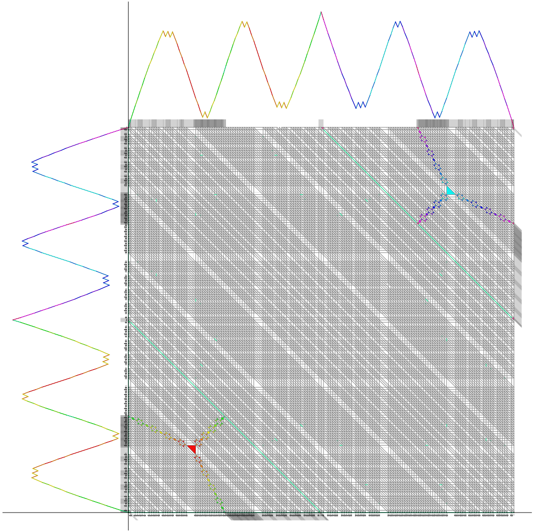
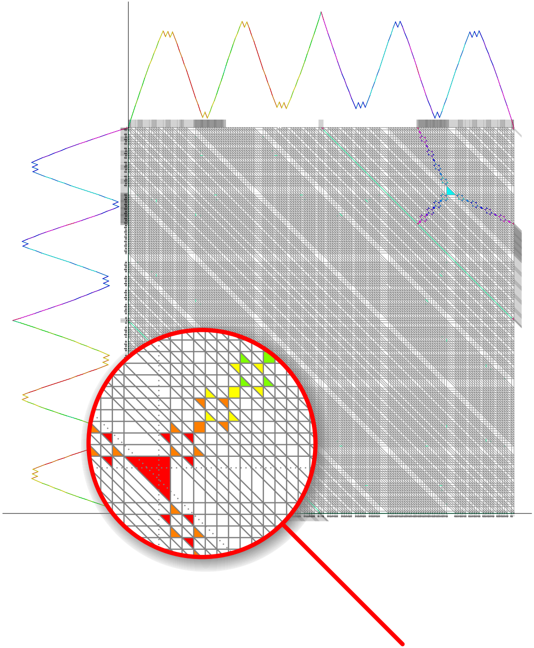


Discrete geometry of functional equations in cutgeneratingfunctionology

Matthias Köppe

University of California, Davis,
Mathematics



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Cut-generating functions in the Gomory–Johnson infinite group relaxation

Let $G = \mathbb{Q}$ or $G = \mathbb{R}$. Consider

$$\min \quad \langle \eta, y \rangle \quad \text{s.t.} \quad y \in F \subseteq \mathbb{Z}_+^{(G)},$$

where

- the primal space is the space $\mathbb{R}^{(G)}$ of finite-support functions $y\colon G \to \mathbb{R}$;
- linear functionals η are in the dual space $\mathbb{R}^G$ of arbitrary functions $\eta\colon G \to \mathbb{R}$;
- dual pairing $\langle \eta, y \rangle = \sum_{r \in G} \eta(r) y(r)$;
- $F = \{\, y\colon G \to \mathbb{R} \mid \sum_{r \in G} y(r)\, r \in f + \mathbb{Z} \,\}$ for a constant $f \notin \mathbb{Z}$.

If $G = \mathbb{Q}$, then $R = \operatorname{conv}(F) \subseteq \mathbb{R}_+^{(G)}$ convex set of "blocking type", $\operatorname{rec}(R) = \mathbb{R}_+^{(G)}$; thus can normalize:

Nontrivial **valid inequalities** $\langle \pi, y \rangle \geq 1$, $\pi \geq 0$.

Same holds for $G = \mathbb{R}$!

Basu–Conforti–Di Summa–Paat, IPCO 2017.

