Mixed Integer Second Order Cone Optimization (MISOCO): Conic Cuts, Warm Start, and Rounding

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Outline

Outline

Conic Optimization

- A general conic optimization (CO) problem is defined as

$$\begin{array}{ll} \min & \langle c, x \rangle \\ \text{s.t.} & \langle a^i, x \rangle = b \quad \forall i \\ & x \in \mathcal{K} \end{array}$$

$\mathcal{K}$ denotes a closed pointed convex cone

$\langle c, x \rangle$ denotes the inner product of vectors c and x.

- $\mathcal{K} = \mathbb{R}^n_+$: Linear Optimization (LO)
- $\mathcal{K} = \mathcal{L}^{n1} \times \mathcal{L}^{n2} \times \cdots \times \mathcal{L}^{nm}$: Second Order Cone Optimization (SOCO)
- $\mathcal{K} = \mathcal{S}^n_+$: Semidefinite Optimization (SDO)
  In this case c, x, and a^i are symmetric matrices, and $\langle c, x \rangle = \text{Tr}(cx)$.

The MISOCO problem

- **Mixed Integer Second-Order Conic Optimization (MISOCO) problems**

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \in \mathcal{L} \\ & x \in \mathbb{Z}^d \times \mathbb{R}^{n+1-d}, \end{array}$$

  in which $\mathcal{L}$ is the Cartesian product of second-order cones.

  For simplicity, we assume that $\mathcal{L}$ is a single second-order cone.

- A Second-Order Cone (SOC) is defined as follows

$$\mathcal{L}^{n+1} = \{x = (x_0, x_1, ..., x_n) |\ \|(x_1, x_2, ..., x_n)\|_2 \leq x_0\}.$$

- MISOCO problems can be solved using a branch and cut methodology.

  We can add cuts to strengthen the formulation and reduce the solution time.

- Nonlinear cuts for MISOCO problems have recently received attention

Disjunction on a convex set

- Let $\mathcal{X} \in \mathbb{R}^n, n > 1$ be a full dimensional closed convex set. Consider two half-spaces

$$\mathcal{A} = \{x \in \mathbb{R}^n : a^T x \geq \alpha\}$$

$$\mathcal{B} = \{x \in \mathbb{R}^n : b^T x \leq \beta\},$$

where $a, b \in \mathbb{R}^n$ and $(a^T, \alpha),\ (b^T, \beta)$ are not scalar multiple of each other.

- **Assumptions**:
  - The intersection $\mathcal{A} \cap \mathcal{B} \cap \mathcal{X}$ is empty.
  - The intersections $\mathcal{X} \cap \mathcal{A}^=$ and $\mathcal{X} \cap \mathcal{B}^=$ are nonempty.

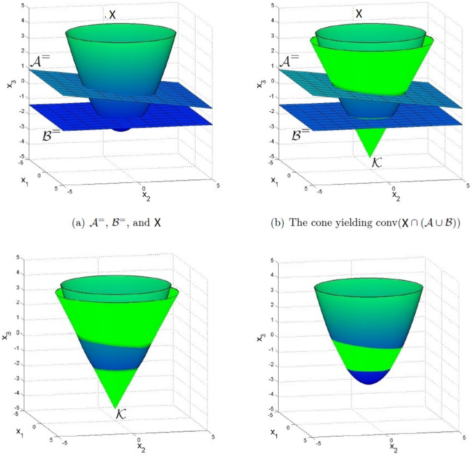
- **Disjunctive Conic Cut (DCC)**:
  A closed convex cone $\mathcal{K} \in \mathbb{R}^n$ with $\dim(\mathcal{K}) > 1$ is called a *DCC* for $\mathcal{X}$ and the disjunction $\mathcal{A} \cup \mathcal{B}$ if

$$\text{conv}(\mathcal{X} \cap (\mathcal{A} \cup \mathcal{B}) = \mathcal{X} \cap \mathcal{K}.$$

- DCCs always exist for MISOCO problems (Belotti et al.).

