which solves at most $\boldsymbol{n}$ auxiliary problems where $\boldsymbol{c}$ is replaced by a vector of rounded integers, whose sizes are bounded by $\boldsymbol{n}^2\boldsymbol{\Delta}$.
- If all the auxiliary problems are nondegenerate, the total number of iterations is bounded by

$$\boldsymbol{m}^2\boldsymbol{n}^4\boldsymbol{\Delta}^3\log(\boldsymbol{m}\boldsymbol{n}^3\boldsymbol{\Delta}^3).$$

- If $\boldsymbol{A}$ is totally unimodular ($\boldsymbol{\Delta} = \boldsymbol{1}$), then the algorithm is strongly polynomial.

Contents

This section

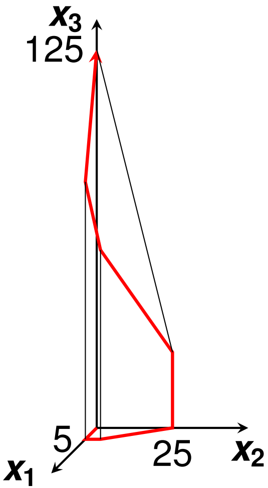
- In this section, we introduce two simple degenerate LPs, for which ℓ is small but the simplex method with Dantzig's rule requires exponential number of iterations.
- The LPs are degenerate variants of Klee-Minty LP.

Klee-Minty LP

Klee and Minty show that the simplex method generates an exponential number $(2^m - 1)$ of vertices for a special LP on a perturbed cube:

$$
\begin{array}{ll}
\max & \sum_{i=1}^{m} 2^{m-i} x_i \\
\text{s. t.} & x_1 \leq 5 \\
& 2^k x_1 + \cdots + 4x_{k-1} + x_k \leq 5^k \\
& \qquad \text{for } k = 2, 3, \ldots, m \\
& x \geq 0
\end{array}
$$

$$\max\ 4x_1 + 2x_2 + x_3$$



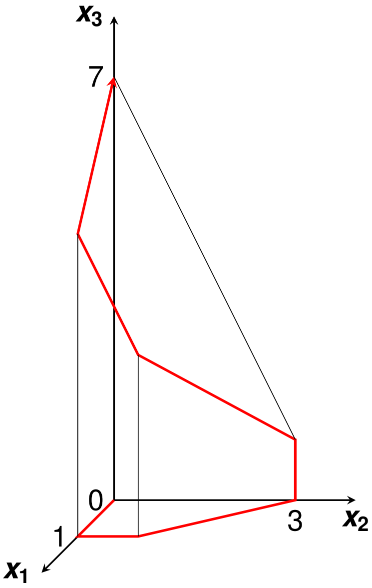
(from Klee Minty cube by Wikipedia.)

A simple variant of Klee-Minty LP

A simple variant of Klee-Minty LP by Ki-M is

$$\begin{array}{ll} \max & \sum_{i=1}^{m} x_i \\ \text{s. t.} & x_1 \leq 1 \\ & 2x_1 + \cdots + 2x_{k-1} + x_k \leq 2^k - 1 \\ & \qquad \text{for } k = 2, 3, \ldots, m \\ & x \geq 0 \end{array}$$

$$\max \; x_1 + x_2 + x_3$$



Two degenerate bad LPs

- When a linear programming problem is degenerate, the number of iterations could be much bigger than the length of the path ℓ even if a cycling does not occur. So it is usually very difficult to get a good upper bound for the number of iterations.
- We will show two simple and small data instances for which the simplex method requires exponential number of iterations.