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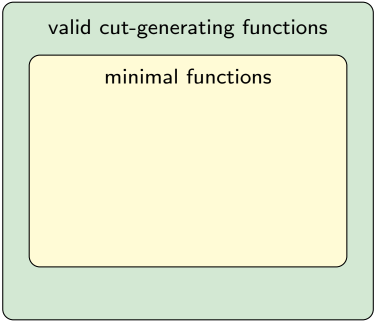

Gomory–Johnson (1972) characterize minimal functions π:

π is **periodic** modulo 1,
$\pi(r) = 0$ for $r \in \mathbb{Z}$,
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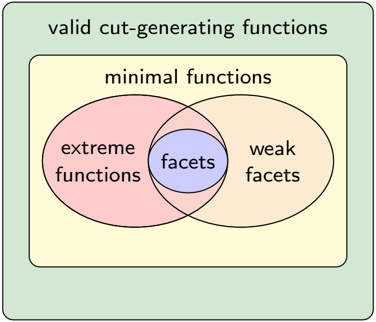

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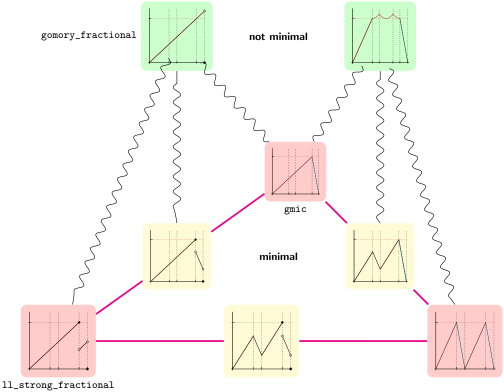
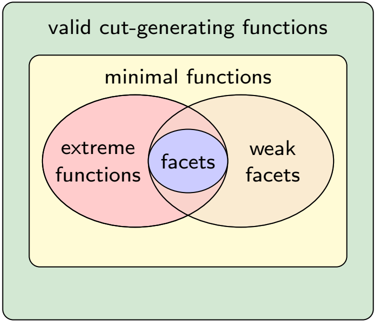
A hierarchy of functions: valid, minimal, extreme / facet

For minimal π define the vector space $\tilde{\Pi}^{\pi}$ of effective perturbation functions $\tilde{\pi}\colon G \to \mathbb{R}$:

$$\exists \epsilon > 0,\ \pi \pm \epsilon\tilde{\pi} \text{ minimal}.$$

Say π is extreme if $\tilde{\Pi}^{\pi} = \{0\}$.





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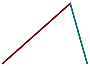
An electronic compendium of extreme functions

Kö.–Zhou (2014–); available at `https://github.com/mkoeppe/cutgeneratingfunctionology`



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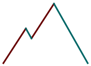
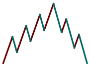


gj_2_slope



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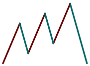
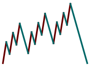
repeat



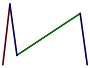
dg_2_step_mir



kf_n_step_mir

bccz_
counterexample

gj_forward_3_
slope



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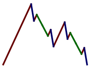

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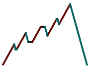
merge_3_slope



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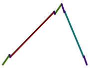
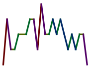


chen_4_slope



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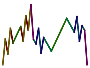

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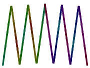

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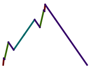

bcdsp_arbitrary_
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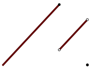

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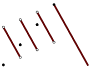

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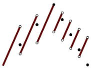
limit



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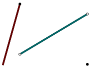

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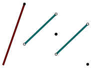

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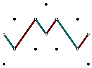

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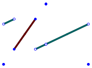
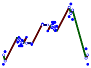


zhou_two_sided_
discontinuous_
cannot_assume_
any_continuity



kzh_minimal_
has_only_crazy_
perturbation_1

bcds_
discontinuous_
everywhere

Effective perturbations of minimal functions

Given a minimal function π, what properties does an effective perturbation $\tilde{\pi} \in \tilde{\Pi}^{\pi}$ necessarily have?

