On Moser's Shadow Problem

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reporting joint work with

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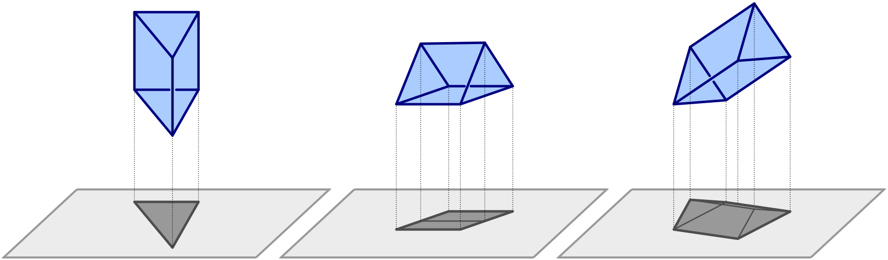
Moser's Shadow Problem

A mimeographed collection of 50 problems collected by Leo Moser titled "Poorly formulated unsolved problems in combinatorial geometry" circulated during the 60's.
Problem 35 was the following:

Moser's Shadow Problem

Estimate the largest $\mathfrak{sh} = \mathfrak{sh}(n)$ such that every convex polyhedron of n vertices has an orthogonal projection onto the plane with $\mathfrak{sh}(n)$ vertices on the 'outside'.



A well-known problem

Moser's shadow problem was a popular open problem in combinatorial geometry which has been...

... restated:

- Geoffrey C. Shephard, *Twenty Problems on Convex Polyhedra, Part II*, 1968.

... publicized:

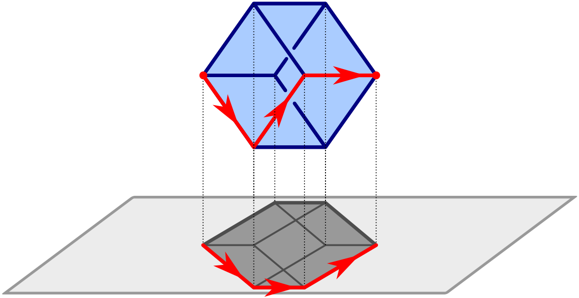
- William O. J. Moser *Problems, Problems, Problems*, 1991.
- Hallard T. Croft, Kenneth J. Falconer and Richard K. Guy, *Unsolved Problems in Geometry*, 1991.

... and mentioned in relation to the silhouette-span problem:

- Bernard Chazelle, Herbert Edelsbrunner and Leonidas J. Guibas, *The Complexity of Cutting Complexes*, 1989.

Shadows in optimization

Shadow sizes are relevant for the complexity of the shadow vertex simplex algorithm (Gass and Saaty 1955).



Used in the study of

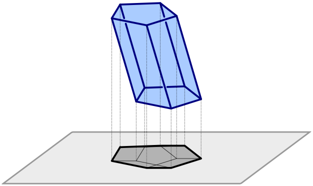
- average complexity of the simplex algorithm (Borgwardt 1982), and
- smoothed analysis of the simplex algorithm (Spielman and Teng 2004).

Shadows and silhouettes

A shadow of a 3-dimensional polyhedron P a 2-dimensional linear projection $\pi(P)$.

The shadow number $\mathfrak{sh}(P)$ is the maximum number of vertices of a shadow.

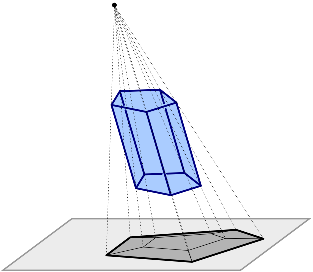
$$\mathfrak{sh}(n) := \min\{\mathfrak{sh}(P) : \ P \text{ 3-polytope with } n \text{ vertices}\}$$



A silhouette of a 3-dimensional polyhedron P a 2-dimensional central projection from a point.