Illustration of a disjunctive conic cut for a MISOCO problem

- Illustration of DCC for a MISOCO problem



(a) $\mathcal{A}^=$, $\mathcal{B}^=$, and X

(b) The cone yielding conv(X ∩ ($\mathcal{A} \cup \mathcal{B}$))

(c) X ∩ $\mathcal{K}$

(d) conv(X ∩ ($\mathcal{A} \cup \mathcal{B}$))

The figure is from Góez

Uni-parametric family of quadrics

Definition 1

Let $P \in \mathbb{R}^{\ell\times\ell}$, $p, w \in \mathbb{R}^{\ell}$ and $\rho \in \mathbb{R}$, then the quadric $\mathcal{Q}$ is the set defined as

$$\mathcal{Q} = \{w \in \mathbb{R}^{\ell} \mid w^\top P w + 2p^\top w + \rho \leq 0\}.$$

Theorem 2

Let (P, p, ρ) be a quadric and consider two hyperplanes

$$\mathcal{A}^{=} = \{z \mid a^\top z = \alpha\} \text{ and } \mathcal{B}^{=} = \{z \mid d^\top z = \beta\}.$$

The family of quadrics $(P(\tau), p(\tau), \rho(\tau))$ parameterized by $\tau \in \mathbb{R}$ b having the same intersection with $\mathcal{A}^{=}$ and $\mathcal{B}^{=}$ as the quadric (P, p, ρ) is given by

$$P(\tau) = P + \tau\frac{ad^T + da^T}{2}$$
$$p(\tau) = p - \tau\frac{\beta a + \alpha d}{2}$$
$$\rho(\tau) = \rho + \tau\alpha\beta.$$

MISOCO Solution Approaches

- **Linear Approximation of SOCOs** ;
  (GaTech: Nemhauser, Savelsberg, ...; Lehigh: Ralphs, Bulut)
  ++ Allows to use advanced MILO methodology
  – Inferior for MISOCO with higher dimensional cones
- **Use IPMS fof MISOCO with B&DCCs**
  (Lehigh and descendants)
  ++ New powerful cuts, power of IPMs for SOCO
  – Novel methodology Conic-MILO methodology needed

Outline

Mean variance portfolio optimization model

Round Lot constraints

$$
\begin{array}{llcll}
\text{minimize:} & x^\top \Sigma x & & & \\
\text{subject to:} & \mu_0 x_0 + \mu^\top x & \geq & r & \\
& x_0 + e^\top x & = & 1 & \\
& x_i & = & a_i z_i & i = 1, \ldots, n \\
& 0 \leq x_i & \leq & 1 & i = 1, \ldots, n \\
& z & \in & \mathbb{Z}^n_+, &
\end{array}
\tag{1}
$$

- Denote $\hat{\Sigma} = \mathrm{diag}(a)^\top \Sigma \mathrm{diag}(a)$.
- Denote $\hat{\mu} = -\mathrm{diag}(\mu) a + \mu_0 a$, and $\hat{r} = \mu_0 - r$

- Send quadratic objective function to the constraint and define new variable t

Revised RL-MVPO

$$\begin{array}{lrcl} \text{minimize:} & t & & \\ \text{subject to:} & \hat{\mu}^\top z & \leq & \hat{r} \\ & a^\top z & \leq & 1 \\ & 0 \leq a_i z_i & \leq & 1 \\ & z^\top \hat{\Sigma} z & \leq & t \\ & z & \in & \mathbb{Z}_+^N \end{array} \tag{RL-MVPO}$$

Comparison of solution approaches

- For round lot problems, we compared BB, BCC-I and MOSEK

	Number of nodes			Solution Time		
Data	BB	BCC-I	MOSEK	BB	BCC-I	MOSEK
AA	47	15	50	2.254	1.074	0.418
RD0	135	119	73	4.822	3.021	0.196
RD1	83	71	80	3.370	7.353	0.526
RD2	223	127	206	4.336	8.776	0.393
RD3	27	29	143	0.619	1.146	0.444
RD4	6	3	4	0.223	0.205	0.077
RD5	35	36	168	0.668	2.363	0.480
RD6	32	29	51	0.440	1.020	0.217
RD7	17	9	26	0.222	0.472	0.139
RD8	167	183	219	5.460	11.171	0.576
RD9	12	3	4	0.348	0.096	0.076

Comparison of number of nodes and solution time of solution approaches for round-lot AAPs.