For a (possibly discontinuous) piecewise linear function π (on partition $\mathcal{P}$), define a **polyhedral complex** $\Delta\mathcal{P}$ on $\mathbb{R} \times \mathbb{R}$ with faces

$$F(I, J, K) = \{\, (x, y) \in \mathbb{R} \times \mathbb{R} \mid x \in I,\ y \in J,\ x + y \in K \,\}$$

where I, J, K are breakpoints or subintervals of $\mathcal{P}$.

- subadditivity slack

$$\Delta\pi(x, y) = \pi(x) + \pi(y) - \pi(x + y)$$

  is affine-linear on $\operatorname{rel\,int}(F)$ for $F \in \Delta\mathcal{P}$.
- Green faces have $\Delta\pi = 0$ on $\operatorname{rel\,int}(F)$
- By convexity, because

$$\left.\begin{aligned} \pi^+ &= \pi + \epsilon\tilde{\pi} \\ \pi & \\ \pi^- &= \pi - \epsilon\tilde{\pi} \end{aligned}\right\} \text{subadditive,}$$

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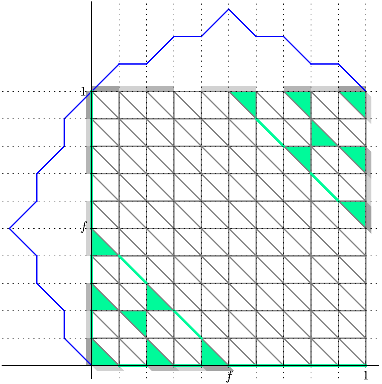

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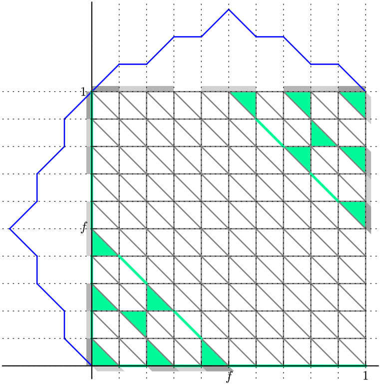

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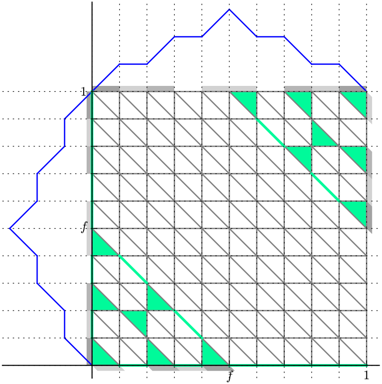

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Cauchy's and Pexider's functional equation on simple bounded domains

Basu–Hildebrand–**Kö.**, Equivariant Perturbation III, MPA 2017

Program: Infer properties of effective perturbations $\tilde\pi$
from the fact that they are bounded solutions to:

Cauchy's functional equation on a domain F:

$$\tilde\pi(u) + \tilde\pi(v) = \tilde\pi(u+v) \quad \text{for all} \quad (u,v) \in F.$$

- $F = \mathbb{R}^k \times \mathbb{R}^k$: Solutions to Cauchy are homogeneous linear functions $\tilde\pi(x) = ax$
- $F = \mathbb{R}^k \times \mathbb{R}^k$: Solutions to Pexider are affine-linear functions $\tilde\pi_i(x) = ax + b_i$
- **Interval lemma** (Gomory–Johnson, 1973/2003):
  $F = U \times V \subseteq \mathbb{R} \times \mathbb{R}$, where U, $V \subseteq \mathbb{R}$ proper intervals: Solutions to Pexider's equation are functions $\tilde\pi_i(x)$ whose restrictions to $p_1(F) = U$, $p_2(F) = V$, $p_3(F) = U + V$ are affine-linear functions $ax + b_i$.
- **Convex additivity domain lemma** (Basu–Hildebrand–**Kö.**, 2014):
  F a full-dimensional convex set of $\mathbb{R}^k$: Solutions to Pexider's equation are functions $\tilde\pi_i(x)$ whose restrictions to $\operatorname{int}(p_1(F))$, $\operatorname{int}(p_2(F))$, $\operatorname{int}(p_3(F))$ are affine-linear functions $ax + b_i$
- **Open:** Characterization of full-dimensional polyhedra $F \subset \mathbb{R}^k$ for which affine linearity extends to boundary of $p_i(F)$.