The silhouette span number $\mathfrak{si}(P)$ is the maximum number of vertices of a silhouette.

$$\mathfrak{si}(n) := \min\{\mathfrak{si}(P) : \ P \text{ 3-polytope with } n \text{ vertices}\}$$



By taking points far enough:

$$\mathfrak{sh}(P) \leq \mathfrak{si}(P)$$

The silhouette span problem

Theorem (Chazelle-Edelsbrunner-Guibas 1989)

The silhouette span number $\mathfrak{si}(n)$ *satisfies*

$$\mathfrak{si}(n) = \Theta\left(\frac{\log n}{\log(\log(n))}\right).$$

And hence

$$\mathfrak{sh}(n) \leq \mathfrak{si}(n) = \mathcal{O}\left(\frac{\log n}{\log(\log(n))}\right).$$

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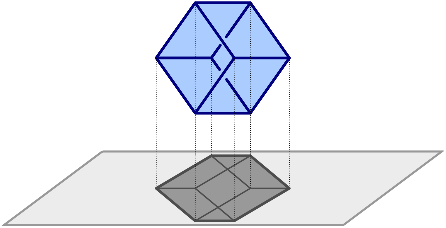
But actually, they also (have all the ingredients to) prove

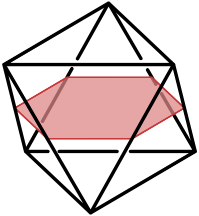
$$\mathfrak{sh}(n) = \Omega\left(\frac{\log n}{\log(\log(n))}\right).$$

A dual approach: cross-section span

The C-E-G proof of the lower bound exploits projective duality:

- points $p \leftrightsquigarrow$ hyperplanes H_p (preserving incidences)
- k-flats $\leftrightsquigarrow$ $(d-k-1)$-flats (preserving incidences)
- polyhedron $P \leftrightsquigarrow$ polar polyhedron P°
- silhouette of P from $p \leftrightsquigarrow$ section $P^\circ \cap H_p$



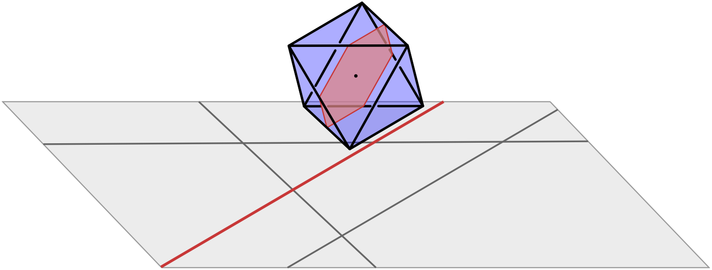


Theorem (Cross-section span problem (C-E-G 1989))

Every 3-polytope with n facets has a planar section with $\Omega\left(\frac{\log n}{\log(\log(n))}\right)$ vertices.

Line span problem

Using central projection of (half of the) polytope onto a plane.



planes through the origin $\leftrightsquigarrow$ lines

Theorem (Line span problem (C-E-G 1989))

For every planar convex subdivision with n regions there is a line stabbing at least $\Omega\left(\frac{\log n}{\log(\log(n))}\right)$ regions.

Moser's shadow problem

By duality,

- shadows of $P \leftrightsquigarrow$ sections $P^\circ \cap H$ with $0 \in H$

hence, the silhouettes for the lower bound are actually shadows:
The C-E-G proof also shows that

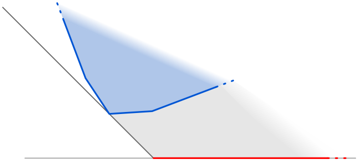
Theorem

The shadow number $\mathfrak{sh}(n)$ satisfies

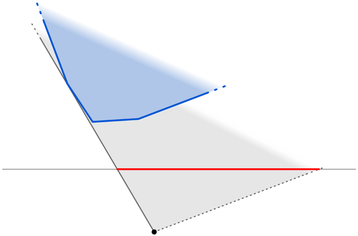
$$\mathfrak{sh}(n) = \Theta\left(\frac{\log n}{\log(\log(n))}\right).$$

Shadows and silhouettes for possibly unbounded polyhedra

$\mathfrak{sh}_u(n) := \min\{\mathfrak{sh}(P) :\ P$ 3-polyhedron (possibly unbounded) with n vertices$\}$



$\mathfrak{si}_u(n) := \min\{\mathfrak{si}_u(P) :\ P$ 3-polyhedron (possibly unbounded) with n vertices$\}$



