Flexibility of Semiparametric Choice Models in Traffic Equilibrium

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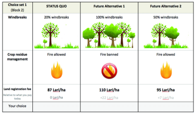
April 26th, 2018

Agenda

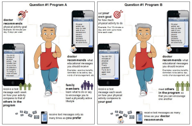
- Discrete Choice Models
- Route/Link Choice in Traffic: (Stochastic) User Equilibrium
- New SUE models arising from DRO framework

Discrete Choice Models

Which route is being used?



Which policy is sustainable?



Which message is effective?

Which product is popular?

Discrete Choice Models

- Random Utility Model (RUM)
- Representative Agent Model (RAM)
- Semiparametric Choice Model (SCM)
- Relations: RUM $\subsetneq$ RAM $=$ SCM

Random Utility Model (RUM)

Let $\mathcal{N} = \{1, 2, \ldots, n\}$ be the set of alternatives.

The random utility of alternative k is defined as:

$$\tilde{U}_k = \mu_k + \tilde{\epsilon}_k, \quad \forall k \in \mathcal{N}.$$

The deterministic/systemic component of the utility captures many observable attributes affecting the choice.

Often a linear-in-parameters model is used to model the deterministic component in terms of observed attributes.

$\tilde{\epsilon}_k$ accounts for the unobserved/random component.

Random utilities form a random vector $\tilde{\boldsymbol{\epsilon}}$ that follow a known joint distribution θ.

Random Utility Model (RUM)

When θ is absolutely continuous,

$$p_k \quad = \quad \mathbb{P}_{\tilde{\epsilon}\sim\theta}\left(k = \arg\max_{l\in\mathcal{N}} \mu_l + \tilde{\epsilon}_l\right), \quad \forall k \in \mathcal{N},$$

is the probability of alternative k to be the best choice.

We refer p_k as the choice probability of alternative k.
(For general integer programs, p_k is the persistency of binary variable x_j.)

Choice probabilities depend on the choice of the distribution θ.

For given θ, we can calculate the expected utility as

$$Z^{\theta}(\mu) \quad = \quad \mathbb{E}_{\tilde{\epsilon}\sim\theta}\left(\max_{k\in\mathcal{N}} \mu_k + \tilde{\epsilon}_k\right).$$

Representative Agent Model (RAM)

A representative agent chooses between products in $\mathcal{N}$ to maximize the expected utility while keeping some level of diversity.

He solves the following optimization problem:

$$\max_{x \in \Delta_{n-1}} \mu^T x - V(x).$$

$V(x)$ is a (strictly) convex regularization term promoting diversification.

Optimal solution, if it is unique, gives the choice probabilities.

Semiparametric Choice Model (SCM)

RUM is a special case of SCM, where the distribution θ of $\tilde{\epsilon}$ is not given but it is known to lie in a set of distributions, say Θ.

Under this model, maximum expected utility is defined as

$$Z^{\Theta}(\mu) \quad = \quad \sup_{\theta \in \Theta} \mathbb{E}_{\tilde{\epsilon} \sim \theta} \left(\max_{k \in \mathcal{N}} \mu_k + \tilde{\epsilon}_k \right).$$

The corresponding choice probabilities are calculated using the extremal distribution θ^*.

$$p_k \quad = \quad \mathbb{P}_{\tilde{\epsilon} \sim \theta^*} \left(k = \arg\max_{l \in \mathcal{N}} \mu_l + \tilde{\epsilon}_l \right), \quad \forall k \in \mathcal{N}.$$

Relationships between different choice models

- MNL $\subset$ RAM (Anderson et al., 1988)
- RUM $\subsetneq$ RAM (Hofbauer and Sandholm, 2002)
- RAM $=$ SCM (Feng et al., 2017)

  Two special cases where Θ and $V(x)$ are given explicitly:
  - MMM, MDM $\subset$ RAM (Natarajan et al., 2009)
  - CMM $\subset$ RAM (Ahipasaoglu et al., 2018)

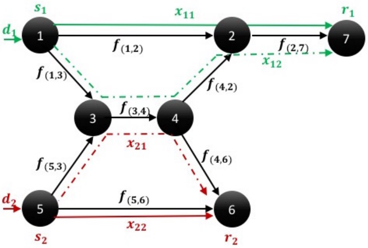
Choice Models (that are of interest to us)

Model	θ	$V(x)$	Θ	p_k
MNL (logit)	i.i.d Gum(θ)	$\frac{1}{\theta}\sum_i x_i \log x_i$	Marginals: Exp$(0, \frac{1}{\theta})$	$\frac{e^{\theta\mu_k}}{\sum_{l\in\mathcal{N}} e^{\theta\mu_l}}$
MNP (probit)	N$(0,\Sigma)$	exists	N$(0,\Sigma)$	$\int\int\int\ldots$ (simulation)
MMM (marginal moment)	none	$-\sum_i \sigma_i\sqrt{x_i(1-x_i)}$	Marginals: mean 0, std σ_i	$\frac{1}{2}+\frac{\mu_k-\lambda}{2\sqrt{(\mu_k-\lambda)^2+\sigma_k^2}})$ (bisection over $\mathbb{R}$)
MDM (marginal distr.)	none	$-\sum_i \int_{1-x_i}^{1} F_i^{-1}(t)dt$	Marginals: $F_i(.)$	$1 - F_k(\lambda - \mu_k)$ (bisection over $\mathbb{R}$)
CMM (cross moment)	none	$-\mathrm{tr}\left(\Sigma^{1/2}\ S(x)\ \Sigma^{1/2}\right)^{1/2}$	Mean 0, cov Σ	Gradient descent (locally linear)

