Disjunctive conic cuts:
The good, the bad, and implementation

Optimization and Discrete Geometry: Theory and Practice

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Motivation

Mixed integer second order cone optimization (MISOCO)

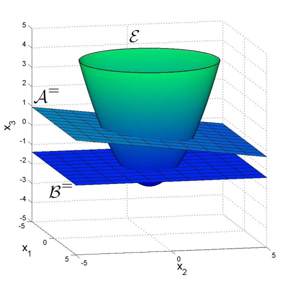
$$
\begin{aligned}
\text{min: } & c^T x \\
\text{s.t.: } & Ax = b \qquad\qquad \text{(MISOCO)} \\
& x \in \mathbb{L}^n \\
& x \in \mathbb{Z}^d \times \mathbb{R}^{n-d},
\end{aligned}
$$

where:

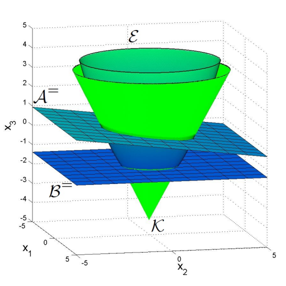
- $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$,
- $\mathbb{L}^n = \{x \in \mathbb{R}^n | x_1 \geq \|x_{2:n}\|\}$,
- Rows of A are linearly independent.
- Here $\|x\|$ denotes Euclidean norm of x.

Disjunctive conic cuts: General theory

- Study the intersection of a convex set $\mathcal{E}$ and a disjunctive set $\mathcal{A} = \{x \in \mathbb{R}^n \mid a^\top x \geq \alpha\} \cup \mathcal{B} = \{x \in \mathbb{R}^n \mid b^\top x \leq \beta\}$[1].
- Show that under some mild assumptions $\text{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B}))$ can be characterized using a convex cone $\mathcal{K}$.



(A)



(B)

[1] $\mathcal{A}^= = \{x \in \mathbb{R}^n \mid a^\top x = \alpha\}$ and $\mathcal{B}^= = \{x \in \mathbb{R}^n \mid b^\top x = \beta\}$

Disjunctive conic cuts (DCCs): Definition

Definition

A closed convex cone $\mathcal{K} \in \mathbb{R}^n$ with $\dim(\mathcal{K}) > 1$ is called a *Disjunctive Conic Cut* (DCC) for $\mathcal{E}$ and the disjunctive set $\mathcal{A} \cup \mathcal{B}$ if

$$\text{conv}(\mathcal{E} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{E} \cap \mathcal{K}.$$

Assumption

The intersection $\mathcal{A} \cap \mathcal{B} \cap \mathcal{E}$ is empty.

Assumption

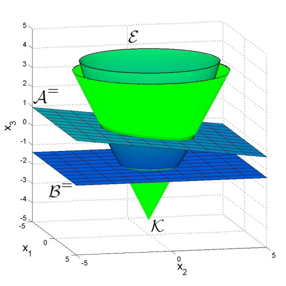
The intersections $\mathcal{E} \cap \mathcal{A}^=$ and $\mathcal{E} \cap \mathcal{B}^=$ are nonempty and bounded.

Disjunctive conic cuts: Characterization

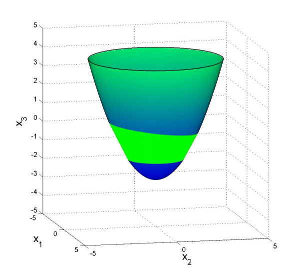
Proposition

A closed convex cone $\mathcal{K} \in \mathbb{R}^n$ *with* $\dim(\mathcal{K}) > 1$ *is a DCC for* $\mathcal{E}$ *and the disjunctive set* $\mathcal{A} \cup \mathcal{B}$*, if*

$$\mathcal{K} \cap \mathcal{A}^= = \mathcal{E} \cap \mathcal{A}^= \quad \text{and} \quad \mathcal{K} \cap \mathcal{B}^= = \mathcal{E} \cap \mathcal{B}^=.$$



(A)



(B)

Quadratic sets and disjunctions

$$\begin{aligned}\mathcal{Q} &= \{x \in \mathbb{R}^n \mid x^T P x + 2p^T x + \rho \leq 0\}\\ \mathcal{A} &= \{x \in \mathbb{R}^n \mid a^\top x \geq \alpha\}\\ \mathcal{B} &= \{x \in \mathbb{R}^n \mid b^\top x \leq \beta\}\end{aligned}$$

- $P \in \mathbb{R}^{n \times n}$ with $n-1$ positive and exactly one non-positive eigenvalues, $p \in \mathbb{R}^n$, $\rho \in \mathbb{R}$.
- $\|a\| = \|b\| = 1$ and $\beta < \alpha$.
- The intersection $\mathcal{Q} \cap \mathcal{A} \cap \mathcal{B}$ results in the disjunctive sets $\mathcal{Q} \cap \mathcal{A}$ and $\mathcal{Q} \cap \mathcal{B}$
- We assume that $\mathcal{A}^= \cap \mathcal{Q} \neq \emptyset$, $\mathcal{B}^= \cap \mathcal{Q} \neq \emptyset$, and $\mathcal{B}^= \cap \mathcal{A}^= \cap \mathcal{Q} = \emptyset$.