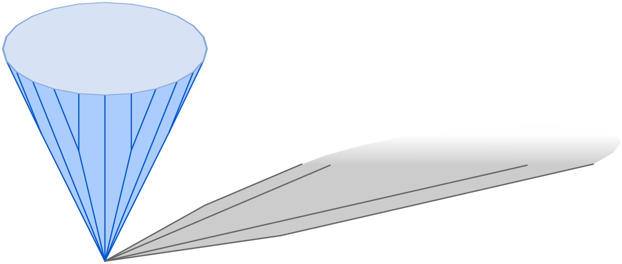
Unbounded shadow number

Theorem

The unbounded n-vertex shadow number $\mathfrak{sh}_u(n)$ for 3-dimensional convex polyhedra satisfies

$$\mathfrak{sh}_u(n) = \Theta(1).$$

In fact $\mathfrak{sh}_u(n) = 3$ for all $n \geq 3$ (and $\mathfrak{sh}_u(n) = n$ for $n \leq 3$).



Unbounded silhouette span

Theorem

The unbounded n-vertex silhouette span number $\mathfrak{si}_u(n)$ *for 3-dimensional convex polyhedra satisfies*

$$\mathfrak{si}_u(n) = \Theta\left(\frac{\log n}{\log(\log(n))}\right).$$

A three parameter question

Question

What is the growth rate of the maximal number $\mathfrak{sh}_b(n,d,k)$ (resp. $\mathfrak{si}_b(n,d,k)$) such that every d-polytope with n vertices has a k-dimensional shadow (resp. silhouette) with $\mathfrak{sh}_b(n,d,k)$ (resp. $\mathfrak{si}_b(n,d,k)$) vertices?

Stabbing with lines and planar shadows

Theorem (Csaba D. Tóth 2008)

For any subdivision of $\mathbb{R}^d$ into n convex cells there is a line stabbing

$$\Omega\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{d-1}}$$

cells. And this is the best possible.

Corollary

Every d-polytope with n vertices has a 2-dimensional shadow with

$$\Omega\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{d-2}}$$

vertices.

Theorem (Hubard-P. 2018+)

For any subdivision of $\mathbb{R}^d$ into n convex cells there is a k-flat stabbing

$$\Omega\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{d-k}}$$

cells.

Corollary

Every d-polytope with n vertices has a k-dimensional shadow with

$$\Omega\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{d-k}}$$

vertices.

Sketch of the proof

Lemma

Let $\mathcal{S}$ be convex subdivision of $\mathbb{R}^d$ into $n > \ell^{(2\ell)^{d-k}}$ cells, where $0 < k < d$ and $\ell \geq 2$. Then for any generic direction v,

- *either there is a line parallel to v stabbing ℓ cells,*
- *or there is hyperplane transversal to v stabbing at least $\ell^{(2\ell)^{d-k-1}}$ cells.*

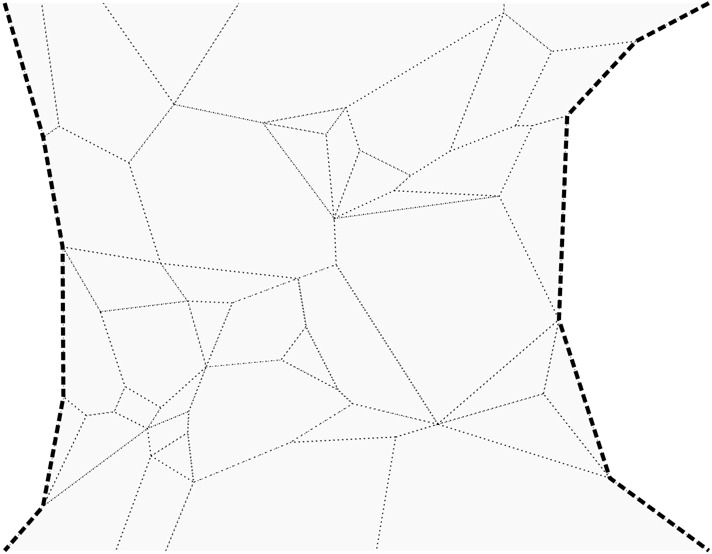
This implies the Theorem by induction on d: $\forall$ subdivision of $\mathbb{R}^d$ into $n > \ell^{(2\ell)^{d-k}}$ convex cells, $\exists$ a k-flat intersecting $\geq \ell$ cells.

