# Chapter 7
# Central Path Curvature and Iteration-Complexity for Redundant Klee–Minty Cubes

Antoine Deza, Tamás Terlaky, and Yuriy Zinchenko

**Summary.** We consider a family of linear optimization problems over the $n$-dimensional Klee–Minty cube and show that the central path may visit all of its vertices in the same order as simplex methods do. This is achieved by carefully adding an exponential number of redundant constraints that forces the central path to take at least $2^n - 2$ sharp turns. This fact suggests that any feasible path-following interior-point method will take at least $O(2^n)$ iterations to solve this problem, whereas in practice typically only a few iterations (e.g., 50) suffices to obtain a high-quality solution. Thus, the construction potentially exhibits the worst-case iteration-complexity known to date which almost matches the theoretical iteration-complexity bound for this type of methods. In addition, this construction gives a counterexample to a conjecture that the total central path curvature is $O(n)$.

**Key words:** Linear programming, central path, interior-point methods, total curvature

## 7.1 Introduction

Consider the following linear programming problem: $\min c^T x$ such that $Ax \geq b$ where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c, x \in \mathbb{R}^n$.

In theory, the so-called feasible path-following interior-point methods exhibit polynomial iteration-complexity: starting at a point on the central path they take at most $O(\sqrt{m} \ln \nu)$ iterations to attain a $\nu$-relative decrease in the duality gap. Moreover, if $L$ is the bit-length of the input data, it takes

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at most $O(\sqrt{m}L)$ iterations to solve the problem exactly; see, for instance, [11]. However, in practice typically only a few iterations, usually less than 50, suffices to obtain a high-quality solution. This remarkable difference stands behind the tremendous success of interior-point methods in applications.

Let $\psi : [\alpha, \beta] \to \mathbb{R}^n$ be a $C^2$ map with nonzero derivative $\forall t \in [\alpha, \beta]$. Denote its arc length by

$$l(t) := \int_\alpha^t \|\dot{\psi}(\tau)\| d\tau,$$

its parametrization by the arc length by $\psi_{\text{arc}}(l) : [0, l(\beta)] \to \mathbb{R}^n$, and its curvature at the point $l$,

$$\kappa(l) := \frac{d}{dl}\ddot{\psi}_{\text{arc}}(l).$$

The total curvature $K$ is defined as

$$K := \int_0^{l(\beta)} \|\kappa(l)\| dl.$$

Intuitively, the total curvature is a measure of how far off a certain curve is from being a straight line. Thus, it has been hypothesized that the total curvature of the central path is positively correlated with the number of iterations that any Newton-like path following method will take to traverse this curve, in particular, the number of iterations for feasible path-following interior-point methods, for example, long-step or predictor-corrector.

The worst-case behavior for path-following interior-point methods has already been under investigation, for example, Todd and Ye [13] gave a lower iteration-complexity bound of order $\sqrt[3]{m}$ necessary to guarantee a fixed decrease in the central path parameter and consequently in the duality gap. At the same time, different notions for the curvature of the central path have been examined. The relationship between the number of approximately straight segments of the central path introduced by Vavasis and Ye [14] and a certain curvature measure of the central path introduced by Sonnevend, Stoer, and Zhao [12] and further analyzed in [15], was further studied by Monteiro and Tsuchiya in [9]. Dedieu, Malajovich, and Shub [1] investigated a properly averaged total curvature of the central path. Nesterov and Todd [10] studied the Riemannian curvature of the central path in particular relevant to the so-called short-step methods. We follow a constructive approach originated in [4, 5] which is driven by the geometrical properties of the central path to address these questions.

We consider a family of linear optimization problems over the $n$-dimensional Klee–Minty cube and show that the central path may visit all of its vertices in the same order as simplex methods do. This is achieved by carefully adding an exponential number of redundant constraints that forces the central path to take at least $2^n - 2$ sharp turns. We derive explicit formulae for the number of the redundant constraints needed. In particular, we give a bound of

$O(n2^{3n})$ on the number of redundant constraints when the distances to those are chosen uniformly. When these distances are chosen to decay geometrically, we give a slightly tighter bound of the same order $n^3 2^{2n}$ as in [5].