The first degenerate LP

We consider the following LP:

$$
\begin{array}{ll}
\max & \sum_{i=1}^{m} x_i \\
\text{s. t.} & 2x_1 \leq 2 \\
 & 2x_1 + \cdots + 2x_{k-1} + x_k \leq 2 \quad \text{for } k = 2, 3, \ldots, m \\
 & x \geq 0
\end{array}
$$

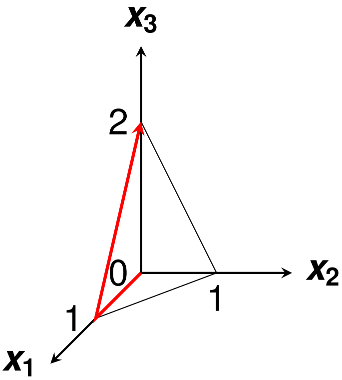
One can check that the first $m - 1$ inequalities are redundant, and that the feasible region is the simplex

$$P = \{x | 2x_1 + \cdots + 2x_{m-1} + x_m \leq 2, x \geq 0\}.$$

The vertices of this simplex are

$$\{0, e_1, e_2, \ldots, e_{m-1}, 2e_m\}.$$

$$\max \; x_1 + x_2 + x_3$$



$\mathbf{2}\boldsymbol{m}\ \mathbf{-\ 1}$ faces intersect at $(\mathbf{1}, \mathbf{0}, \mathbf{0})^T$, which is highly degenerated.

Degenerate LP on a simplex

- The entries of the constraint matrix, the right-hand-side vector, and the cost vector are $\{0, 1, 2\}$-valued.
- The feasible region is a full dimensional simplex including a highly degenerate vertex $\boldsymbol{x} = \boldsymbol{e}_1$. ($(2m - 1)$ faces intersect at $\boldsymbol{x} = \boldsymbol{e}_1$)
- Starting from $\boldsymbol{x} = \mathbf{0}$, the simplex method with Dantzig's pivoting rule visits exactly $\mathbf{3}$ distinct vertices, and makes $2^{m-1} + 1$ iterations, including $2^{m-1} - 1$ at a highly degenerate vertex $\boldsymbol{x} = \boldsymbol{e}_1$.

The second degenerate LP

We consider the following LP:

$$
\begin{array}{ll}
\max & \sum_{i=1}^{m} x_i \\
\text{s. t.} & x_1 \leq 0 \\
 & 2x_1 + \cdots + 2x_{k-1} + x_k \leq 0 \quad \text{for } k = 2, 3, \ldots, m \\
 & x \geq 0
\end{array}
$$

One can check that the feasible region is reduced to the origin $\mathbf{0}$ which forms the unique and highly degenerate optimal vertex.

Degenerate LP on a single point

- The entries of the constraint matrix, the right-hand-side vector, and the cost vector are $\{0, 1, 2\}$-valued.
- The feasible region is reduced to a highly degenerate point.
- Starting from $\boldsymbol{x} = \mathbf{0}$, the simplex method with Dantzig's pivoting rule visits exactly 1 vertex, and makes $2^m - 1$ iterations at this highly degenerate vertex.

Contents

This section

- We discuss ℓ (the length of the path in the graph $\boldsymbol{G}$ of the feasible region $\boldsymbol{P}$), when $\boldsymbol{P}$ is a polytope of integer vertices.
- It is difficult to get a good upper bound for ℓ in this case too.

Polytope of integer vertices

Let $P \subset R^d$ be a convex hull of vetices in $\{0, 1, 2, \ldots, k\}^d$, and consider LP

$$\min \ c^T x, \text{subject to } x \in P.$$

Then

$$\begin{aligned} \text{the length of the shortest path} &\leq kd, \\ \ell &\leq M(M+d)k\log(Mk), \\ \text{the length of the longest path} &\leq (k+1)^d, \end{aligned}$$

where M is the number of faces (inequalities).

Summary

Table: Lower and Upper bounds

Polytope	shortest	ℓ	Ki-M	longest
Cube	d	d	$2d^2\log(d)$	2^d
Klee-M	d	2^d	(*)	2^d
$\{0,\ldots,k\}^d$	kd	?	(**)	$(1+k)^d$

(*) is $2d^2 2^d \log(d2^d)$.
(**) is $M(M+d)k\log(Mk)$.
(M is a number of inequalities)

Contents

Conclusion

- The length of the shortest path of a polytope may not be a good estimate of the number of vertices generated by the simplex method. The length of the longest monotone path could be its upper bound, but it is usually very difficult to estimate the length.
- The simplex method is strongly polynomial for TU-LP, if we use Tardos' basic algorithm and all the auxiliary problems are non-degenerate.
- We presented two simple and small data instances of degenerate LPs for which the simplex method requires exponential number of iterations.