- $d = r$ $\checkmark$
- Otherwise, either
  - line stabbing ℓ cells

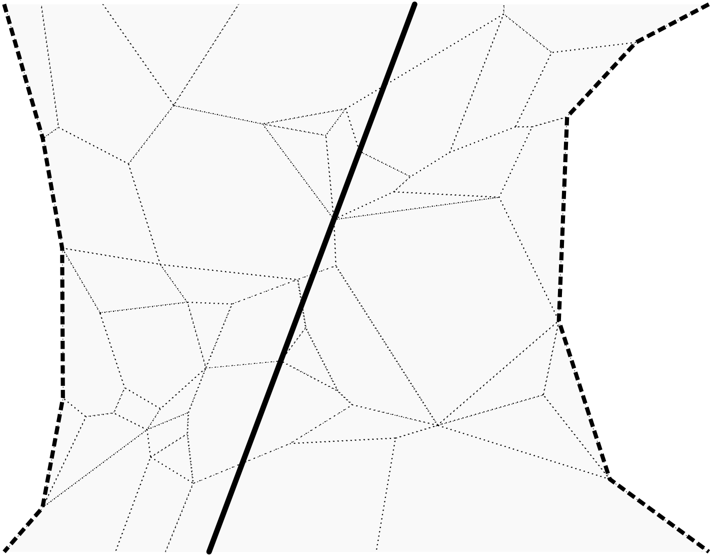
    $\longrightarrow$ take any k-flat containing it $\checkmark$
  - $\exists$ hyperplane H intersecting $\ell^{(2\ell)^{d-r-1}}$ cells

    $\longrightarrow$ apply induction and find k-flat in H $\checkmark$

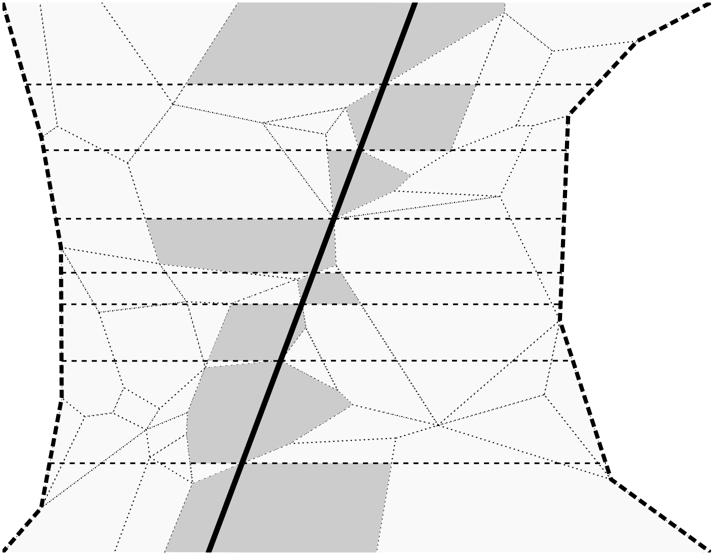
Sketch of the proof of the lemma



Sketch of the proof of the lemma



Sketch of the proof of the lemma



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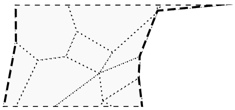

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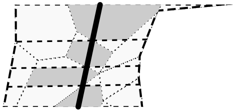

Sketch of the proof of the lemma

Work in progress: converse statements

Conjecture → Proposition

For each n, there is a subdivision of $\mathbb{R}^d$ into n convex cells such that no k-flat stabs more than

$$\mathcal{O}\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{d-k}}$$

cells.

Work in progress: converse statements

Conjecture — Proposition

For each n, there is a d-polytope with n vertices no k-shadow has more than

$$\mathcal{O}\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{d-k}}$$

vertices.

Work in progress: converse statements

Conjecture Proposition

For each n, there is a d-polytope with n vertices no k-silhouette has more than

$$\mathcal{O}\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{d-k}}$$

vertices.

Work in progress: converse statements

Conjecture **Proposition**

For each n, there is a d-polytope with n vertices no k-silhouette has more than

$$\mathcal{O}\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{d-k}}$$

vertices.

THANK YOU!