A family of quadratic inequalities

Let $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ be a family of quadratic sets having the same intersection with $\mathcal{A}^=$ and $\mathcal{B}^=$, with

$$\begin{aligned}\mathcal{Q}(\tau) &= \left\{x \in \mathbb{R}^n \mid \left(x^\top P x + 2p^\top x + \rho\right) + \tau\left(a^\top x - \alpha\right)\left(b^\top x - \beta\right) \leq 0\right\} \\ &= \{x \in \mathbb{R}^n \mid x^\top P(\tau)x + 2p(\tau)^\top x + \rho(\tau) \leq 0\}.\end{aligned}$$

where

- $P(\tau) = P + \tau\frac{ab^\top + ba^\top}{2}$
- $p(\tau) = p - \tau\frac{\beta a + \alpha b}{2}$
- $\rho(\tau) = \rho + \tau\alpha\beta$

Parallel hyperplanes

Let $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ be a family of quadratic sets having the same intersection with $\mathcal{A}^=$ and $\mathcal{B}^=$, with

$$\begin{aligned}\mathcal{Q}(\tau) &= \left\{x \in \mathbb{R}^n \mid \left(x^\top P x + 2p^\top x + \rho\right) + \tau \left(a^\top x - \alpha\right)\left(a^\top x - \beta\right) \leq 0\right\} \\ &= \{x \in \mathbb{R}^n \mid x^\top P(\tau) x + 2p(\tau)^\top x + \rho(\tau) \leq 0\}\end{aligned}$$

where

- $P(\tau) = P + \tau a a^\top$
- $p(\tau) = p - \tau \frac{\beta+\alpha}{2} a$
- $\rho(\tau) = \rho + \tau\alpha\beta$

Non-convex quadratic cones in the family

Rewrite $Q(\tau)$ as:

$$\mathcal{Q}(\tau) = \{x \in \mathbb{R}^n \mid \left(x + P^{-1}(\tau)p(\tau)\right) P(\tau) \left(x + P^{-1}(\tau)p(\tau)\right) \\ \leq p(\tau)P^{-1}(\tau)p(\tau) - \rho(\tau)\}$$

and we obtain

$$p(\tau)^\top P(\tau)^{-1}p(\tau) - \rho(\tau) = \frac{\left(1-2a_1^2\right)\frac{(\alpha-\beta)^2}{4}\tau^2 - (\rho(1-2a_1^2)+\alpha\beta)\tau - \rho}{1+\tau(1-2a_1^2)}.$$

The roots $\bar{\tau}_1 \leq \bar{\tau}_2$ of the numerator are

$$2\left(\frac{\rho(1-2a_1^2)+\alpha\beta \pm \sqrt{(\rho(1-2a_1^2)+\beta^2)(\rho(1-2a_1^2)+\alpha^2)}}{(1-2a_1^2)(\alpha-\beta)^2}\right)$$

Non-convex quadratic cones in the family

Lemma

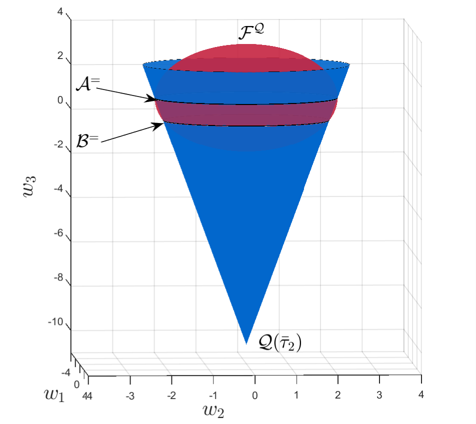
Let $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ be one of the quadratic sets in our list. We have the following cases:

- *If $a_1^2 > \frac{1}{2}$ and the set $Q(\bar{\tau}_2)$ is a non-convex quadratic cone, then its vertex v is either in $\mathcal{A}$ or $\mathcal{B}$.*
- *If $a_1^2 = \frac{1}{2}$ and $\alpha\beta \geq 0$, then the set $Q(-\frac{\rho}{\alpha\beta})$ is a non-convex quadratic cone with its vertex v is either in $\mathcal{A}$ or $\mathcal{B}$.*
- *If $a_1^2 < \frac{1}{2}$ and the set $Q(\bar{\tau}_1)$ is a non-convex quadratic cone, then its vertex v is either in $\mathcal{A}$ or $\mathcal{B}$.*