Route Choice and Traffic Equilibrium

Assumptions:

- Multiple origin-destination pairs with fixed demand
- Multiple available routes (possibly overlapping) for each OD pair
- Arc costs as a function of arc flows
- Additive model for path/route costs



Setup and Notation

$\mathcal{G} = (N, \mathcal{A})$	Directed graph with nodes N and arcs/links $\mathcal{A}$
$\mathcal{W}$	Set of origin-destination (OD) pairs in $\mathcal{G}$
(r_w, s_w)	The wth OD pair
$\mathcal{K}_w$	Directed simple paths between r_w and s_w
$\mathcal{K}$	Set of all simple paths in $\mathcal{G}$, i.e., $\bigcup_{w \in \mathcal{W}} \mathcal{K}_w$
d_w	Demand associated with the wth OD pair
$\boldsymbol{x} = (x_{kw})_{k \in \mathcal{K}_w, w \in \mathcal{W}}$	The path flow vector
$\boldsymbol{f} = (f_a)_{a \in \mathcal{A}}$	The arc flow vector
$c_a(f_a)$	The det. cost of arc $a \in \mathcal{A}$ with f_a units[1]
$\boldsymbol{c}(\boldsymbol{f}) = (c_{kw}(\boldsymbol{f}))_{k \in \mathcal{K}_w, w \in \mathcal{W}}$	The path cost vector
$\boldsymbol{x}_w(\boldsymbol{f}) = (x_{kw}(\boldsymbol{f}))_{k \in \mathcal{K}_w}$	The wth path flow vector
$\boldsymbol{c}_w(\boldsymbol{f}) = (c_{kw}(\boldsymbol{f}))_{k \in \mathcal{K}_w}$	The wth path cost vector

[1] We slightly abuse the notation and use the same symbol for arc costs and path costs. We assume that $c_a(f_a)$'s are non-decreasing and continuous.

Traffic Equilibrium

Introduced by Wardrop (1952):

- Deterministic Wardropian User Equilibrium: The travel costs on all routes that are actually used are equal to or less than those which would be experienced by a user on any unused route.
- System Optimum: Traffic is distributed (by a central planner) to minimize the average journey time.

Wardropian User Equilibrium

Convex formulation by Beckman, McGuire, and Winsten (1956):

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{f}} \quad & \sum_{a\in\mathcal{A}} \int_0^{f_a} c_a(t)dt \\
\text{s.t.} \quad & \sum_{k\in\mathcal{K}_w} x_{kw} = d_w, && \forall w \in \mathcal{W}, \\
& x_{kw} \geq 0, && \forall k \in \mathcal{K}_w, w \in \mathcal{W}, \\
& f_a = \sum_{w\in\mathcal{W}} \sum_{k\in\mathcal{K}_w : k\ni a} x_{kw}, && \forall a \in \mathcal{A}.
\end{aligned}
$$

Equilibrium flows can be calculated by a linearisation algorithm based on Frank-Wolfe.

Stochastic User Equilibrium

Introduced by Daganzo and Sheffi (1977):

- Stochastic User Equilibrium: No user can improve his *perceived* travel time by unilateraly changing routes.

Route choice by passengers can be modelled as a discrete choice problem.

Potentially, it can also be viewed as a game between infinitely many agents in which agents select routes to minimize their *perceived* cost.

Stochastic Route Choice with RUM

Traditionally, stochastic choice is modelled using a random utility model:

$$\tilde{U}_{kw} = -c_{kw}(\boldsymbol{f}) + \tilde{\epsilon}_{kw}, \forall k \in \mathcal{K}_w, \forall w \in \mathcal{W},$$

where $\tilde{\boldsymbol{\epsilon}}_{\boldsymbol{w}} \sim \theta_w$.

Additive cost model is used to calculate the path costs:

$$c_{kw}(\boldsymbol{f}) = \sum_{a \in k} c_a(f_a),$$

c_a is estimated from characteristics of link a.

Stochastic Route Choice with RUM

We define

$$p_{kw}(\boldsymbol{f}) \quad = \quad \mathbb{P}_{\theta_w}\left(-c_{kw}(\boldsymbol{f}) + \tilde{\epsilon}_{kw} \geq -c_{lw}(\boldsymbol{f}) + \tilde{\epsilon}_{lw}, \quad \forall l \neq k, l \in \mathcal{K}_w\right),$$

to be the route choice probabilities. I.e., the probability of route k to be the best choice among all possible routes for OD pair w.

Choice probabilities depend on the choice of the distribution θ_w.