The behavior of the central path suggests that any feasible path-following interior-point method will take at least order $2^n$ iterations to solve this problem. Thus, the construction potentially exhibits the worst-case iteration-complexity known to date which almost matches the theoretical iteration-complexity bound for this type of methods. However, state-of-the art linear optimization solvers that include preprocessing of the problem as described in [6, 7] are expected to recognize and remove the redundant constraints in no more than two passes. This underlines the importance of the implementation of efficient preprocessing algorithms.

We show that the total curvature of the central path for the construction is at least exponential in $n$ and, therefore, provides a counterexample to a conjecture of Dedieu and Shub [2] that it can be bounded by $O(n)$. Also, the construction may serve as an example where one can relate the total curvature and the number of iterations almost exactly.

The chapter is organized as follows. In Section 7.2 we introduce a family of linear programming problems studied along with a set of sufficient conditions that ensure the desired behavior for the central path and give a lower bound on the total curvature of the central path, in Section 7.3 we outline the approach to determine the number of the redundant constraints required, and Sections 7.4 and 7.5 contain a detailed analysis of the two distinct models for the distances to the redundant constraints. We give a brief conclusion in Section 7.6.

## 7.2 Sufficient Conditions for Bending the Central Path and the Total Curvature

Let $x \in \mathbb{R}^n$. Consider the following optimization problem.

$$\left\{\begin{array}{rclll} & \min & x_n & & \\ 0 & \le & x_1 & \le 1 & \\ \varepsilon x_{k-1} & \le & x_k & \le 1 - \varepsilon x_{k-1} & k = 2, \ldots, n \\ 0 & \le & d_1 + x_1 & & \text{repeated } h_1 \text{ times} \\ \varepsilon x_1 & \le & d_2 + x_2 & & \text{repeated } h_2 \text{ times} \\ & & \vdots & & \\ \varepsilon x_{n-1} & \le & d_n + x_n & & \text{repeated } h_n \text{ times.} \end{array}\right.$$

The feasible region is the Klee–Minty $n$-cube and is denoted by $\mathcal{C} \subset \mathbb{R}^n$. Denote $d := (d_1, \ldots, d_n) \in \mathbb{R}^n_+$ – the vector containing the distances to the redundant constraints from $\mathcal{C}$, $h := (h_1, \ldots, h_n) \in \mathbb{N}^n$ – the vector containing the number of the redundant constraints.

By analogy with the unit cube $[0,1]^n$, we denote the vertices of $\mathcal{C}$ as follows. For $S \subset \{1,\ldots,n\}$, a vertex $v^S$ of $\mathcal{C}$ satisfies

$$v_1^S = \begin{cases} 1 & \text{if } 1 \in S \\ 0 & \text{otherwise} \end{cases}$$
$$v_k^S = \begin{cases} 1 - \varepsilon v_{k-1}^S & \text{if } k \in S \\ \varepsilon v_{k-1}^S & \text{otherwise} \end{cases} \qquad k = 2,\ldots,n.$$

Define $\delta$-neighborhood $\mathcal{N}_\delta(v^S)$ of a vertex $v^S$, with the convention $x_0 = 0$, by

$$\mathcal{N}_\delta(v^S) := \left\{ x \in \mathcal{C} : \begin{cases} 1 - x_k - \varepsilon x_{k-1} \le \varepsilon^{k-1}\delta & \text{if } k \in S \\ x_k - \varepsilon x_{k-1} \le \varepsilon^{k-1}\delta & \text{otherwise} \end{cases} \quad k = 1,\ldots,n \right\}.$$

**Remark 7.1.** Observe that $\forall S \subseteq \{1,\ldots,n\}$ for $\mathcal{N}_\delta(v^S)$ to be pairwise-disjoint it suffices $\varepsilon + \delta < 1/2$: given $\varepsilon, \delta > 0$, the shortest amongst all $n$ coordinates' distance between the neighborhoods, equal to $(1 - 2\varepsilon - 2\varepsilon\delta)$, is attained along the second coordinate and must be positive, which is readily implied.