DCCs for MISOCO when intersections are bounded

Theorem

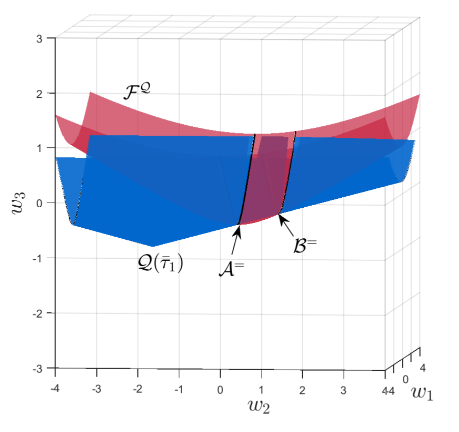
Let $\mathcal{A}^= = \{w \in \mathbb{R}^\ell | a^\top w = \alpha\}$ and $\mathcal{B}^= = \{w \in \mathbb{R}^\ell | a^\top w = \beta\}$ be given. If the sets $\mathcal{A}^= \cap \mathcal{F}^\mathcal{Q}$ and $\mathcal{B}^= \cap \mathcal{F}^\mathcal{Q}$ are bounded, then the quadric $\mathcal{Q}(\bar{\tau}_2)$ contains a DCC for MISOCO.



DCCs for MISOCO when intersections are unbounded

Theorem

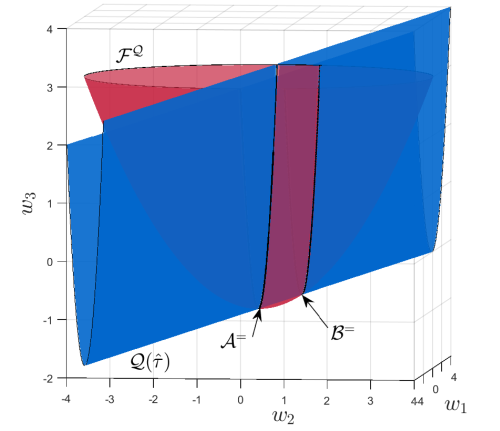
Let $\mathcal{A}^= = \{w \in \mathbb{R}^\ell | a^\top w = \alpha\}$ and $\mathcal{B}^= = \{w \in \mathbb{R}^\ell | a^\top w = \beta\}$ be given. If P is non-singular and the sets $\mathcal{A}^= \cap \mathcal{F}^\mathcal{Q}$ and $\mathcal{B}^= \cap \mathcal{F}^\mathcal{Q}$ are unbounded, then the quadric $\mathcal{Q}(\bar{\tau}_1)$ contains a DCC for MISOCO.



DCCs for MISOCO when intersections are unbounded

Theorem

Let $\mathcal{A}^= = \{w \in \mathbb{R}^\ell \mid a^\top w = \alpha\}$ and $\mathcal{B}^= = \{w \in \mathbb{R}^\ell \mid a^\top w = \beta\}$ be given. If P is singular and the sets $\mathcal{A}^= \cap \mathcal{F}^\mathcal{Q}$ and $\mathcal{B}^= \cap \mathcal{F}^\mathcal{Q}$ are unbounded, then the quadric $\mathcal{Q}(\hat{\tau})$ is a DCC for MISOCO.



Applications and related work

- Infrastructure planning for electric vehicles
  - Mak, Rong, and Shen (2013)
- Sequencing appointments for service systems
  - Mak, Rong, and Zhang (2014)
- The design of service systems with congestion
  - Góez and Anjos (2017)

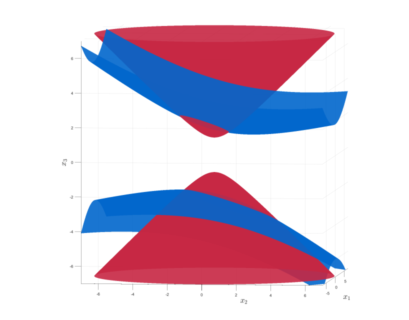
- A Complete Characterization of Disjunctive Conic Cuts for Mixed Integer Second Order Cone Optimization
  - Belotti, Góez, Pólik, Ralphs, Terlaky (2017).
- Intersection cuts for nonlinear integer programming: Convexification techniques for structured sets
  - Modaresi, Kılınç, Vielma (2016).
- Two-term disjunctions on the second-order cone
  - Kılınç-Karzan, Yıldız (2015).
- Disjunctive cuts for cross-sections of the second-order cone
  - Yıldız, Cornuéjols (2015).