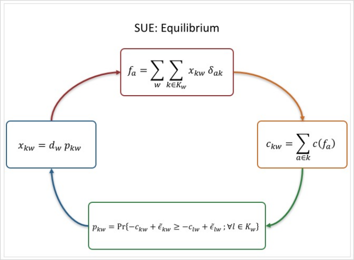
The corresponding flow on a path is given by:

$$x_{kw} \quad = \quad d_w p_{kw}(\boldsymbol{f}), \quad \forall k \in \mathcal{K}_w, w \in \mathcal{W}.$$

Stochastic User Equilibrium with RUM

In a SUE model, the equilibrium arc flow vector $\boldsymbol{f}$ is the solution to the fixed point equation:

$$f_a = \sum_{w \in \mathcal{W}} d_w \sum_{k \in \mathcal{K}_w : k \ni a} p_{kw}(\boldsymbol{f}), \quad \forall a \in \mathcal{A}, \quad \forall w \in \mathcal{W}.$$



Stochastic User Equilibrium with RUM

SUE can be calculated using one of the following formulations:

In terms of arc flow variables (Sheffi and Powell, 1982):

$$\min_{\boldsymbol{f}} \quad \sum_{w \in \mathcal{W}} d_w \mathbb{E}_{\theta_w} \left(\max_{k \in \mathcal{K}_w} (-c_{kw}(\boldsymbol{f}) + \tilde{\epsilon}_{kw}) \right) + \sum_{a \in \mathcal{A}} f_a c_a(f_a) - \sum_{a \in \mathcal{A}} \int_0^{f_a} c_a(t) dt.$$

In terms of arc cost variables (Daganzo, 1982):

$$\min_{\boldsymbol{c}} \quad \sum_{w \in \mathcal{W}} d_w \mathbb{E}_{\theta_w} \left(\max_{k \in \mathcal{K}_w} \left(- \sum_{a \in \mathcal{A}: k \ni a} c_a + \tilde{\epsilon}_{kw} \right) \right) + \sum_{a \in \mathcal{A}} \int_{\underline{c}_a}^{c_a} f_a(c) dc.$$

Stochastic User Equilibrium with RUM

Two well-known SUE models based on RUM are:

- MNL-SUE: Multinomial Logit Model

  Suffers from IIA property, needs to be extended to beyond i.i.d. for successful applications

- MNP-SUE: Multinomial Probit Model

  Captures correlations but not practical

MNL-SUE

- Earliest and most widely studied choice model in transportation.
- $\tilde{\epsilon}_{kw}$'s are iid Gumbel with θ (with mean 0 and variance $\frac{\pi^2}{6}$).
- The maximum of iid Gumbel variables is also a Gumbel variable.
- Closed-form choice probabilities:

$$p_{kw}^{\text{mnl}}(\boldsymbol{f}) = \frac{e^{-\theta c_{kw}(\boldsymbol{f})}}{\sum_{l \in \mathcal{K}_w} e^{-\theta c_{lw}(\boldsymbol{f})}}, \quad \forall k \in \mathcal{K}_w, w \in \mathcal{W}.$$

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- Convex optimization formulation by Fisk (1980).
- Advantage: Efficient link-based algorithms (Akamatsu, 2001) even for large scale networks.
- Disadvantage: Cannot capture correlations between paths, suffers from IIA property.
- Fix: Extensions of MNL such as C-logit, path-based logit, nested-logit etc.

MNL-SUE

Fisk's entropy-based optimization formulation for computing MNL-SUE:

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{f}} \quad & \sum_{a\in\mathcal{A}} \int_0^{f_a} c_a(t)dt + \frac{1}{\theta} \sum_{w\in\mathcal{W}} \sum_{k\in\mathcal{K}_w} x_{kw} \log x_{kw} \\
\text{s.t.} \quad & \sum_{k\in\mathcal{K}_w} x_{kw} = d_w, && \forall w \in \mathcal{W}, \\
& x_{kw} \geq 0, && \forall k \in \mathcal{K}_w, w \in \mathcal{W}, \\
& f_a = \sum_{w\in\mathcal{W}} \sum_{k\in\mathcal{K}_w : k\ni a} x_{kw}, && \forall a \in \mathcal{A}.
\end{aligned}
$$

IIA Property

In MNL, the ratio of the choice probabilities of any two alternatives is unaffected by the systematic utilities of other alternatives:

$$\frac{p_{kw}^{\text{mnl}}(\boldsymbol{f})}{p_{lw}^{\text{mnl}}(\boldsymbol{f})} = \frac{e^{\theta c_{lw}(\boldsymbol{f})}}{e^{\theta c_{kw}(\boldsymbol{f})}}.$$

IIA Property

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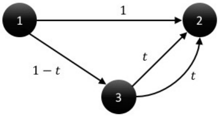
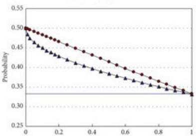




- MNL assigns equal probability to routes for all values of t.
- We expect the probability to get close to $1/2$ as t gets small.
- Many variants of MNL are developed that ‘somehow’ incorporate correlations to overcome the IIA property.