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$$\tilde{\pi}(u) + \tilde{\pi}(v) = \tilde{\pi}(u + v) \quad \text{for all} \quad (u, v) \in F.$$

- $F = \mathbb{R}^k \times \mathbb{R}^k$: Solutions to Cauchy are homogeneous linear functions $\tilde{\pi}(x) = ax$
- $F = \mathbb{R}^k \times \mathbb{R}^k$: Solutions to Pexider are affine-linear functions $\tilde{\pi}_i(x) = ax + b_i$
- **Interval lemma** (Gomory–Johnson, 1973/2003):
  $F = U \times V \subseteq \mathbb{R} \times \mathbb{R}$, where U, $V \subseteq \mathbb{R}$ proper intervals: Solutions to Pexider's equation are functions $\tilde{\pi}_i(x)$ whose restrictions to $p_1(F) = U$, $p_2(F) = V$, $p_3(F) = U + V$ are affine-linear functions $ax + b_i$.
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- **Open:** Characterization of full-dimensional polyhedra $F \subset \mathbb{R}^k$ for which affine linearity extends to boundary of $p_i(F)$.

Cauchy's and Pexider's functional equation on simple bounded domains

Basu–Hildebrand–**Kö.**, Equivariant Perturbation III, MPA 2017

Program: Infer properties of effective perturbations $\tilde{\pi} = \tilde{\pi}_i$ from the fact that they are bounded solutions to:

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Effective perturbations of minimal functions

Hong–**Kö.**–Zhou, Equivariant Perturbation V, OMS 2017

Lemma (Dey, Richard, Li, Miller, MPA 2010; Hong–Kö.–Zhou, OMS 2017)

Let $\pi\colon G \to \mathbb{R}_+$ be a piecewise linear minimal function that is continuous from the right at 0 or continuous from the left at 1. Let $\tilde\pi \in \tilde\Pi^\pi$ be an effective perturbation function.

If π is continuous on a proper interval $I \subset G$, then $\tilde\pi$ is Lipschitz continuous on I.

In particular, limits $\tilde\pi(x^-)$ and $\tilde\pi(x^+)$ exist for any $x \in G$.

Proof.

WLOG, π is continuous from the right at 0. $\exists\, s, b > 0$ s.t. $\pi(x) = sx$ for $x \in [0, 2b]$.

$\exists\, \epsilon > 0$ such that $\pi^+ = \pi + \epsilon\tilde\pi$ and $\pi^- = \pi - \epsilon\tilde\pi$ are minimal.

For $x, y \in [0, b]$, we have $\pi(x) + \pi(y) = \pi(x + y)$; so $\tilde\pi(x) + \tilde\pi(y) = \tilde\pi(x + y)$.

$\tilde\pi(0) = 0$. By the **Interval Lemma**, $\exists\, \tilde s \in \mathbb{R}$ such that $\tilde\pi(x) = \tilde s x$ for $x \in [0, b]$.

Then π^+ and π^- have slopes $s^+ := s + \epsilon\tilde s$ and $s^- := s - \epsilon\tilde s$ on $[0, b]$, respectively.

$\exists\, s_I \in \mathbb{R}$ such that $\pi(x) - \pi(y) \geq s_I(x - y)$ for $x, y \in I$.