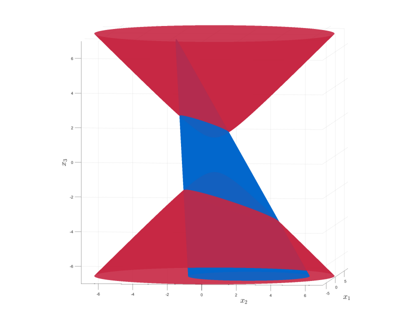
The good

A family of valid quadratic inequalities

Corollary

Given a quadratic set $\mathcal{Q}$ *and two half spaces* $\mathcal{A}$ *and* $\mathcal{B}$*, any quadratic set in the family* $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ *is a valid quadratic inequality for* $\mathcal{Q} \cap (\mathcal{A}^{=} \cup \mathcal{B}^{=})$.

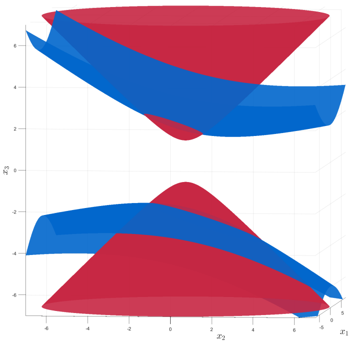




A family of valid quadratic inequalities

Lemma

Given a quadratic set $\mathcal{Q}$ and two half spaces $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{B}^= \cap \mathcal{A}^= \cap \mathcal{Q} = \emptyset$, a quadratic set in the family $\{\mathcal{Q}(\tau) \mid \tau \in \mathbb{R}\}$ is a valid quadratic inequality for $\mathcal{Q} \cap (\mathcal{A} \cup \mathcal{B})$ if and only if $\tau \leq 0$.

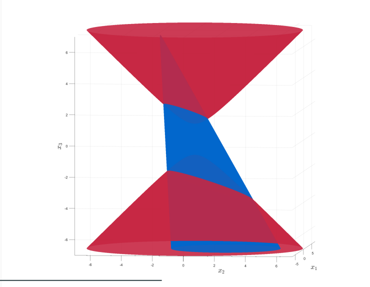


A family of valid quadratic inequalities

Lemma

Consider a quadratic set $\mathcal{Q}$ and two half-spaces $\mathcal{A}$ and $\mathcal{B}$. If there exists a $\bar{\tau}$ such that $\mathcal{Q}(\bar{\tau}) = \mathcal{Q}_1(\bar{\tau}) \cup \mathcal{Q}_2(\bar{\tau})$ is a non-convex quadratic cone, and its vertex v is contained in $\mathcal{A}$ or $\mathcal{B}$ but not in $\mathcal{A} \cap \mathcal{B}$, then each branch $i = 1, 2$ of $\mathcal{Q}(\bar{\tau})$ is a valid quadratic inequality for $\mathcal{Q} \cap (\mathcal{A}_i^{=}(\bar{\tau}) \cup \mathcal{B}_i^{=}(\bar{\tau}))^2$, such that

$$\operatorname{conv}(\mathcal{Q} \cap (\mathcal{A}_i^{=}(\bar{\tau}) \cup \mathcal{B}_i^{=}(\bar{\tau}))) = \operatorname{conv}(\mathcal{Q}_i(\bar{\tau}) \cap (\mathcal{A}_i^{=}(\bar{\tau}) \cup \mathcal{B}_i^{=}(\bar{\tau}))) \subseteq \mathcal{Q}_i(\bar{\tau}).$$



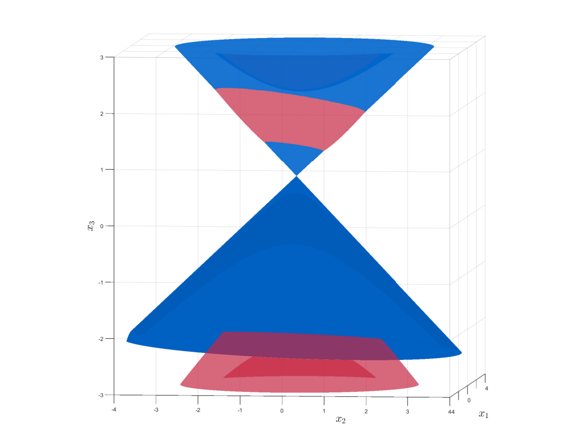
[2] $\mathcal{A}_1^{=}(\bar{\tau}) = \mathcal{A}^{=} \cap \mathcal{Q}_1(\bar{\tau})$, $\mathcal{A}_2^{=}(\bar{\tau}) = \mathcal{A}^{=} \cap \mathcal{Q}_2(\bar{\tau})$, and similarly we define $\mathcal{B}_1^{=}(\bar{\tau})$, $\mathcal{B}_2^{=}(\bar{\tau})$

Does this approach work beyond MISOCO?