Outline

Pathological Disjunctions

See Julio Góez's presentation!

Outline

Rounding - Jordan Frames

Rounding

Jordan frames and values

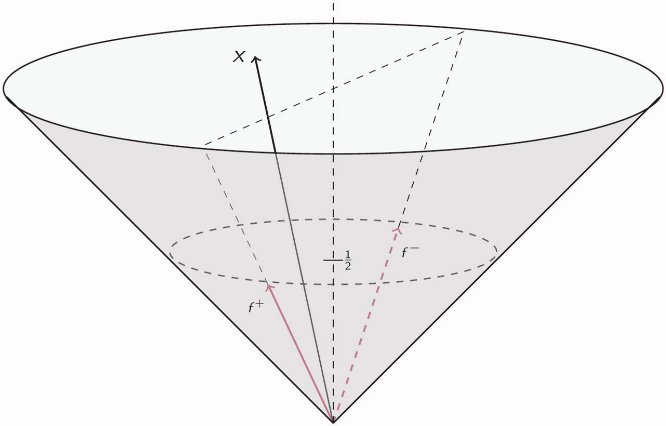


Figure: Jordan frames in a cone

Jordan Frame LO

Let $x^i \in \mathbb{L}_{n_i}$, then the the eigenvalues and Jordan vectors are given as:

$$\lambda_i^+ = x_1^i + \|x_{2:n_i}^i\| \qquad \lambda_i^- = x_1^i - \|x_{2:n_i}^i\|$$

$$f_i^+ = \frac{1}{2}\begin{pmatrix} 1 \\ \frac{x_{2:n_i}^i}{\|x_{2:n_i}^i\|} \end{pmatrix} \qquad f_i^- = \frac{1}{2}\begin{pmatrix} 1 \\ -\frac{x_{2:n_i}^i}{\|x_{2:n_i}^i\|} \end{pmatrix}$$

and we have:

$$x^i = \lambda_i^+ f_i^+ + \lambda_i^- f_i^-.$$

Jordan Frame LO

Rounding

Notation

Denote $F = \begin{bmatrix} F^+ & F^- \end{bmatrix}$ where

$$F^+ = \begin{bmatrix} f_1^+ & 0 & \dots & 0 \\ 0 & f_2^+ & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_k^+ \end{bmatrix}, \qquad F^- = \begin{bmatrix} f_1^- & 0 & \dots & 0 \\ 0 & f_2^- & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_k^- \end{bmatrix}$$

Also, $\lambda = \begin{bmatrix} \lambda^+ \\ \lambda^- \end{bmatrix}$ where

$$\lambda^+ = \begin{bmatrix} \lambda_1^+ \\ \vdots \\ \lambda_k^+ \end{bmatrix}, \qquad \lambda^- = \begin{bmatrix} \lambda_1^- \\ \vdots \\ \lambda_k^- \end{bmatrix}.$$

Let $x = F_P\lambda$ and $z = F_D\kappa$ are Jordan frames and values for primal and dual problems, respectively.

The Rounding LO Problem

Rounding

Rounding problems

Idea

Fix Jordan frames and optimize over Jordan values λ and κ

Rounding on fixed Jordan frames

$$
\begin{array}{lrcl}
\text{minimize:} & c^\top F_P \lambda & & \\
\text{subject to:} & A F_P \lambda & = & b \\
 & \lambda & \geq & 0
\end{array}
$$

$$
\begin{array}{lrcl}
\text{maximize:} & b^\top y & & \\
\text{subject to:} & A^\top y + F_D \kappa & = & c \\
 & \kappa & \geq & 0
\end{array}
$$