Extensions of Logit

Two assumptions of the MNL model have received particular criticism in the traffic assignment literature (Sheffi, 1985).

1. Random terms are independently distributed: overlapping problem.

   Path-size Logit:

$$p_{kw}^{\text{mnl}}(\boldsymbol{f}) = \frac{PS_{kw} e^{-\theta c_{kw}(\boldsymbol{f})}}{\sum_{l \in \mathcal{K}_w} PS_{lw} e^{-\theta c_{lw}(\boldsymbol{f})}}, \quad \forall k \in \mathcal{K}_w, w \in \mathcal{W},$$

   where $PS_{kw} := \sum_{a \in k} \frac{l_a}{L_k} \frac{1}{\sum_{l \in K_w} \delta_{al}}$.

Extensions of Logit

Two assumptions of the MNL model have received particular criticism in the traffic assignment literature (Sheffi, 1985).

2. Random terms are identically distributed: equal variance problem.

   Path-size Logit(s) by Chen et al (2012):

$$p_{kw}^{\text{mnl}}(\boldsymbol{f}) = \frac{PS_{kw} e^{-\theta SF_w c_{kw}(\boldsymbol{f})}}{\sum_{l \in \mathcal{K}_w} PS_{lw} e^{-\theta SF_w c_{lw}(\boldsymbol{f})}}, \quad \forall k \in \mathcal{K}_w, w \in \mathcal{W},$$

   where $SF_w := \frac{\pi}{\sqrt{6\nu c_{\bar{k}w}(0)}}$ and $\sigma_{kw}^2 = \nu c_{\bar{k}w}(0)/\theta^2$.

   The underlying idea of the scaling approach is to assign higher perception variances to longer routes.

MNP-SUE

- $\tilde{\epsilon}_{\boldsymbol{w}}$'s are multivariate normal with mean $\mathbf{0}$ and covariance $\mathbf{\Sigma}_w \succ 0$.
- No closed-form choice probabilities.
- No convex optimization formulation.
- Sheffi and Powell's MSA (Method of Successive Averages), a *gradient-based* method, is very popular.
- Advantage: Captures correlations between paths.
- Disadvantage: Very hard to compute, requires a large scale optimization-simulation approach.
- Fix: Distributed computing.
- Question: Do we need normality?

Semiparametric Route Choice Models

Choice probabilities are evaluated under an extremal distribution:

$$p_{kw}(\boldsymbol{f}) \;\;=\;\; \mathbb{P}_{\theta_w^*}\left(-c_{kw}(\boldsymbol{f}) + \tilde{\epsilon}_{kw} \geq -c_{lw}(\boldsymbol{f}) + \tilde{\epsilon}_{lw}, \quad \forall l \neq k, l \in \mathcal{K}_w\right),$$

where

$$\theta_w^* = \arg\max_{\theta \in \Theta_w} E_\theta \left(\max_{k \in \mathcal{K}} \{\tilde{U}_{kw}\}\right).$$

Choice of the uncertainty set Θ_w leads to different choice models and, therefore, different SUE models.

RAM/SCM - SUE

The distributionally robust counterpart of Daganzo's arc cost formulation:

$$\min_{\boldsymbol{c}} \sum_{w \in W} d_w \max_{\theta_w \in \Theta_w} E_{\theta_w} \left[\max_{k \in K_w} \left(- \sum_{a \in \mathcal{A}: k \ni a} c_a + \tilde{\epsilon}_{kw} \right) \right) + \sum_{a \in \mathcal{A}} \int_{\underline{c}_a}^{c_a} f_a(c) dc.$$

Under this approach, the system planner assumes only limited distributional information and uses a 'worst-case' potential function in computing the equilibrium.

RAM/SCM - SUE

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{f}} \quad & \sum_{a\in\mathcal{A}} \int_0^{f_a} c_a(t)dt + \sum_{w\in\mathcal{W}} d_w V(x_{kw}) && \\
\text{s.t.} \quad & \sum_{k\in\mathcal{K}_w} x_{kw} = d_w, && \forall w \in \mathcal{W}, \\
& x_{kw} \geq 0, && \forall k \in \mathcal{K}_w, w \in \mathcal{W}, \\
& f_a = \sum_{w\in\mathcal{W}} \sum_{k\in\mathcal{K}_w : k\ni a} x_{kw}, && \forall a \in \mathcal{A}.
\end{aligned}
$$

This is a generalization of the convex formulation of Wardropian UE (Beckman et al., 1956) and MNL - SUE (Fisk, 1980).

RAM/SCM - SUE

Classical RUM - SUE models are:

- MNL-SUE: Multinomial Logit Model
- MNP-SUE: Multinomial Probit Model
- MNW-SUE: Multinomial Weibit Model (relatively new)

Two new RAM/SCM - SUE models are introduced recently:

- CMM-SUE: Cross Moment Model (Ahipasaoglu et al., 2015)
- MDM-SUE: Marginal Distributions Model (Ahipasaoglu et al., 2016)

Summary: CMM-SUE

- Captures correlations between routes (due to overlapping arcs).
- CMM choice probabilities can be calculated efficiently as an SDP or using first-order methods.
- Using the representative agent version of the CMM model, a convex-concave min-max reformulation gives the CMM-SUE flows.
- CMM-SUE flows exist and are unique.
- CMM-SUE flows can be calculated by a gradient-descent type algorithm, similar to the MSA for MNP-SUE.
- CMM-SUE provides a practical alternative to the MNP-SUE model.

