Integer Programming in Strongly Polynomial Oracle Time

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joint work with Martin Koutecký and Asaf Levin, coming ICALP

Integer Programming

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The bit size of the data is denoted by [A,w,b,l,u]

Outline

The Strongly Polynomial Oracle Algorithm

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Effective Realization of Oracles

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Programs with Block Structure

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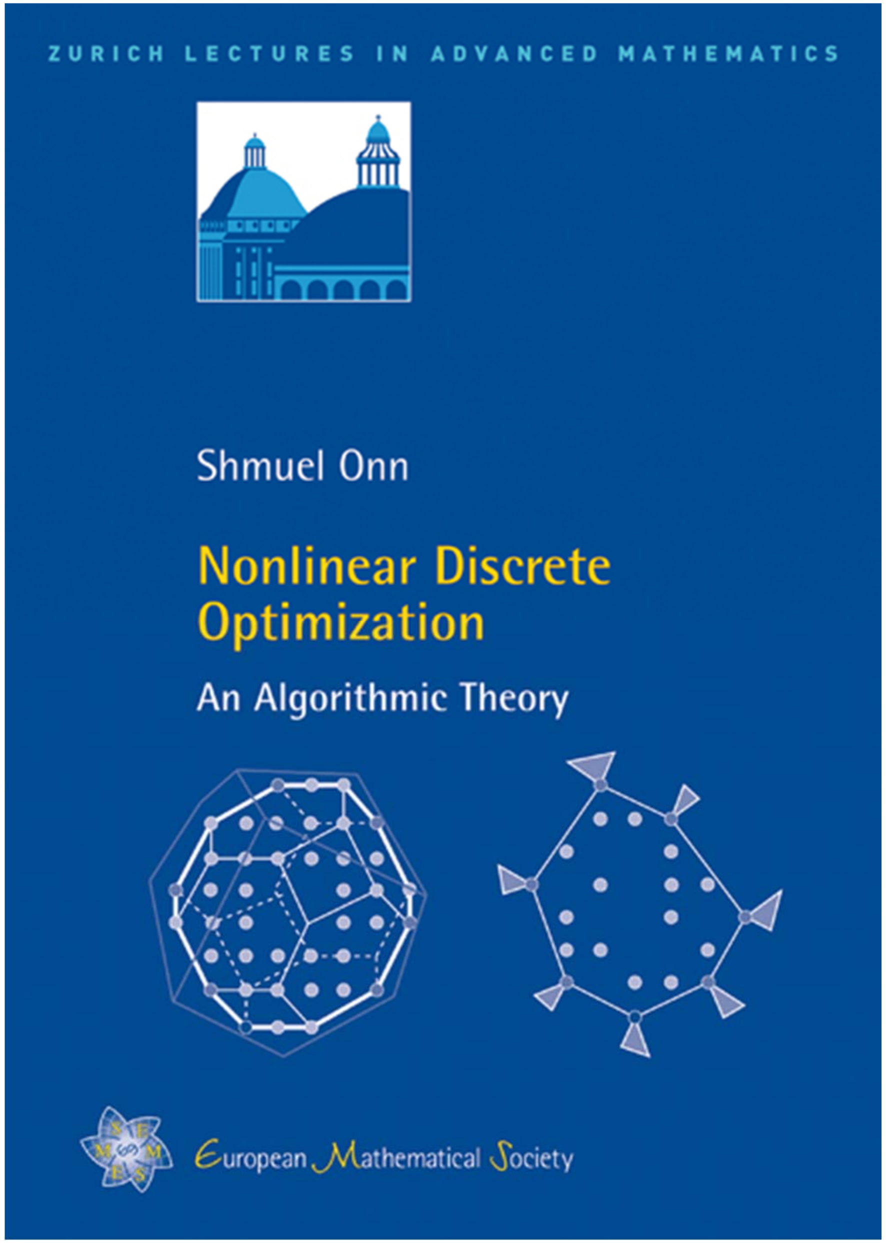
Programs with Bounded Tree-Depth

The Strongly Polynomial Oracle Algorithm



Background in my Book:

Theory of Graver bases
for integer programming

(and more)

Available electronically
from my homepage

(with kind permission of EMS)

Graver Bases

The Graver basis of an integer matrix A is the finite set $G(A)$ of conformal-minimal nonzero integer vectors x satisfying $Ax = 0$.

x is conformal to y if $x_i y_i \geq 0$ (same orthant) and $|x_i| \leq |y_i|$ for all i

Graver-Best Oracle

Let x be a feasible point of the integer program

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A Graver-best oracle for an integer matrix A is one that queried on w,b,l,u and feasible x returns a Graver-best step h for x.