Let us consider

- Hyperboloids of two sheets and non-convex quadratic cones.
- Hyperboloids of one sheet.

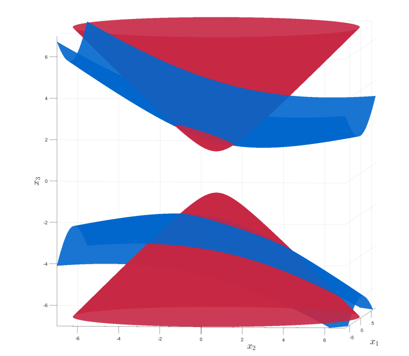
The sets $\mathcal{Q} \cap \mathcal{A}^{=}$ and $\mathcal{Q} \cap \mathcal{B}^{=}$ are bounded, $a_1 > \frac{1}{2}$



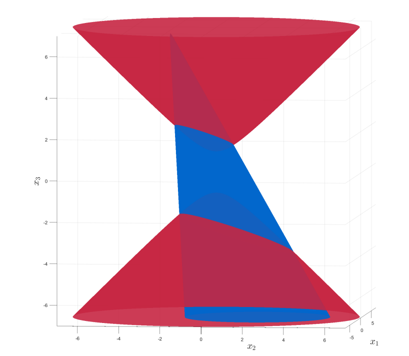
Valid conic inequality $\mathcal{Q}(\bar{\tau}_2)$ when both hyperplanes intersecting the same branch of $\mathcal{Q}$

The sets $\mathcal{Q} \cap \mathcal{A}^{=}$ and $\mathcal{Q} \cap \mathcal{B}^{=}$ are bounded, $a_1 > \frac{1}{2}$

Both hyperplanes intersecting different branches of $\mathcal{Q}$

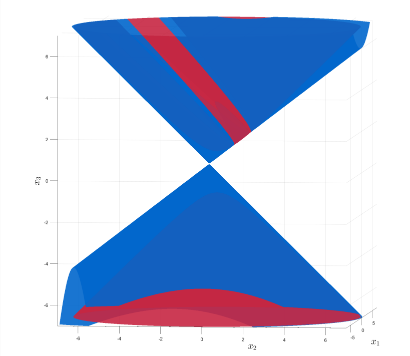


Valid conic inequality $\mathcal{Q}(\bar{\tau}_2)$



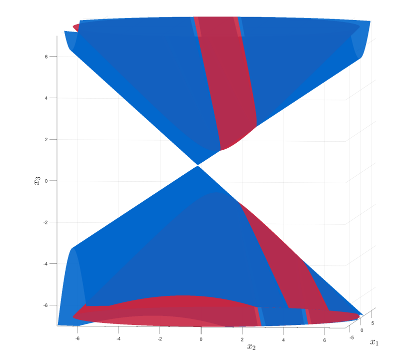
Valid conic inequality $\mathcal{Q}(\bar{\tau}_1)$

The sets $\mathcal{Q} \cap \mathcal{A}^=$ and $\mathcal{Q} \cap \mathcal{B}^=$ are unbounded, $a_1 = \frac{1}{2}$



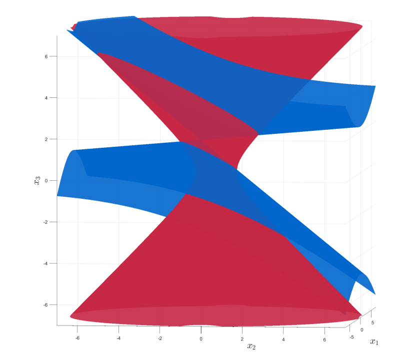
Valid conic inequality $\mathcal{Q}(\bar{\tau}_2)$, in this case
$\text{conv}(\mathcal{Q}_1 \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{Q}_1 \cap \mathcal{Q}(\bar{\tau})$

The sets $\mathcal{Q} \cap \mathcal{A}^{=}$ and $\mathcal{Q} \cap \mathcal{B}^{=}$ are unbounded, $a_1 < \frac{1}{2}$

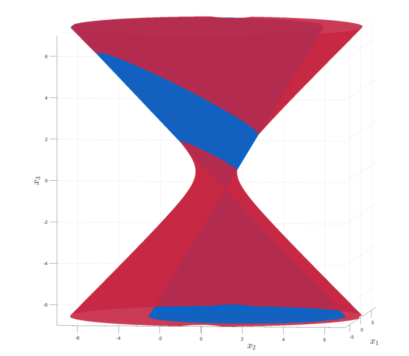


Valid conic inequality $\mathcal{Q}(\bar{\tau}_1)$, $\text{conv}(\mathcal{Q}_1 \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{Q}_1 \cap \mathcal{Q}(\tau_1)$ and $\text{conv}(\mathcal{Q}_2 \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{Q}_2 \cap \mathcal{Q}(\tau_1)$