Let $x, y \in I$ such that $x > y$. By **subadditivity**, we have $\pi^+(x) - \pi^+(y) \leq s^+(x - y)$ and $\pi^-(x) - \pi^-(y) \leq s^-(x - y)$. It follows from $\epsilon\tilde\pi = \pi^+ - \pi = \pi - \pi^-$ that

$$(s_I - s^-)(x - y) \leq \epsilon(\tilde\pi(x) - \tilde\pi(y)) \leq (s^+ - s_I)(x - y).$$

Therefore, $|\tilde\pi(x) - \tilde\pi(y)| \leq C\,|x - y|$, where $C = \frac{1}{\epsilon}\max(|s^+ - s_I|, |s^- - s_I|)$. □

Effective perturbations of minimal functions

Hong–**Kö.**–Zhou, Equivariant Perturbation V, OMS 2017

Lemma (Dey, Richard, Li, Miller, MPA 2010; Hong–Kö.–Zhou, OMS 2017)

Let $\pi\colon G \to \mathbb{R}_+$ be a piecewise linear minimal function that is continuous from the right at 0 or continuous from the left at 1. Let $\tilde\pi \in \tilde\Pi^\pi$ be an effective perturbation function.

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$\exists\ \epsilon > 0$ such that $\pi^+ = \pi + \epsilon\tilde\pi$ and $\pi^- = \pi - \epsilon\tilde\pi$ are minimal.
For $x, y \in [0, b]$, we have $\pi(x) + \pi(y) = \pi(x + y)$; so $\tilde\pi(x) + \tilde\pi(y) = \tilde\pi(x + y)$.
$\tilde\pi(0) = 0$. By the **Interval Lemma**, $\exists\ \tilde s \in \mathbb{R}$ such that $\tilde\pi(x) = \tilde s x$ for $x \in [0, b]$.
Then π^+ and π^- have slopes $s^+ := s + \epsilon\tilde s$ and $s^- := s - \epsilon\tilde s$ on $[0, b]$, respectively.
$\exists\ s_I \in \mathbb{R}$ such that $\pi(x) - \pi(y) \geq s_I(x - y)$ for $x, y \in I$.
Let $x, y \in I$ such that $x > y$. By **subadditivity**, we have $\pi^+(x) - \pi^+(y) \leq s^+(x - y)$ and $\pi^-(x) - \pi^-(y) \leq s^-(x - y)$. It follows from $\epsilon\tilde\pi = \pi^+ - \pi = \pi - \pi^-$ that

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Therefore, $|\tilde\pi(x) - \tilde\pi(y)| \leq C\,|x - y|$, where $C = \frac{1}{\epsilon}\max(\left|s^+ - s_I\right|, \left|s^- - s_I\right|)$. □

Effective perturbations of minimal functions

Hong–**Kö.**–Zhou, Equivariant Perturbation V, OMS 2017

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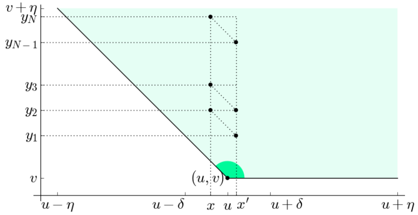
Cauchy–Pexider in the limit

Basu–Hildebrand–**Kö.**, Equivariant Perturbation I, MOR 2012; **Kö.**–Zhou, Equivariant Perturbation VI, arXiv: 1605.03975v3, 2018

Program: Additional properties of effective perturbations follow from Cauchy–Pexider's equation holding only in the limit near some points.

("stability of functional equations")



Theorem (**Kö.**–Zhou, 2018)

Let F be a two-dimensional face of $\Delta\mathcal{P}$, where $\mathcal{P}$ is the one-dimensional polyhedral complex of a piecewise linear function. Let $(u, v) \in F$.
For $i = 1, 2, 3$, let $\tilde{\pi}_i \colon \mathbb{R} \to \mathbb{R}$ be a function that is bounded near

$$u = p_1(u, v), \quad v = p_2(u, v), \quad u + v = p_3(u, v).$$

If

$$\Delta\tilde{\pi}_F(u, v) = \lim_{\substack{(x,y)\to(u,v)\\(x,y)\in\operatorname{int}(F)}} \tilde{\pi}_1(x) + \tilde{\pi}_2(y) - \tilde{\pi}_3(x + y) = 0,$$

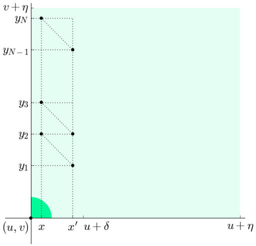
then for $i = 1, 2, 3$, the limit $\lim_{t\to p_i(u,v),\ t\in\operatorname{int}(p_i(F))} \tilde{\pi}_i(t)$ exists.