For brevity of the notation we introduce slack variables corresponding to the constraints in the problem above as follows:

$$\begin{aligned} s_1 &= x_1 \\ s_k &= x_k - \varepsilon x_{k-1} && k = 2,\ldots,n \\ \bar{s}_1 &= 1 - x_1 \\ \bar{s}_k &= 1 - \varepsilon x_{k-1} - x_k && k = 2,\ldots,n \\ \tilde{s}_1 &= d_1 + x_1 \\ \tilde{s}_k &= d_k + (x_n - \varepsilon x_{n-1}) && k = 2,\ldots,n. \end{aligned}$$

Recall that the analytic center $\chi$ corresponds to the unique maximizer

$$\arg\max_x \sum_{i=1}^n (\ln s_i + \ln \bar{s}_i + h_i \ln \tilde{s}_i).$$

Also, recall that the primal central path $\mathcal{P}$ can be characterized as the closure of the set of maximizers

$$\left\{ x \in \mathbb{R}^n : x = \arg\max_{x : x_n = \alpha} \sum_{i=1}^n (\ln s_i + \ln \bar{s}_i + h_i \ln \tilde{s}_i), \text{ for some } \alpha \in (0, \chi_n) \right\}.$$

Therefore, setting to 0 the derivatives of $\sum_{i=1}^n (\ln s_i + \ln \bar{s}_i + h_i \ln \tilde{s}_i)$ with respect to $x_n$,

$$\frac{1}{s_n} - \frac{1}{\bar{s}_n} + \frac{h_n}{\tilde{s}_n} = 0, \tag{7.1}$$

and with respect to $x_k$,

$$\frac{1}{s_k} - \frac{\varepsilon}{s_{k+1}} - \frac{1}{\bar{s}_k} - \frac{\varepsilon}{\bar{s}_{k+1}} + \frac{h_k}{\tilde{s}_k} - \frac{\varepsilon h_{k+1}}{\tilde{s}_{k+1}} = 0, \qquad k = 1, \ldots, n-1, \quad (7.2)$$

combined give us necessary and sufficient conditions for $x = \chi$. Furthermore, (7.2) combined with $x_n = \alpha \in (0, \chi_n)$ gives us necessary and sufficient conditions for $x \in \mathcal{P} \setminus (\{\mathbf{0}\} \cup \{\chi\})$ where $\mathbf{0} \in \mathbb{R}^n$ denotes the origin.

Given $\varepsilon, \delta > 0$, the sufficient conditions for $h = h(d, \varepsilon, \delta)$ to guarantee that the central path $\mathcal{P}$ visits the (disjoint) $\delta$-neighborhoods of each vertex of $\mathcal{C}$ may be summarized in the following proposition. We write $\mathbf{1}$ for the vector of all ones in $\mathbb{R}^n$.

**Proposition 7.1.** *Fix $\varepsilon, \delta > 0$. Denote for $k = 2, \ldots, n$*

$$\mathcal{I}_\delta^k := \{x \in \mathcal{C} : \bar{s}_k \geq \varepsilon^{k-1}\delta, s_k \geq \varepsilon^{k-1}\delta\}$$

*and*

$$\mathcal{B}_\delta^k := \{x \in \mathcal{C} : \bar{s}_{k-1} \leq \varepsilon^{k-2}\delta, s_{k-2} \leq \varepsilon^{k-3}\delta, \ldots, s_1 \leq \delta\}.$$

*If $h = h(d, \varepsilon, \delta) \in \mathbb{N}^n$ satisfies*

$$Ah \geq \frac{3}{\delta}\mathbf{1} \quad (7.3)$$

*and*

$$\frac{h_k}{d_k + 1}\varepsilon^{k-1} \geq \frac{h_{k+1}}{d_{k+1}}\varepsilon^k + \frac{3}{\delta}, \qquad k = 1, \ldots, n-1, \quad (7.4)$$

*where*

$$A = \begin{pmatrix} \frac{1}{d_1+1} & \frac{-\varepsilon}{d_2} & 0 & 0 & \cdots & 0 & 0 \\ \frac{-1}{d_1} & \frac{2\varepsilon}{d_2+1} & \frac{-\varepsilon^2}{d_3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \frac{-1}{d_1} & 0 & \cdots & \frac{2\varepsilon^{k-1}}{d_k+1} & \frac{-\varepsilon^k}{d_{k+1}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{-1}{d_1} & 0 & 0 & 0 & \cdots & \frac{2\varepsilon^{n-2}}{d_{n-1}+1} & \frac{-\varepsilon^{n-1}}{d_n} \\ \frac{-1}{d_1} & 0 & 0 & 0 & \cdots & 0 & \frac{2\varepsilon^{n-1}}{d_n+1} \end{pmatrix}$$

*then*

$$\mathcal{I}_\delta^k \cap \mathcal{P} \subset \mathcal{B}_\delta^k.$$

*Proof.* Fix $k \geq 2$ and let $x \in \mathcal{I}_\delta^k \cap \mathcal{P}$.

