A notion of Total Dual Integrality for Convex, Semidefinite and Extended Formulations

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Theorem

(Edmonds-Giles [1977]) If $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$ satisfy $\sup\{\langle c, x\rangle : \ Ax \leq b\} \in \mathbb{Z} \cup \{\pm\infty\}$ *for each $c \in \mathbb{Z}^n$, then the polyhedron $\{x \in \mathbb{R}^n : \ Ax \leq b\}$ is integral.*

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Corollary

(Hoffman [1974]) Let $A \in \mathbb{Q}^{m \times n}$ *and* $b \in \mathbb{Q}^m$*. If* $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ *is bounded and* $\max_{x \in P} \langle c, x\rangle \in \mathbb{Z}$ *for each* $c \in \mathbb{Z}^n$*, then* P *is integral.*

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Definition

Let S be feasible in the dual SDP. We say that "S is *integral*" if S is a sum

$$S = \sum_{k=1}^{N} S_k$$

of rank-one matrices $S_1, \ldots, S_N \in \mathbb{S}_+^{n+1}$ such that, for each $k \in [N]$, we have

$$\begin{aligned} (S_k)_{00} &= 1, & \\ (S_k)_{0j} + (S_k)_{jj} &= 0 & \forall j \in [n]. \end{aligned}$$

Theorem

If $\mathcal{C} \subset \mathbb{R}^n$ *is a compact convex set, then* $\mathcal{C} = \{x \in \mathbb{R}^n : \ \langle w, x \rangle \leq \delta^*(\mathcal{C}|w) \, \forall w \in \mathbb{Z}^n\}$.

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Using a bit more from IP theory, we conclude the following generalization of Hoffman's Theorem.

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Using a bit more from IP theory, we conclude the following generalization of Hoffman's Theorem.

Corollary

If $\mathcal{C} \subset \mathbb{R}^n$ is a nonempty compact convex set, then $\mathcal{C} = \mathcal{C}_I$ if and only if $\delta^*(\mathcal{C}|w) \in \mathbb{Z}$ for every $w \in \mathbb{Z}^n$.

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(v) $\exists \bar{x} \in \mathcal{C}$ *such that for each $w \in \mathbb{Z}^n$,*

$$\langle w, \bar{x}\rangle + \inf\left\{\eta \in \mathbb{R}_{++} \ : \ \frac{1}{\eta} w \in (\mathcal{C} - \bar{x})^o\right\} \in \mathbb{Z}.$$

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Let $\mathcal{L} : \mathbb{R}^k \to \mathbb{S}^{n+1}$ be a linear map. The system

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Note,

$$\hat{X} \text{ is of the form } \begin{bmatrix} 1 & x^\top \\ x & X \end{bmatrix}$$

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and all suprema and infima are attained.

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- Obtain systematic, primal-dual symmetric conditions for exactness in SDP relaxations for continuous problems.

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This presentation was based on:

- M. K. de Carli Silva and L. T., A notion of total dual integrality for convex, semidefinite, and extended formulations, arXiv:1801.09155
- M. K. de Carli Silva and L. T., Pointed closed convex sets are the intersection of all rational supporting closed halfspaces, arXiv:1802.03296