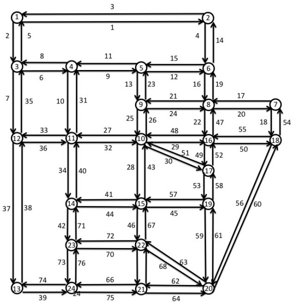


Figure: Sioux Falls Network: 24 nodes, 76 links, 552 OD pairs,

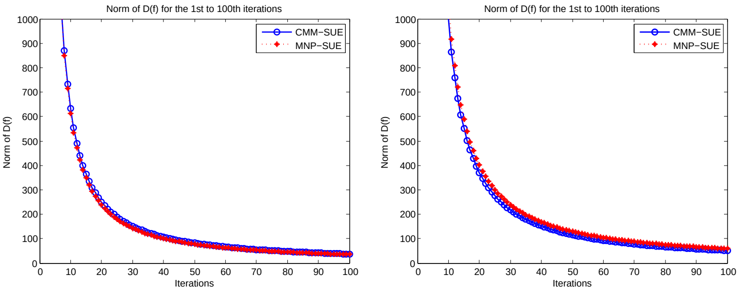
$$c_a = \bar{c}_a \left(1 + B\left(\frac{f_a}{s_a}\right)^t\right).$$

Numerical results - Real Network

Table: Computational times and relative difference of total costs.

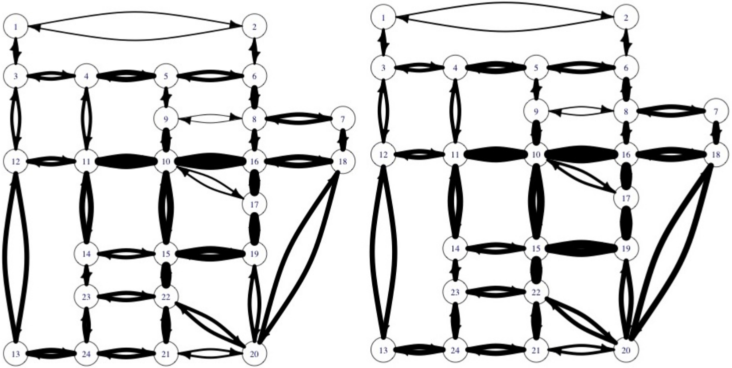
OD	No. of paths	3 paths	4 paths	5 paths	6 paths	7 paths	8 paths	9 paths
10	time CMM	0.07	0.10	0.12	0.15	0.17	0.23	0.27
	time MNP	0.16	0.20	0.26	0.32	0.36	0.40	0.46
	time ratio	2.11	2.08	2.21	2.16	2.13	1.76	1.75
	cost difference	4.80e-8	6.19e-8	9.25e-8	9.51e-8	1.31e-7	1.78e-7	2.29e-7
50	time CMM	0.30	0.59	0.54	0.64	0.78	0.93	1.06
	time MNP	0.81	1.02	1.29	1.53	1.83	2.07	2.54
	time ratio	2.68	1.73	2.40	2.39	2.34	2.23	2.41
	cost difference	8.25e-6	6.86e-6	7.75e-6	5.09e-6	6.51e-6	8.92e-6	1.01e-5
100	time CMM	0.75	1.01	1.29	1.65	2.04	2.71	3.44
	time MNP	2.30	3.05	3.11	5.92	6.59	8.42	10.05
	time ratio	3.04	3.02	2.42	3.59	3.23	3.11	2.92
	cost difference	9.13e-6	1.38e-5	1.14e-5	1.75e-5	2.82e-5	3.49e-5	3.64e-5
200	time CMM	3.48	6.06	8.50	11.02	16.48	21.07	26.34
	time MNP	20.61	34.61	44.71	61.80	81.30	94.36	116.90
	time ratio	5.92	5.71	5.26	5.61	4.93	4.48	4.44
	cost difference	1.78e-4	3.24e-4	4.44e-4	5.16e-4	6.11e-4	6.75e-4	7.36e-4
400	time CMM	18.88	34.25	52.31	73.87	104.61	129.41	172.64
	time MNP	136.16	262.13	367.83	506.45	610.91	781.89	957.77
	time ratio	7.21	7.65	7.03	6.86	5.84	6.04	5.55
	cost difference	2.58e-3	4.88e-3	6.96e-3	8.60e-3	1.03e-2	1.19e-2	1.31e-2
552	time CMM	26.58	47.82	84.11	119.44	156.10	210.10	286.46
	time MNP	186.64	342.60	483.98	685.20	881.98	1250.48	1535.29
	time ratio	7.02	7.16	5.75	5.74	5.65	5.95	5.36
	cost difference	5.16e-3	9.41e-3	1.43e-2	1.79e-2	2.16e-2	2.51e-2	2.81e-2

Numerical results - Real Network



Convergence of the algorithms. (Left: 3 paths, Right: 5 paths)

Numerical results - Real Network



The CMM-SUE (left) and MNP-SUE (right) flows.