Illustration: Primal and Dual Rounding

Rounding

Illustrations

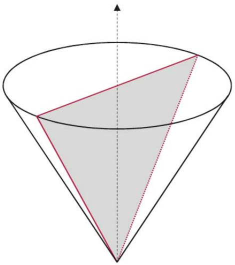


Figure: Primal rounding

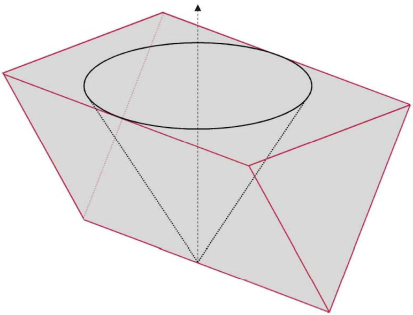


Figure: Dual of dual rounding

Duality in Rounding Schemes

Rounding

Relations between problems

$$\begin{array}{llcll} \min & (c^\top F_P)\lambda & & & \\ \text{s.t.} & (AF_P)\lambda & = & b, & \text{(PR)} \\ & \lambda & \geq & 0 & \end{array} \quad [m]\times[2k] \qquad \overset{\text{dual}}{\Leftrightarrow} \qquad \begin{array}{llcll} \max & b^\top y & & & \\ \text{s.t.} & F_P^\top A^\top y + u & = & F_P^\top c, & \text{(D-PR)} \\ & u & \geq & 0 & \end{array} \quad [2k]\times[m+2k]$$

$$(\text{for } x = F_P\lambda)\ \rotatebox{90}{$\subseteq$} \qquad\qquad\qquad\qquad \rotatebox{90}{$\supseteq$}$$

$$\begin{array}{llcll} \min & c^\top x & & & \\ \text{s.t.} & Ax & = & b, & \text{(P-SOCO)} \\ & x & \in & \mathcal{K} & \end{array} \quad [m]\times[n] \qquad \overset{\text{dual}}{\Leftrightarrow} \qquad \begin{array}{llcll} \max & b^\top y & & & \\ \text{s.t.} & A^\top y + z & = & c, & \text{(D-SOCO)} \\ & z & \in & \mathcal{K} & \end{array} \quad [n]\times[m+n]$$

$$\rotatebox{90}{$\subseteq$} \qquad\qquad\qquad\qquad (\text{for } z = F_D\kappa)\ \rotatebox{90}{$\supseteq$}$$

$$\begin{array}{llcll} \min & c^\top x & & & \\ \text{s.t.} & Ax & = & b, & \text{(D-DR)} \\ & F_D^\top x & \geq & 0 & \end{array} \quad [m+2k]\times[n] \qquad \overset{\text{dual}}{\Leftrightarrow} \qquad \begin{array}{llcll} \max & b^\top y & & & \\ \text{s.t.} & A^\top y + F_D\kappa & = & c, & \text{(DR)} \\ & \kappa & \geq & 0 & \end{array} \quad [n]\times[m+2k]$$

Implications: Weak duality, optimality, infeasibility

Primal Penalty and Rounding problems

$$
\begin{aligned}
\text{minimize: } & c^\top x \\
\text{subject to: } & Ax = b, \\
& x = F^* \lambda, \\
& x_1^i \geq 0, \qquad i \in 1, \ldots, k \\
& x_j \in \mathbb{Z}, \qquad j \in J \subseteq N \\
& \lambda \in \mathbb{R}_+^{2k}.
\end{aligned}
\tag{MIPR}
$$

$$
\begin{aligned}
\text{minimize: } & c^\top x \\
\text{subject to: } & Ax = b, \\
& x_j = x_j^* \quad \forall j \in J \subseteq N, \\
& x \in \mathcal{K}.
\end{aligned}
\tag{FR}
$$

$$
\begin{aligned}
\text{minimize: } & \varphi \frac{c^\top}{\|c\|} x + (1 - \varphi) \sum_\ell \left\| \hat{F}_\ell^\top x \right\| \\
\text{subject to: } & Ax = b, \\
& x \in \mathcal{K},
\end{aligned}
\tag{PEN}
$$

Primal Rounding Algorithm

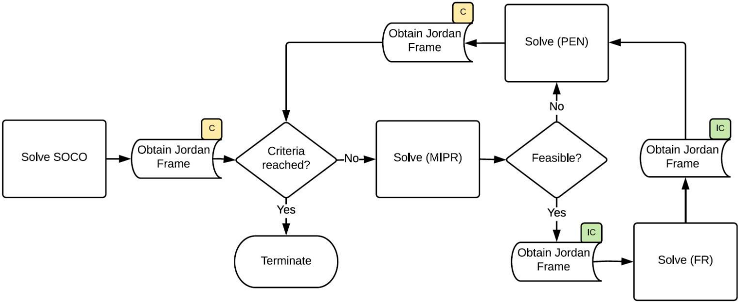
Algorithm 1 The primal rounding heuristic for MISOCO

