A notion of Total Dual Integrality for Convex, Semidefinite and Extended Formulations

Marcel de Carli Silva    Levent Tunçel

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A vector in $\mathbb{R}^n$ is *integral* if each of its components is an integer, and a rational system of linear inequalities $Ax \leq b$ is called *totally dual integral (TDI)* if, for every integral vector $c \in \mathbb{Z}^n$, the LP problem dual to $\sup \{ \langle c, x \rangle : Ax \leq b \}$ has an integral optimal solution whenever it has an optimal solution at all.

In this case, the polyhedron $P$ determined by $Ax \leq b$ is integral, i.e., each nonempty face of $P$ has an integral vector; thus, (under the assumption that both primal and dual are feasible) equality holds throughout in the chain from the board. This was proved in seminal work of Edmonds and Giles [1977] as a consequence of the following fundamental result:

**Theorem** (Edmonds-Giles [1977]) If $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$ satisfy $\sup \{ \langle c, x \rangle : Ax \leq b \} \in \mathbb{Z} \cup \{\pm \infty\}$ for each $c \in \mathbb{Z}^n$, then the polyhedron $\{ x \in \mathbb{R}^n : Ax \leq b \}$ is integral.
A vector in $\mathbb{R}^n$ is **integral** if each of its components is an integer, and a rational system of linear inequalities $Ax \leq b$ is called **totally dual integral (TDI)** if, for every integral vector $c \in \mathbb{Z}^n$, the LP problem dual to $\sup \{ \langle c, x \rangle : Ax \leq b \}$ has an integral optimal solution whenever it has an optimal solution at all.

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*(Edmonds-Giles [1977])* If $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$ satisfy

$$\sup \{ \langle c, x \rangle \mid Ax \leq b \} \in \mathbb{Z} \cup \{ \pm \infty \} \text{ for each } c \in \mathbb{Z}^n,$$

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for each $c \in \mathbb{Z}^n$, then the polyhedron \(\{x \in \mathbb{R}^n : Ax \leq b\}\) is integral.

Corollary

(Hoffman [1974]) Let $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$. If
\[
P := \{x \in \mathbb{R}^n : Ax \leq b\}
\]
is bounded and \(\max_{x \in P} \langle c, x \rangle \in \mathbb{Z}\) for each $c \in \mathbb{Z}^n$, then $P$ is integral.
To define the *integrality constraint* for the dual SDP, we shall consider the dual slack.

Definition

Let $S$ be feasible in the dual SDP. We say that "$S$ is integral" if $S$ is a sum $S = \sum_{k=1}^{N} S_k$ of rank-one matrices $S_1, \ldots, S_N \in S_{n+1}$ such that, for each $k \in [N]$, we have $(S_k)^{00} = 1$, $(S_k)^{0j} + (S_k)^{jj} = 0 \forall j \in [n]$. 

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Total Dual Integrality
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$$\begin{align*}
(S_k)_{00} &= 1, \\
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\end{align*}$$
Theorem

If $C \subset \mathbb{R}^n$ is a compact convex set, then

$$C = \{ x \in \mathbb{R}^n : \langle w, x \rangle \leq \delta^*(C|w) \forall w \in \mathbb{Z}^n \}.$$

Let $C_I := \operatorname{conv}(C \cap \mathbb{Z}^n)$.

Using a bit more from IP theory, we conclude the following generalization of Hoffman's Theorem.

Corollary

If $C \subset \mathbb{R}^n$ is a nonempty compact convex set, then $C = C_I$ if and only if $\delta^*(C|w) \in \mathbb{Z}$ for every $w \in \mathbb{Z}^n$. 
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If $C \subset \mathbb{R}^n$ is a compact convex set, then

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Theorem

Let $C \subseteq \mathbb{R}^n$ be a nonempty compact convex set. Then, TFAE

(i) $C = C_{\text{i}}$;
(ii) every nonempty face of $C$ contains an integral point;
(iii) for every $w \in \mathbb{R}^n$, $\max \{ \langle w, x \rangle : x \in C \}$ is attained by an integral vector;
(iv) every rational supporting hyperplane for $C$ contains integral points;
(v) $\exists \bar{x} \in C$ such that for each $w \in \mathbb{Z}^n$, $\langle w, \bar{x} \rangle + \inf \{ \eta \in \mathbb{R}^+ : 1/\eta w \in (C - \bar{x}) \}$ is integrally attainable.
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$$\langle w, \bar{x} \rangle + \inf \left\{ \eta \in \mathbb{R}_{++} : \frac{1}{\eta} w \in (C - \bar{x})^o \right\} \in \mathbb{Z}.$$
Definition

Let $\mathcal{L} : \mathbb{R}^k \rightarrow \mathbb{S}^{n+1}$ be a linear map. The system

$$\mathcal{A}(\hat{X}) \leq b, \hat{X} \succeq 0$$

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is **TDI through $\mathcal{L}$** if for every $c \in \mathbb{Z}^k$, the SDP dual to

$$\sup \left\{ \langle \mathcal{L}(c), \hat{\mathbf{X}} \rangle : \mathcal{A}(\hat{\mathbf{X}}) \leq \mathbf{b}, \hat{\mathbf{X}} \succeq \mathbf{0} \right\}$$

has an “integral” optimal solution, whenever it has an optimal solution.
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Note,

$$\hat{X} \text{ is of the form } \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix}$$
Let $A(\hat{X}) \leq b$, $\hat{X} \succeq 0$ be TDI through a linear map $L : \mathbb{R}^k \to \mathbb{S}^{n+1}$. If $b$ is integral, $C$ is compact, and $\hat{C}$ has a positive definite matrix, then $C = C_I$. 