Let $j \leq k-2$. Summing up all of the $i$th equations of (7.2) over $i = j, \ldots, (k-2)$, each multiplied by $\varepsilon^{i-1}$, and then subtracting the $(k-1)$st equation multiplied by $\varepsilon^{k-2}$, we have

$$-\frac{2h_{k-1}\varepsilon^{k-2}}{\tilde{s}_{k-1}} + \frac{h_j\varepsilon^{j-1}}{\tilde{s}_j} + \frac{h_k\varepsilon^{k-1}}{\tilde{s}_k} + \frac{\varepsilon^{j-1}}{s_j} + \frac{\varepsilon^{k-1}}{s_k} + \frac{\varepsilon^{k-1}}{\bar{s}_k}$$

$$= \frac{2\varepsilon^{k-2}}{s_{k-1}} + \frac{\varepsilon^{j-1}}{\bar{s}_j} + 2\sum_{i=j}^{k-3}\frac{\varepsilon^i}{\bar{s}_{i+1}}.$$

Because $\tilde{s}_{k-1} < d_{k-1}+1$, $\tilde{s}_k > d_k$, $\tilde{s}_j > d_j$, and $s_k \geq \varepsilon^{k-1}\delta$, $\bar{s}_k \geq \varepsilon^{k-1}\delta$ as $x \in \mathcal{I}_\delta^k$, from the above we get

$$\frac{2h_{k-1}\varepsilon^{k-2}}{d_{k-1}+1} - \frac{h_j\varepsilon^{j-1}}{d_j} - \frac{h_k\varepsilon^{k-1}}{d_k} \leq \frac{\varepsilon^{j-1}}{s_j} + \frac{2}{\delta}.$$

From (7.4) it follows that $(h_1/d_1) \geq \left(h_j\varepsilon^{j-1}/d_j\right)$, thus we can write

$$-\frac{h_1}{d_1} + \frac{2h_{k-1}\varepsilon^{k-2}}{d_{k-1}+1} - \frac{h_k\varepsilon^{k-1}}{d_k} \leq \frac{\varepsilon^{j-1}}{s_j} + \frac{2}{\delta};$$

that is, as $(3/\delta) \leq -(h_1/d_1) + \left(\left(2h_{k-1}\varepsilon^{k-2}\right)/(d_{k-1}+1)\right) - \left(h_k\varepsilon^{k-1}/d_k\right)$ by (7.3), we have

$$s_j^n \leq \varepsilon^{j-1}\delta, \qquad \forall j \leq k-2.$$

In turn, the $(k-1)$st equation of (7.2),

$$\frac{h_{k-1}\varepsilon^{k-2}}{\tilde{s}_{k-1}} - \frac{h_k\varepsilon^{k-1}}{\tilde{s}_k} = \frac{\varepsilon^{k-2}}{\bar{s}_{k-1}} + \frac{\varepsilon^{k-1}}{s_k} + \frac{\varepsilon^{k-1}}{\bar{s}_k} - \frac{\varepsilon^{k-2}}{s_{k-1}}$$

implies

$$\frac{h_{k-1}\varepsilon^{k-2}}{d_{k-1}+1} - \frac{h_k\varepsilon^{k-1}}{d_k} \leq \frac{\varepsilon^{k-2}}{\bar{s}_{k-1}} + \frac{2}{\delta}$$

and since $(3/\delta) \leq \left(\left(h_{k-1}\varepsilon^{k-2}\right)/(d_{k-1}+1)\right) - \left(h_k\varepsilon^{k-1}/d_k\right)$ by (7.4), we have $\bar{s}_{k-1} \leq \varepsilon^{k-2}\delta$. □

**Proposition 7.2.** *Fix* $\varepsilon, \delta > 0$. *If* $h \in \mathbb{N}^n$ *satisfies* (7.3) *and* (7.4), *then* $\chi \in \mathcal{N}_\delta(v^{\{n\}})$.