Input: A MISOCO instance (1),
maximum number of iterations t

Output: A feasible solution $\tilde{x}$ to MISOCO, if found

```
Set c̃ = ∞, φ = 0.5
Solve the continuous relaxation of MISOCO, obtain its solution x^s
Add F^s to the Jordan frame pool
while i ≤ t do
    Solve (MIPR), obtain its solution x* if exists
    if (MIPR) is feasible then
        Add F* to the Jordan frame pool
        Solve (FR) using x*, obtain its solution x^r
        Add F^r to the Jordan frame pool
        if c^T x^r ≤ c̃ then
            c̃ = c^T x^r, x̃ = x^r
        φ = (1+φ)/2
    else
        φ = φ/2
    Solve the penalty problem (PEN), obtain its solution x^p
    Add F^p to the Jordan frame pool
    i = i + 1
return x̃
```

Flow of Primal Rounding Algorithm



Dual Penalty and Rounding problems

$$
\begin{aligned}
\text{minimize:}\quad & c^\top x \\
\text{subject to:}\quad & Ax = b, \\
& F^\top x \geq 0, \\
& x_1^i \geq 0, \quad i \in 1, \ldots, k \\
& x_j \in \mathbb{Z}, \quad j \in J \subseteq N.
\end{aligned}
\tag{MIDR}
$$

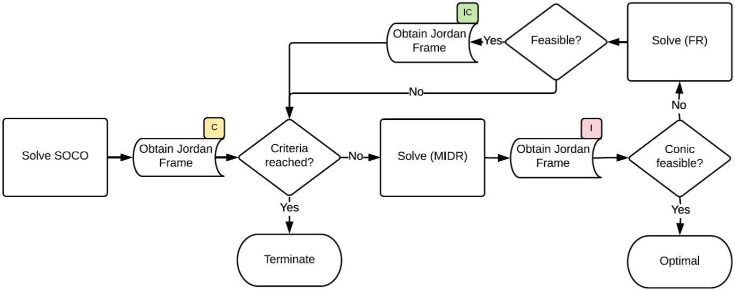
Dual Rounding Algorithm

Algorithm 2 The dual rounding heuristic for MISOCO

Input: A MISOCO instance (1),
maximum number of iterations t
Output: A feasible solution $\tilde{x}$ to MISOCO if found, a global lower bound c_L

Set $\tilde{c} = \infty$
Solve the continuous relaxation of MISOCO, obtain its solution x^s, set $c_L = c^\top x^s$
Add F^s to the Jordan frame pool
while $i \leq t$ **do**
  Solve (MIDR), obtain solution x^*
  Add F^* to the Jordan frame pool
  if $c^\top x^* \geq c_L$ **then**
    $c_L = c^\top x^*$
  if $x^* \in \mathcal{K}$ **then**
    $\tilde{x} = x^*$, $\tilde{c} = c^\top x^*$
    Terminate with an optimal solution to MISOCO x^*.
  else
    Solve (FR) using x^*, obtain its solution x^r if exists
    if (FR) is feasible **then**
      Add F^r to the Jordan frame pool
      if $c^\top x^r \leq \tilde{c}$ **then**
        $\tilde{c} = c^\top x^r$, $\tilde{x} = x^r$
  if $c_L = \tilde{c}$ **then**
    Terminate with an optimal solution to MISOCO $\tilde{x}$.
  $i = i + 1$
return $\tilde{x}, c_L$