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Theorem

Let $A(\hat{X}) \leq b$, $\hat{X} \succeq 0$ be TDI through a linear map $\mathcal{L}: \mathbb{R}^k \to \mathbb{S}^{n+1}$. Set $\hat{C} := \{ \hat{X} \in \mathbb{S}^{n+1}_+ : A(\hat{X}) \leq b \}$ and $C := \mathcal{L}^*(\hat{C}) \subseteq \mathbb{R}^k$. If $b$ is integral, $C$ is compact, and $\hat{C}$ has a positive definite matrix, then $C = C^I$. 

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Let $\mathcal{A}(\hat{X}) \leq b$, $\hat{X} \succeq 0$ be TDI through a linear map $\mathcal{L} : \mathbb{R}^k \to \mathbb{S}_+^{n+1}$. Set $\hat{\mathcal{C}} := \left\{ \hat{X} \in \mathbb{S}_+^{n+1} : \mathcal{A}(\hat{X}) \leq b \right\}$ and $\mathcal{C} := \mathcal{L}^*(\hat{\mathcal{C}}) \subseteq \mathbb{R}^k$. If $b$ is integral, $\mathcal{C}$ is compact, and $\hat{\mathcal{C}}$ has a positive definite matrix, then $\mathcal{C} = \mathcal{C}_I$. 
Theorem

Let $A(\hat{X}) \leq b$, $\hat{X} \succeq 0$ be TDI through a linear map $L: \mathbb{R}^k \to S^{n+1}$ such that $b$ is integral.
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has a positive definite matrix and that $C := \mathcal{L}^*(\hat{C}) \subseteq [0, 1]^k$ is compact. If $\hat{C}$ is a rank-one embedding of $C_{I}$ via $\mathcal{L}$,
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Theorem

Let $A(\hat{X}) \leq b$, $\hat{X} \succeq 0$ be TDI through a linear map $\mathcal{L}: \mathbb{R}^k \to \mathbb{S}^{n+1}$ such that $b$ is integral. Suppose that

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has a positive definite matrix and that $\mathcal{C} := \mathcal{L}^*(\hat{\mathcal{C}}) \subseteq [0, 1]^k$ is compact. If $\hat{\mathcal{C}}$ is a rank-one embedding of $\mathcal{C}_I$ via $\mathcal{L}$, then for every $w \in \mathbb{Z}^k$, equality holds throughout in the chain of inequalities on the board, all optimum values are equal to

$$\max \left\{ \langle w, x \rangle : x \in \mathcal{C}_I \right\} ,$$
Theorem

Let $A(\hat{X}) \leq b$, $\hat{X} \succeq 0$ be TDI through a linear map $L: \mathbb{R}^k \rightarrow \mathbb{S}^{n+1}$ such that $b$ is integral. Suppose that

$\hat{C} := \left\{ \hat{X} \in \mathbb{S}_{+}^{n+1} : A(\hat{X}) \leq b \right\}$ has a positive definite matrix and that $C := L^*(\hat{C}) \subseteq [0, 1]^k$ is compact. If $\hat{C}$ is a rank-one embedding of $C_I$ via $L$, then for every $w \in \mathbb{Z}^k$, equality holds throughout in the chain of inequalities on the board, all optimum values are equal to

$$\max \left\{ \langle w, x \rangle : x \in C_I \right\},$$

and all suprema and infima are attained.
Open Problems/research Directions

- Obtain a primal-dual symmetric integrality condition for SDPs that applies to arbitrary ILPs, not just binary ones.
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Given $k \geq 1$ and the $LS_+$ operator of Lovász and Schrijver [1991](called $N_+$ in their paper), determine the class of graphs for which the $k$th iterate of the $LS_+$ operator applied to the system

$$x \geq 0, \quad x_i + x_j \leq 1 \quad \forall ij \in E$$

yields a TDI system through the appropriate lifting, leading to a family of combinatorial min-max theorems involving maximum weight stable sets in such graphs.
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yields a TDI system through the appropriate lifting, leading to a family of combinatorial min-max theorems involving maximum weight stable sets in such graphs.

Obtain systematic, primal-dual symmetric conditions for exactness in SDP relaxations for continuous problems.
Theorem

Every pointed closed convex set is the intersection of all closed rational halfspaces containing it.
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Every pointed closed convex set is the intersection of all closed rational halfspaces containing it.

This presentation was based on:

- M. K. de Carli Silva and L. T., Pointed closed convex sets are the intersection of all rational supporting closed halfspaces, arXiv:1802.03296