Numerical results - Real Network

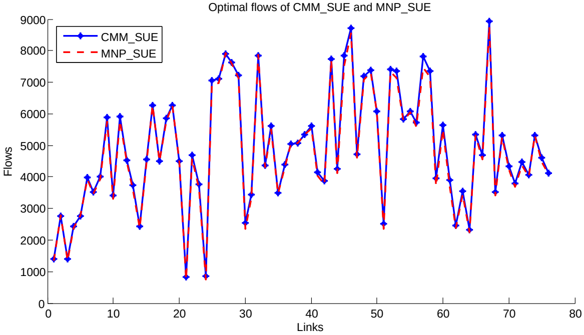


Figure: CMM-SUE and MNP-SUE flows when there are 552 OD pairs.

Summary: MDM-SUE

- MDM assumes that the marginal distributions are given, but not the general distribution.

- MDM-SUE exists and is unique when
  - $E|\widetilde{U}_{kw}| < \infty$.
  - $\widetilde{U}_{kw}$ has support on $(-\infty, \infty)$ or $[\underline{u}_{kw}, \infty)$.
  - The cumulative distribution function $F_{kw}(\cdot)$ is assumed to be strictly increasing and continuous with a pdf $f_{kw}(\cdot) > 0$ on the support.
  - $c_a(f_a)$ is nondecreasing in f_a and continuous.

- Since $V(x)$ is separable, MDM-SUE flows can be easily calculated.

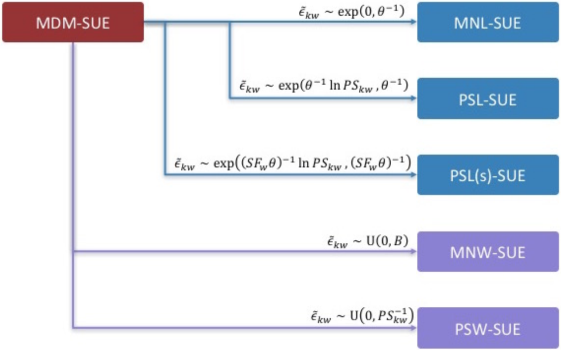
Summary: MDM-SUE

MDM-SUE is very flexible in terms of capturing the user behaviour:

- A generalization of some important logit and weibit-based models.
- Handles the overlapping and equal variance problems simultaneously.
- Generalizes the scaling approach by allowing route-level scaling.
- Allows assigning perception variances independent of the route costs.
- A practical approach to incorporate normal random variables.
- Allows to use skewness to model different route choice behaviors.
- Extended to use bounded and discrete marginal distributions.
- Allows to distinguish between the used and unused routes.

MDM-SUE

MDM-SUE can generate existing logit and weibit based models.



MDM-SUE with exponential marginals

- MDM-SUE provides modelling flexibility beyond the capabilities of existing extensions of MNL.
- It can extend Chen's OD-level scaling approach to route-level scaling by setting $\theta_{kw} = \theta\pi/\sqrt{6\eta c_{kw}(0)}$.

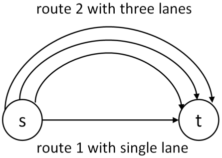
  Route-specific perception variances become:

$$\sigma_{kw}^2 = \eta c_{kw}(0)/\theta^2, \quad \forall k \in K_w, w \in W.$$

- MDM can scale dispersion variances more smoothly, especially when the routes in OD pairs have significantly varying lengths.

MDM-SUE with normal marginals

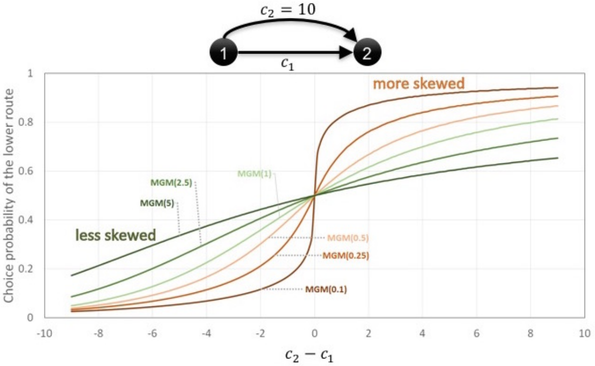
- Normal traffic random variables have several nice properties: location and scale stable, and reproductive (Castillo et al., 2014).
- MNP model is expensive.
- MDM-SUE provides a practical approach to use normal distribution.
- MDM allows assigning perception variance independent of the cost.



$\sigma_1 > \sigma_2$

MDM-SUE with gamma marginals

- Shifted gamma random variables are location-scale stable (Castillo et al., 2014), and share nice properties with normal random variables such as being reproductive.
- Moreover, it can have positive skewness as the observed traffic flows.