Weakly Polynomial Solution

IP: $\max\{\, wx \;:\; Ax = b,\;\; l \le x \le u,\;\; x \in Z^n \,\}$

Lemma: IP is solvable with Graver-best oracle for A in time $O(n\,[A,w,b,l,u])$

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(Hemmecke, Onn, Weismantel and my book Theorem 3.12)

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The algorithm iteratively adds Graver-best steps till optimum is reached

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Theorem 1: IP is solvable with Graver-best oracle for A in time poly($n[A]$)

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(our postdoc Koutecký, Levin, Onn, coming ICALP)

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Talk Wednesday morning by Martin Koutecký:
breakthroughs in computational social choice

Shmuel Onn

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Proof:

1. Solve the LP-relaxation and get real optimal y^* (Tardos)
2. By proximity lemma (HS,HKW) can search for integer optimal x^* close to y^* and reduce data to $\underline{b},\underline{l},\underline{u}$ with $[\underline{b},\underline{l},\underline{u}]$ polynomial in $[A]$
3. Reduce data to $\underline{w}$ with $[\underline{w}]$ polynomial in $[A]$ (Frank-Tardos)
4. Use lemma to solve the reduced program in $O(n[A,\underline{w},\underline{b},\underline{l},\underline{u}])=\text{poly}(n[A])$

Effective Realization

of

Graver-Best Oracles

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Primal Lemma (KLO): If the primal graph of A has bounded tree-width and

$$\max\{\, |g|_\infty \;:\; g \in G(A) \,\}$$

is bounded then get an effective Graver-best oracle for A by DP.

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Dual Lemma (KLO): If the dual graph of A has bounded tree-width and

$$\max \{ \|g\|_1 \ : \ g \in G(A) \}$$

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Programs with Block Structure

N-Fold Integer Programming

The n-fold product of $r \times t$ block A_1 and $s \times t$ block A_2 is

$$A = \underbrace{\begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix}}_{n} .$$

Parameterization of N-Fold IP

Consider n-fold integer programming over $r \times t$ block A_1 and $s \times t$ block A_2

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The input is n and the bit size [w,b,l,u]

Complexity of N-Fold Integer Programming

-- Polynomial time $n^{f(r,s,t,a)}[w,b,l,u]$ (De Loera, Hemmecke, Onn, Weismantel)

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-- Parameter tractable $f(r,s,t,a)n^3[w,b,l,u]$ (Hemmecke, Onn, Romanchuk)

Led to several recent breakthroughs in the theory of parameterized complexity

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-- Strongly polynomial $n^{f(r,s,t,a)}$ (De Loera, Hemmecke, Lee)

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Theorem 2 (Koutecký, Levin, Onn) :

Parameter tractable and strongly polynomial $f(r,s,a)\,\mathrm{poly}(n,t)$

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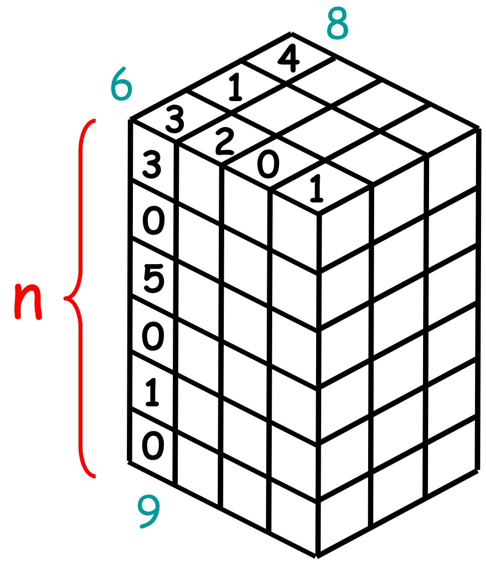
Proof: Use dual lemma to get parameter tractable Graver-best oracle and use Theorem 1 to get parameter tractable strongly polynomial time.

Example - Multiway Tables

Optimization over $l \times m \times n$ tables with given line sums:



Corollary: Parameter tractable and strongly polynomial $f(l,m)\,\mathrm{poly}(n)$

Tree-Fold Integer Programming

(Chen, Marx, SODA 2018): Tree-fold integer programs have a matrix with several blocks in tree structure, parameterized by s_i,t,a=max A

$$
A = \begin{bmatrix}
A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 \\
A_2 & A_2 & A_2 & A_2 & A_2 & A_2 & A_2 & A_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_2 & A_2 & A_2 & A_2 \\
A_3 & A_3 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_3 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_3 & A_3 & A_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_3 & A_3 & A_3 & A_3 \\
A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4
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Improves on Chen, Marx, SODA 2018 - strongly polynomial and t variable

Multistage Stochastic Integer Programming

These programs have a matrix the transpose A^T of a tree-fold matrix with several blocks in tree structure, parameterized by s_i,t,a=max A_i

$$A = \begin{bmatrix}
A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 \\
A_2 & A_2 & A_2 & A_2 & A_2 & A_2 & A_2 & A_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_2 & A_2 & A_2 & A_2 \\
A_3 & A_3 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_3 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
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Theorem 5 (Koutecký, Levin, Onn) :
Integer programs with matrix A parameterized by a=max A and by the tree-depth d of the dual graph of A can be solved in parameter tractable and strongly polynomial time $f(a,d)$ poly(n)

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Proof: Embed such programs in tree-fold programs and use Theorem 3

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Proof: Embed them in multistage stochastic programs and use Theorem 4

Reference:

A Parameterized Strongly Polynomial Algorithm
for Block Structured Integer Programs

Martin Koutecký, Asaf Levin, Shmuel Onn, coming ICALP

also available at http://ie.technion.ac.il/~onn