Flow of Dual Rounding Algorithm



Test Problem Set

		Variables		Integers		Cones		Cone sizes	
Pr. Types	#P	Min	Max	Min	Max	Min	Max	Min	Max
ck	90	611	3271	25	75	10	20	27	77
classical	399	146	356	20	50	1	1	21	51
estein	9	125	246	9	18	9	18	3	3
pp	3	72	702	10	100	10	100	3	3
robust	400	198	468	21	51	2	2	22	52
shortfall	400	194	464	21	51	2	2	21	51
sssd	14	273	785	72	264	12	24	3	3
turbine	7	121	512	11	56	25	119	3	3
QPLIB	6	3033	13538	20	400	1	1	802	4502
Summary	1328	72	13538	9	400	1	119	3	4502

Results Primal and Dual Rounding

		# Iters											
Heur	P.Type	1	2	3	4	5	6	7	8	9	10	Failed	Total
P		647	628	28	2	3			1		1	18	1328
	ck	90											90
	classical	5	393	1									399
	estein	9											9
	pp		2	1									3
	robust	136	231	24	2	3			1		1	2	400
	shortfall	400											400
	sssd											14	14
	turbine	2	2	2								1	7
	QPLIB	5										1	6
D		445	146	73	80	50	57	52	46	50	44	285	1328
	ck	10	15	13	8	8	5	6	9	3		13	90
	classical	140	42	19	30	15	19	16	9	18	14	77	399
	estein	9											9
	pp		2					1					3
	robust	120	40	24	33	16	16	19	20	17	13	82	400
	shortfall	142	45	17	8	11	17	10	8	12	17	113	400
	sssd	14											14
	turbine	4	2		1								7
	QPLIB	6											6

Results Primal-Dual adn Hybrid Rounding

PD		647	230	8	68		36	2	53	1	23	260	1328
	ck	90											90
	classical	5	138		42		18		30		15	151	399
	estein	9											9
	pp			3									3
	robust	136	73	5	25		18	2	23	1	8	109	400
	shortfall	400											400
	sssd		14										14
	turbine	2	4		1								7
	QPLIB	5	1										6
HS		647	629	27	16	2	2			2		3	1328
	ck	90											90
	classical	5	393	1									399
	estein	9											9
	pp		3										3
	robust	136	231	24	2	2				2		3	400
	shortfall	400											400
	sssd				13		1						14
	turbine	2	2	2			1						7
	QPLIB	5			1								6

Warm Start Methodology

Warm-start methodology

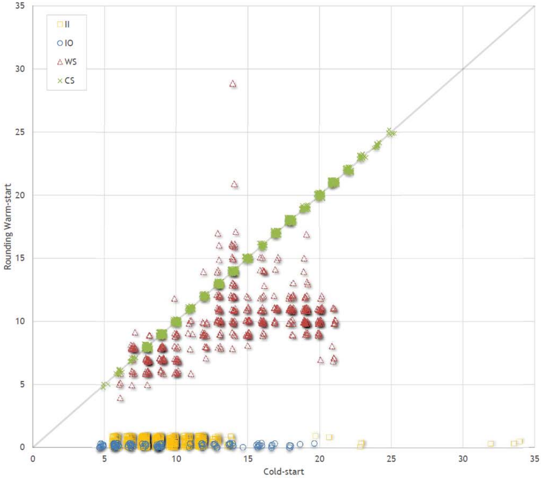
Branching

1. Solve continuous relaxation and obtain solution x^*.
2. Obtain optimal Jordan frames, F_P and F_D.
3. Choose a variable to branch and add the corresponding bound constraint.
4. Solve dual rounding problem.
    - If it is unbounded, then primal SOCO is infeasible.
5. Solve primal rounding problem.
    - If objectives are equal, then the solution is optimal.
6. If both feasible, take the convex combination of an IPM iteration and the rounding solutions.
7. Warm-start self-dual embedding IPM from this initial point.

Numerical Experiences – Proof of the pie

Numerical experiments

Comparison to cold-start



Conclusion

- We presented Disjunctive Conic Cuts (DCCs) for MISOCO.
- We demonstrated the power of DCCs.
- The identification of the pathological cases is important for the efficient implementation of DCCs (see Góez).
- Utilized Jordan frames to developed Primal, Dual, and Prima-Dual Hybrid rounding heuristics for MISOCOs.
- Developed an efficient warm-start method for SOCO.
- Both the rounding and warm-start methodologies are proved to be efficient.

Thanks

Any questions?