The sets $\mathcal{Q} \cap \mathcal{A}^{=}$ and $\mathcal{Q} \cap \mathcal{B}^{=}$ are bounded, $a_1 > \frac{1}{2}$



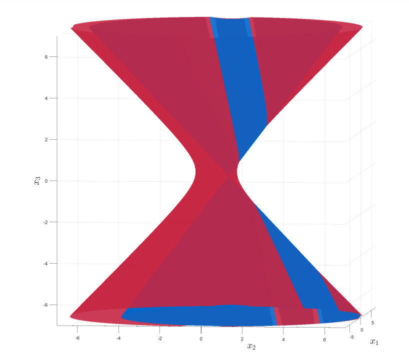
Valid conic inequality $\mathcal{Q}(\bar{\tau}_2)$



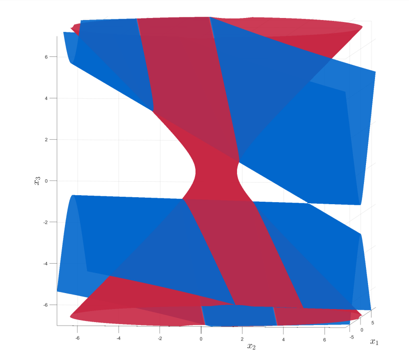
Valid conic inequality $\mathcal{Q}(\bar{\tau}_1)$

The sets $\mathcal{Q} \cap \mathcal{A}^{=}$ and $\mathcal{Q} \cap \mathcal{B}^{=}$ are unbounded

$\beta^2 \geq 1 - 2a_1^2$ and $\alpha^2 \geq 1 - 2a_1^2$



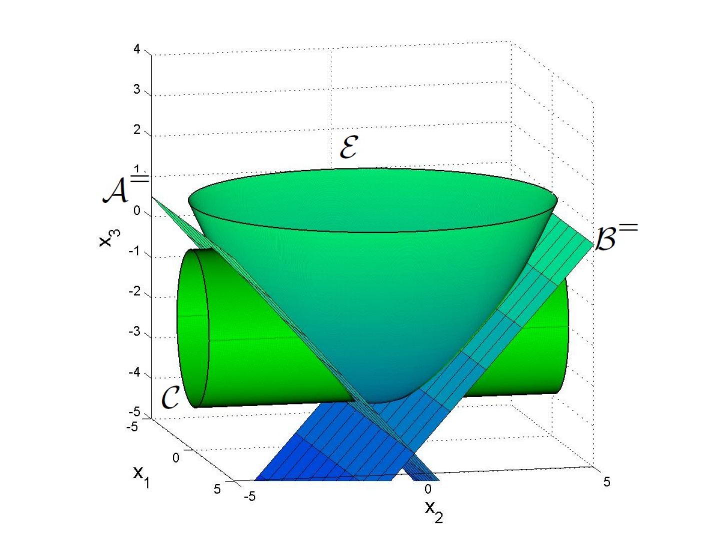
Valid conic inequality $\mathcal{Q}(\bar{\tau}_1)$, $\alpha\beta > 0$



Valid conic inequality $\mathcal{Q}(\bar{\tau}_1)$, $\alpha\beta < 0$

What happens if the hyperplanes are non-parallel?

The results for the bounded intersections still hold.



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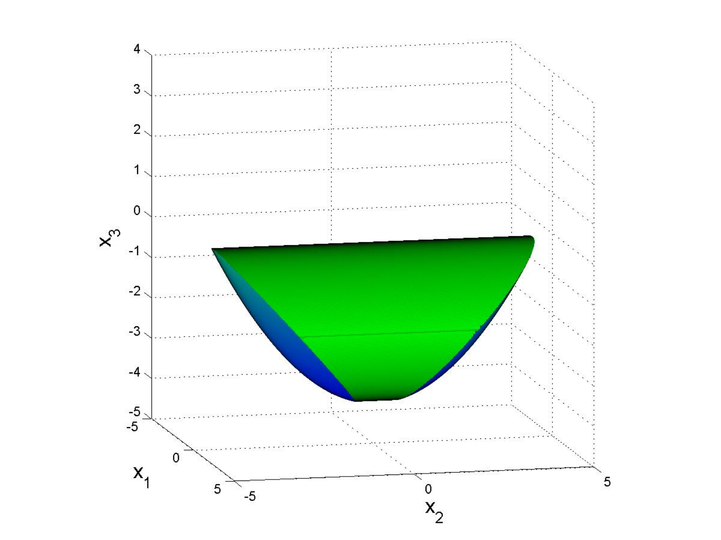
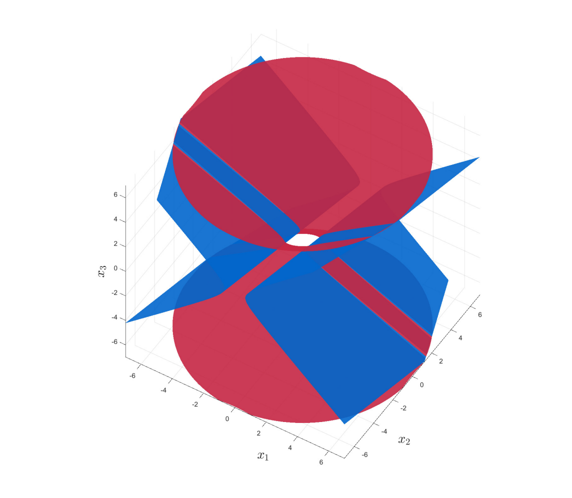