Cauchy–Pexider in the limit. Case 1

Kö.–Zhou, Equivariant Perturbation VI, arXiv:1605.03975

Show: $\forall \epsilon > 0$ $\exists$ a relative neighborhood $U = (u, u + \delta(\epsilon))$ of u in $\operatorname{int}(p_1(F))$ so that

for all $x, x' \in U$, we have $\left|\tilde{\pi}_1(x) - \tilde{\pi}_1(x')\right| \le \varepsilon$.



Pick $\eta > 0$ small enough so that

- $C_\eta = [u, u + \eta] \times [v, v + \eta] \subseteq F$
- $|\Delta\tilde{\pi}(x, y)| < \varepsilon/4$ for $(x, y) \in C_\eta$
- $|\tilde{\pi}_i(t)| \le M$ for $t \in p_i(C_\eta)$ (some M)

Take $N > 4M/\varepsilon + 1$ and $\delta = \eta/(2N)$.

Take $x, x' \in U$, $x < x'$. Define $y_n = v + \delta + (n-1)(x' - x)$ for $1 \le n \le N$. All (x, y_i) and (x', y_i) lie in $C_\eta \cap \operatorname{int}(F)$.

$$|\Delta\tilde{\pi}(x, y_{n+1})| = |\tilde{\pi}_1(x) + \tilde{\pi}_2(y_{n+1}) - \tilde{\pi}_3(x + y_{n+1})| \le \varepsilon/4$$
$$|\Delta\tilde{\pi}(x', y_n)| = \left|\tilde{\pi}_1(x') + \tilde{\pi}_2(y_n) - \tilde{\pi}_3(x' + y_n)\right| \le \varepsilon/4$$

With $x + y_{n+1} = x' + y_n$ for $n \in \{1, 2, \dots, N-1\}$, by the triangle inequality,

$$\left|\tilde{\pi}_1(x) - \tilde{\pi}_1(x') + \tilde{\pi}_2(y_{n+1}) - \tilde{\pi}_2(y_n)\right| \le \varepsilon/2.$$

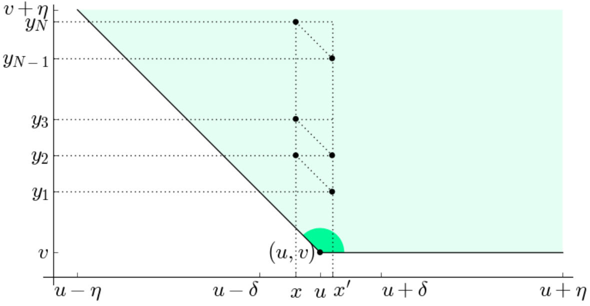
Summing over $n = 1, 2, \dots, N-1$, triangle inequality:

$$\left|(N-1)\big(\tilde{\pi}_1(x) - \tilde{\pi}_1(x')\big) + \tilde{\pi}_2(y_N) - \tilde{\pi}_2(y_1)\right| \le (N-1)\varepsilon/2.$$

Therefore, $|\tilde{\pi}_1(x) - \tilde{\pi}_1(x')| \le |\tilde{\pi}_2(y_N) - \tilde{\pi}_2(y_1)|/(N-1) + \varepsilon/2 \le 2M/(N-1) + \varepsilon/2 \le \varepsilon$.