MDM-SUE with bounded and discrete marginals

Some recent criticism (by Watling et al, 2015) on SUE models are:

- UE models can be considered to be too restrictive as it may assign no flow to a route that is slightly more costly than the used routes.
- SUE models assign a positive flow to each route in the choice set.
- The continuous and unbounded distribution assumption of SUE is unrealistic.
- MDM-SUE formulation that can be extended to handle bounded and discrete distributions by relaxing some of the assumptions.
- Optimal values of the dual variables behave as reference utilities to distinguish between used and unused routes within the SUE framework.
- Equilibrium flows are not unique in this case.

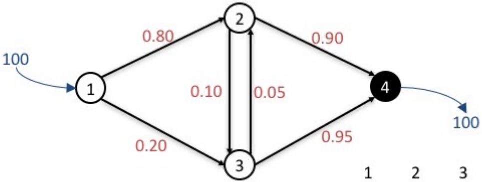
(Link-based) Markovian Traffic Assignment

- Markovian traffic assignment can be considered as a special case of a dynamic discrete choice model for route choice.

- Each route choice is defined as a sequence of link choices, where p_{ij}^d is the probability of choosing link (i, j) at node i if destination is d.

- Link choice at a particular node is independent of the previous choices (Markovian property).

(Link-based) Markovian Traffic Assignment



	1	2	3	4
1	0	0.80	0.20	0
2	0	0	0.10	0.90
3	0	0.05	0	0.95
4	0	0	0	1.00

$\mathbf{M^d} = \left(\mathbf{I} - \mathbf{Q^{d\prime}}\right)^{-1}$: fundamental matrix

$\mathbf{Q}^{\mathbf{d}}_{ij}$, is equal to p^d_{ij} if $(i,j) \in A_d$ and zero otherwise.

$\mathbf{M}^{\mathbf{d}}_{ij}$ is the expected number of times a user entering the system at node i visits node j before reaching the destination.

(Link-based) Markovian Traffic Assignment with RUM

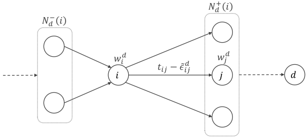
- Link cost of arc (i, j) is defined as $t_{ij} - \tilde{\epsilon}_{ij}^d$.

- Let w_j^d denote the expected minimum cost from node j to destination d.

- A Markovian choice model for destination d solves:

$$w_i^d = E_{\theta_{id}}\left[\min_{j \in N_d^+(i)} \left\{t_{ij} - \tilde{\epsilon}_{ij}^d + w_j^d\right\}\right], \ \forall i \in N_d, \text{and } w_d^d = 0,$$

where θ_{id} is the joint distribution of the error terms $\{\tilde{\epsilon}_{ij}^d; j \in N_d^+(i)\}$.

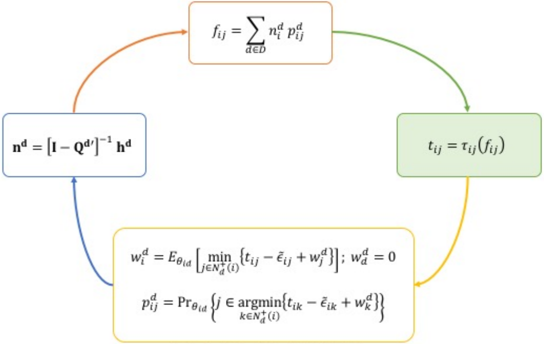


Markovian Traffic Equilibrium with RUM

The MTE (defined by Baillon and Cominetti, 2008) is the solution to the fixed point problem:

$$
\begin{aligned}
t_{ij} &= \tau_{ij}\left(f_{ij}\right), && \forall (i,j) \in A, \\
w_i^d &= E_{\theta_{id}}\left[\min_{j \in N_d^+(i)} \left\{ t_{ij} - \widetilde{\epsilon}_{ij}^d + w_j^d \right\}\right], && \forall i \in N_d, d \in D, \\
w_d^d &= 0, && \forall d \in D, \\
n_i^d &= h_i^d + \sum_{k \in N_d^-(i)} x_{ki}^d, && \forall i \in N_d, d \in D, \\
x_{ij}^d &= n_i^d \cdot \mathrm{P}_{\theta_{id}}\left\{ j = \operatorname*{argmin}_{k \in N_d^+(i)} \left\{ t_{ik} - \widetilde{\epsilon}_{ik}^d + w_k^d \right\} \right\}, && \forall (i,j) \in A_d, d \in D, \\
f_{ij} &= \sum_{d \in D : (i,j) \in A_d} x_{ij}^d, && \forall (i,j) \in A.
\end{aligned}
$$

Markovian Traffic Equilibrium with RUM



Distributionally Robust Markovian Traffic Equilibrium

Markovian Traffic Equilibrium with RUM

Under mild assumptions on $\boldsymbol{\theta}$ and τ, the MTE exists and is unique.