The bad

The sets $\mathcal{Q} \cap \mathcal{A}^{=}$ and $\mathcal{Q} \cap \mathcal{B}^{=}$ are unbounded

$$\beta^2 \leq 1 - 2a_1^2 \text{ and } \alpha^2 \leq 1 - 2a_1^2$$



$\mathcal{Q}(\bar{\tau}_1)$ is a cylinder defined by a hyperboloid of one sheet

Pathological disjunction for MISOCO

Definition (Shahabsafa, G., Terlaky)

Let $\mathcal{X} \in \mathbb{R}n$ be a closed convex set, and consider the disjunction $\mathcal{A} \cup \mathcal{B}$. If $\text{conv}(\mathcal{X} \cap (\mathcal{A} \cup \mathcal{B})) = \mathcal{X}$, then disjunction $\mathcal{A} \cup \mathcal{B}$ is pathological for the set $\mathcal{X}$.

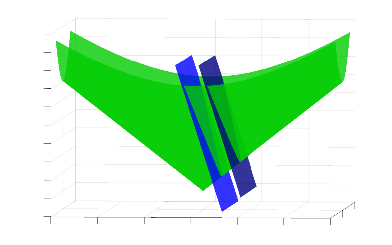
Identification of a redundant DCC for MISOCO

Corollary (Shahabsafa, G., Terlaky)

If the following two conditions are satisfied for the set $\hat{\mathcal{Q}}$ defined, and the disjunctive set, then we have a redundant DCC:

- *the matrix P has exactly $n-1$ positive eigenvalues and one negative eigenvalue, and $p^\top P^{-1} p - \rho = 0$;*
- *the vertex of the cone $v = P^{-1} p$ satisfies either $\hat{a}^\top v \geq \hat{\beta}$, or $\hat{a}^\top v \leq \hat{\alpha}$.*

Identification of a redundant DCC for MISOCO



Hyperboloid intersection (Redundant DCC)

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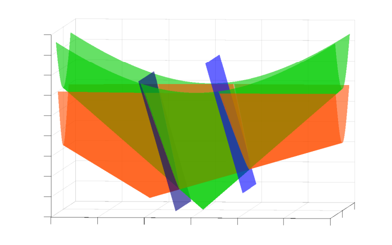
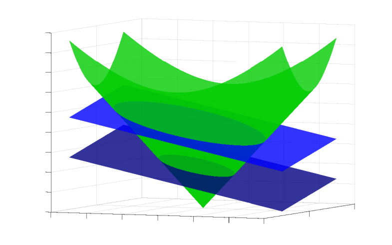
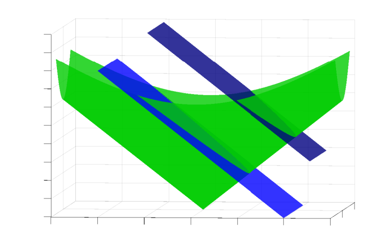


Hyperboloid intersection and the DCC (not a redundant DCC)

Identification of a redundant DCC for MISOCO



Ellipsoid intersection (Redundant DCC)



Paraboloid intersection (Redundant DCC)

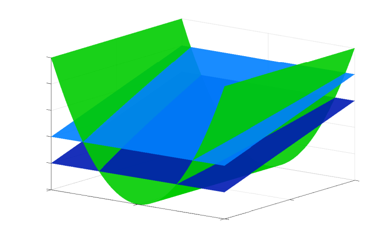
Identification of a redundant DCyC for MISOCO

Corollary (Shahabsafa, G., Terlaky)