Cauchy–Pexider in the limit. Case 2

Kö.–Zhou, Equivariant Perturbation VI, arXiv:1605.03975



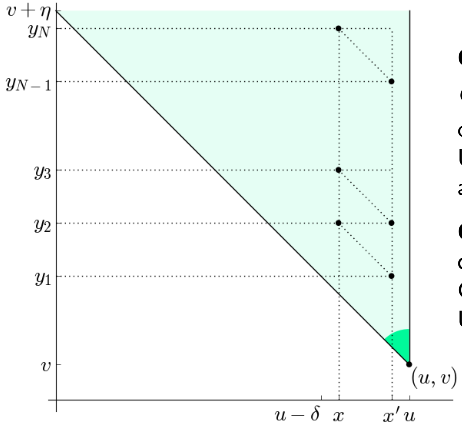
The quadrilateral $C_\eta = \operatorname{conv}\left(\binom{u}{v}, \binom{u-\eta}{v+\eta}, \binom{u+\eta}{v+\eta}, \binom{u+\eta}{v}\right) \subseteq F$.

Use

$$U := (u-\delta, u+\delta),$$
$$V := (v, v+\delta),$$
$$W := (u+v, u+v+\delta).$$

Cauchy–Pexider in the limit. Case 3

Kö.–Zhou, Equivariant Perturbation VI, arXiv:1605.03975



Case 3a (sharp-angle corner): Then $C_\eta = \operatorname{conv}\left(\binom{u}{v}, \binom{u-\eta}{v+\eta}, \binom{u}{v+\eta}\right)$ is contained in F.
Use $U := (u-\delta, u)$, $V := (v, v+\delta)$, and $W := (u+v, u+v+\delta)$.

Case 3b (right-angle corner, second quadrant): The sharp-angle corner of Case 3a appears as a subcone.
Use U and V as in Case 3a and

$$W := (u+v-\delta/2, u+v+\delta/2).$$

Cauchy–Pexider in the limit. Application. Open questions

Conjecture

Theorem is false for arbitrary convex polygons $F \subset \mathbb{R}^2$*.*

Open Question

Generalize the theorem to faces $F = F(I, J, K) \in \Delta\mathcal{P}$, $F \subset \mathbb{R}^k \times \mathbb{R}^k$, where $\mathcal{P}$ is the polyhedral complex of an arbitrary piecewise linear function $\pi\colon \mathbb{R}^k \to \mathbb{R}$.

Main theorem on perturbation spaces / extremality test

Basu–Hildebrand–**Kö.**, Equivariant Perturbation I, MOR 2012; Basu–Hildebrand–**Kö.**, Light on the infinite group relaxation, 4OR 2014; Zhou, dissertation 2017; Hildebrand–**Kö.**–Zhou, 2018+

Theorem (Basu–Hildebrand–**Kö.**, 2012, 2014; Zhou, 2017, Hildebrand–**Kö.**–Zhou, 2018+)

Let π be minimal function for $R_f(\mathbb{R}/\mathbb{Z})$ that is

- *a (possibly discontinuous) piecewise linear function with rational breakpoints in $\frac{1}{q}\mathbb{Z}$*
- *or a piecewise linear function with arbitrary breakpoints that is at least one-sided continuous at 0 and for which the "move completion procedure" terminates.*

Then there is a computable direct sum decomposition of the space $\tilde{\Pi}^{\pi}$ into:

- *a finite-dimensional space of perturbations that are (possibly discontinuous) linear interpolations of values (and limits) at breakpoints ,*
- *a finite number of infinite-dimensional spaces of Lipschitz perturbations $\tilde{\pi}$, equivariant under a computable semigroup action, zero on breakpoints.*