Link costs at equilibrium solve an unconstrained convex optimization;

$$Z(\boldsymbol{\theta}) = \max_{\mathbf{t}} \sum_{d \in D} \sum_{i \in N_d} h_i^d \cdot w_i^d(\mathbf{t}) - \sum_{(i,j) \in A} \int_0^{t_{ij}} \tau_{ij}^{-1}(\omega) d\omega,$$

Only used for the MNL since computing the choice probabilities under RUM is not trivial.

Markovian Traffic Equilibrium with RUM

We can also calculate MTE from a constrained convex optimization problem:

$$Z(\boldsymbol{\theta}) = \max_{\mathbf{t},\mathbf{w}} \sum_{d\in D}\sum_{i\in N_d} h_i^d \cdot w_i^d - \sum_{(i,j)\in A}\int_0^{t_{ij}} \tau_{ij}^{-1}(\omega)d\omega$$

$$\text{s.t. } w_i^d \leq E_{\theta_{id}}\left[\min_{j\in N_d^+(i)}\left\{t_{ij} - \tilde{\epsilon}_{ij}^d + w_j^d\right\}\right], \quad \forall i \in N_d, d \in D,$$

$$w_d^d = 0, \qquad \forall d \in D.$$

Semiparametric Markovian Traffic Equilibrium

MTE with semiparametric choice model is defined as:

$$\max_{\mathbf{t},\mathbf{w}} \sum_{d \in D} \sum_{i \in N_d} h_i^d \cdot w_i^d - \sum_{(i,j) \in A} \int_0^{t_{ij}} \tau_{ij}^{-1}(\omega) d\omega$$

$$\text{s.t. } w_i^d \leq E_{\theta_{id}} \left[\min_{j \in N_d^+(i)} \left\{ t_{ij} - \widetilde{\epsilon}_{ij}^d + w_j^d \right\} \right], \quad \forall \theta_{id} \in \Theta_{id}, \forall i \in N_d, d \in D,$$

$$w_d^d = 0, \qquad \forall d \in D,$$

Markovian Traffic Equilibrium with MDM

- MTE-MDM uses the MDM choice model for link choice.
- MTE-MDM exists and is unique when MDM probabilities are unique.
- It provides a distributionally robust optimization perspective on the MTE from the system planner's view.
- Using the equivalent RAM, we propose equivalent convex optimization formulations to obtain the equilibrium flows.
- The model is flexible in capturing a wide range of user choice behavior and can be calculated efficiently even for large networks.

Numerical results - Winnipeg Network[2]

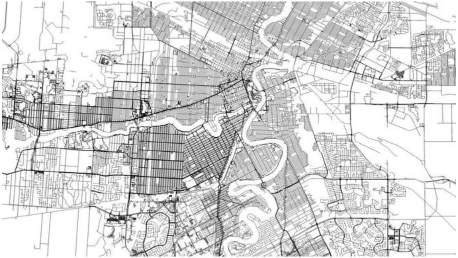


Figure: 1040 nodes (138 destinations), 2836 links, 4344 OD pairs

[2] from Bar-Gera's Traffic Assignment Test Problem website:See https://github.com/bstabler/TransportationNetworks

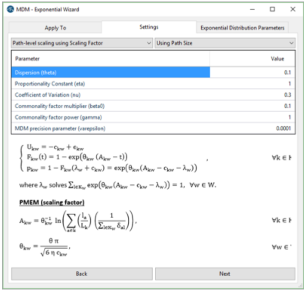
Numerical results - Real Network

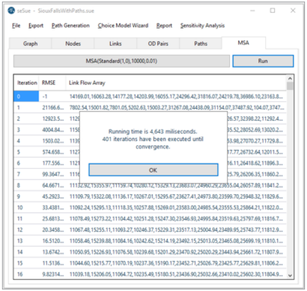
Table: MDM-SUE with exponential marginals

		Sioux Falls		Winnipeg		Winnipeg (NRL)
α_1	α_2	Iter.	Time(s)	Iter.	Time(m)	Time(m)
2.00	-1.0	255	8.19	117	10.88	
	-0.5	530	20.21	154	14.63	
	0.0	430	19.78	188	16.37	12.07
	0.5	357	20.73	229	23.15	
1.75	-1.0	450	16.85	152	11.27	
	-0.5	480	20.25	175	15.16	
	0.0	363	17.42	194	16.96	11.35
	0.5	327	19.29	218	23.99	
1.50	-1.0	567	22.41	168	11.32	
	-0.5	356	16.27	174	15.23	
	0.0	319	16.99	206	17.03	12.50
	0.5	288	17.26	223	24.10	
1.25	-1.0	280	12.71	185	11.09	
	-0.5	308	15.71	194	14.91	
	0.0	273	14.87	204	16.68	13.71
	0.5	264	16.69	223	23.59	

Summary: Software

An extensive sofware package that can calculate (almost) all equilibrium models in the literature!





http://people.sutd.edu.sg/~ugur_arikan/seSue/

Future Work

- Dynamic equilibrium
- Congestion pricing
- Sensitivity analysis
- Price of anarchy
- Parameter estimation (Software package for MDM estimation)

- Product assortment (Challenging combinatorial problem)
- Equivalence results for Integer Linear Programs (Persistency)

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