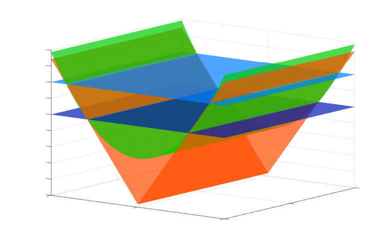
Consider the set $\hat{\mathcal{Q}}$, as defined, and a disjunction. We have a cylindrical redundant DCyC if the following two conditions are satisfied:

- *System* $\begin{bmatrix} P & p \end{bmatrix}^\top d = 0$, *for* $d \neq 0$, *has a solution.*
- *System* $\begin{bmatrix} P & p \end{bmatrix} y = \hat{a}$, *for* $y \in \mathbb{R}^{\ell+1}$, *does not have a solution.*

Identification of a redundant DCyC for MISOCO



A cylindrical redundant DCyC



Not a cylindrical redundant DCyC

Implementation

Implementation challenges

- Generating DCC may mess the structure of the problem, the matrices associated with the cuts are usually dense.
- DCC generation brings numerical challenges.
- Adding DCC may increase the solution time of the linear relaxations.
- No efficient warm start is available for interior point methods.

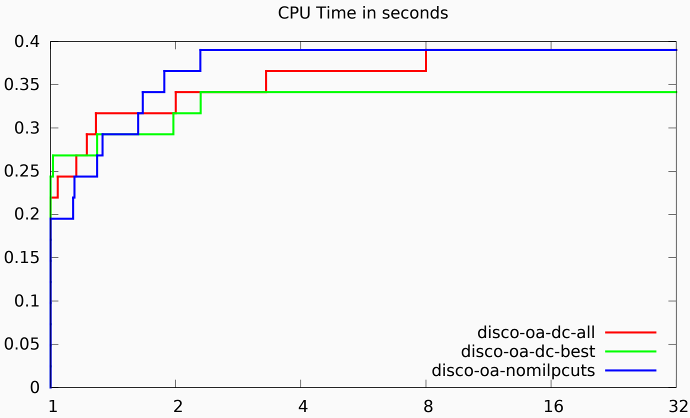
COIN-OR framework - Aykut Bulut and Ted Ralphs

- OsiConic: A generic interface class for SOCP solvers. This interface provides a way to build and solve SOCPs that is uniform across a variety of solvers, as well as a standard interface for querying the results.
- OsiXxxxx: Implementations of the interface for various open source and commercial solvers.
- COLA: A solver for SOCP that implements the cutting-plane Algorithm.
- CglConic: A library of procedures for generating valid inequalities for MISOCP.
- DisCO: A solver library for MISOCP that uses all the libraries mentioned. This library implements classical branch-and-bound type of algorithm and and outer approximation branch-and-cut algorithm.

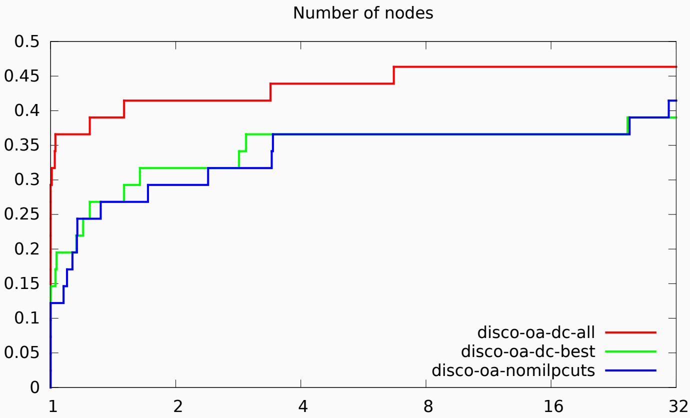
COLA statistics on random instances

instance	NC	LC	NUMLP	CPU
r12c15k5i10	5	3	5	0.01
r14c18k3i9	3	6	16	0.01
r17c30k3i12	3	10	74	0.07
r17c20k5i15	5	4	4	0.0
r22c30k10i20	10	3	8	0.02
r22c40k10i20	10	4	22	0.03
r23c45k3i21	3	15	148	0.25
r27c50k5i25	5	10	77	0.11
r32c45k15i30	15	3	6	0.0
r32c60k15i30	15	4	32	0.02
r52c75k5i35	5	15	74	0.15

Performance Profile of CPU Time using bb-lp with disjunctive cuts



Performance Profile of Number of Nodes Processed using bb-lp with disjunctive cuts



Conclusions and future work

Conclusions and future work

- We provided an extension of disjunctive programming to MISOCO problems.
- We were able to provide closed forms for the derivation of DCCs for MISOCO problems.
- This work gives a full characterization of DCCs for MISOCO problems when using parallel disjunctions.
- We provided valid inequalities for the cross-sections of a non-convex quadratic cone and a one sheet hyperboloid.
- Investigate the potential to use the family of quadrics with some other quadratic sets.
- Investigate the computational potential of this inequalities.

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