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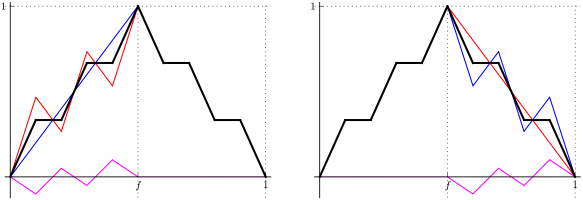
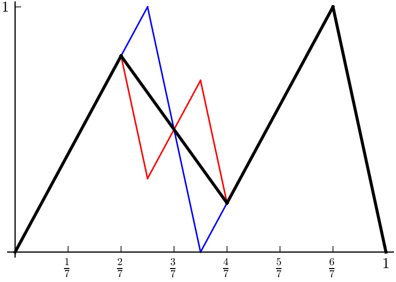
Effective perturbations of minimal functions

Finite-dimensional perturbation subspace: Interpolations of perturbations on ‘breakpoints’



not_extreme_1

Infinite-dimensional perturbation subspace: Equivariant perturbations, 0 on ‘breakpoints’



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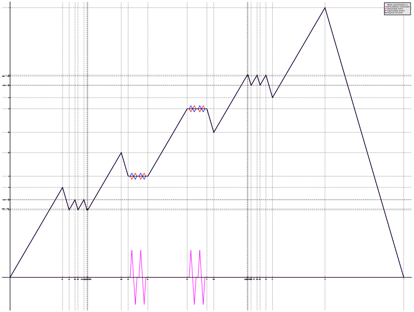
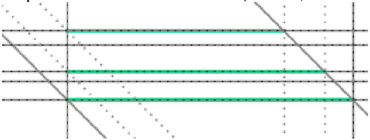


bhk_irrational

Cauchy–Pexider on irrational edges – the strip lemma

Basu–Hildebrand–Köppe, Equivariant Perturbation I, MOR 2012; Hildebrand, Hong, **Kö.**, La Haye, Louveaux, unpublished, 2014; Zhou, dissertation 2017; Hildebrand–**Kö.**–Zhou, Equivariant Perturbation VII, 2018+

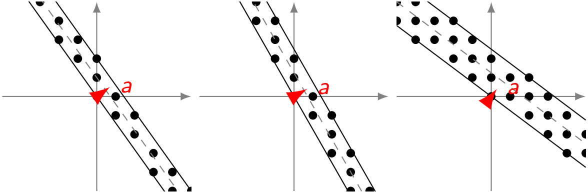
F = union of $n+1$ irrational translates of an edge



Equivalent: Characterize connectivity of lattice points in the strip

$$S_a = \{\, x \in \mathbb{R}^n \mid 0 \leq a \cdot x \leq 1 \,\}$$

by standard unit moves $\{\pm e^1, \pm e^2, \ldots, \pm e^n\}$.



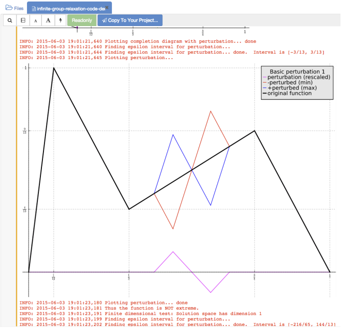
Theorem

Let $n = 2$ and $a_1, a_2 \in (0,1)$ be rationally independent. Then $S_a \cap \mathbb{Z}^2$ is connected if and only if $\|a\|_1 \leq 1$.

SageMath (Python) package cutgeneratingfunctionology

https://github.com/mkoeppe/cutgeneratingfunctionology

Authors: Chun Yu Hong (2013), **Kö.** (2013–), Yuan Zhou (2014–), Jiawei Wang (2016–), contributing undergraduate programmers



Models:

- 1-row Gomory–Johnson model
- Gomory's finite (cyclic) group problem
- superadditive lifting functions
- dual-feasible functions
- multi-row code under development

Functionality:

- electronic compendium of functions
- automatic extremality test (Basu–Hildebrand–**Kö.**, Math. Oper. Res. 2014, Hong–**Kö.**–Zhou, ICMS 2016, Zhou 2017, Hildebrand–**Kö.**–Zhou, 2018+)
- computer-based search for extreme functions (**Kö.**–Zhou, MPC 2016)