*Proof.* Summing up all of the $i$th equations of (7.2) over $i = k, \ldots, (n-1)$, each multiplied by $\varepsilon^{i-1}$, and then subtracting (7.1) multiplied by $\varepsilon^{n-1}$, we have

$$\frac{\varepsilon^{k-1}}{s_k} - \frac{\varepsilon^{k-1}}{\bar{s}_k} + \frac{h_k\varepsilon^{k-1}}{\tilde{s}_k} - \frac{2\varepsilon^{n-1}}{s_n} - 2\sum_{i=k}^{n-2}\frac{\varepsilon^i}{\bar{s}_{i+1}} - \frac{2h_n\varepsilon^{n-1}}{\tilde{s}_n} = 0$$

implying

$$\frac{2h_n\varepsilon^{n-1}}{\tilde{s}_n} - \frac{h_k\varepsilon^{k-1}}{\tilde{s}_k} \leq \frac{\varepsilon^{k-1}}{s_k}.$$

Because $\tilde{s}_n \leq d_n + 1$, $\tilde{s}_k \geq d_k$ and by (7.4), $(h_1/d_1) \geq \left(h_k \varepsilon^{k-1}/d_k\right)$, from the above we get

$$\frac{2h_n\varepsilon^{n-1}}{d_n+1} - \frac{h_1}{d_1} \leq \frac{\varepsilon^{k-1}}{s_k},$$

combined with $\left(\left(2h_n\varepsilon^{n-1}\right)/\left(d_n+1\right)\right) - (h_1/d_1) \geq (3/\delta)$ (from (7.3)) this leads to

$$s_k \leq \frac{\varepsilon^{k-1}}{3}\delta \qquad k = 1, \ldots, n-1.$$

In turn, (7.1) implies $\left(h_n\varepsilon^{n-1}/\tilde{s}_n\right) \leq \left(\varepsilon^{n-1}/\bar{s}_n\right)$. And because $\tilde{s}_n < d_n + 1$ and by (7.4), $\left(h_n\varepsilon^{n-1}/\left(d_n+1\right)\right) \geq (3/\delta)$, we have $(3/\delta) \leq \left(\varepsilon^{n-1}/\bar{s}_n\right)$; that is, $\bar{s}_n \leq \left(\varepsilon^{n-1}/3\right)\delta$. □

**Corollary 7.1.** *Fix* $\varepsilon, \delta > 0$ *such that* $\varepsilon + \delta < 1/2$. *If* $h \in \mathbb{N}^n$ *satisfies* (7.3) *and* (7.4), *then the central path* $\mathcal{P}$ *intersects the disjoint* $\delta$*-neighborhoods of all the vertices of* $\mathcal{C}$. *Moreover,* $\mathcal{P}$ *is confined to a polyhedral tube defined by* $\mathcal{T}_\delta := \bigcup_{k=1}^{n}\left(\left(\bigcap_{j=k+1}^{n}(\mathcal{I}_\delta^j)^c\right) \cap \mathcal{B}_\delta^k\right)$ *with the convention* $\mathcal{B}_\delta^1 = \mathcal{C}$.

**Remark 7.2.** Observe that $\mathcal{T}_0$ is the sequence of connected edges of $\mathcal{C}$ starting from $v^{\{n\}}$ and terminating at $v^{\varnothing}$, and is precisely the path followed by the simplex method on the original Klee–Minty problem as it pivots along the edges of $\mathcal{C}$.

For simplicity of the notation we write $\mathcal{T}$ instead of $\mathcal{T}_\delta$ when the choice of $\delta$ is clear. For a fixed $\delta$, we define a turn of $\mathcal{T}$ adjacent to a vertex $v^S$, or corresponding to $\mathcal{N}_\delta(v^S)$ if the $\delta$-neighborhoods are disjoint because in the latter case $\mathcal{N}_\delta(v^S)$ determines $v^S$ uniquely, to be the angle between the two edges of $\mathcal{C}$ that belong to $\mathcal{T}_0$ and connect at this vertex.

Intuitively, if a smooth curve is confined to a narrow tube that makes a sharp turn, then the curve itself must at least make a similar turn and thus have a total curvature bounded away from zero. It might be worthwhile to substantiate this intuition with a proposition.

**Proposition 7.3.** *Let* $\Psi : [0,T] \to \mathbb{R}^2$ *be* $C^2$, *parameterized by its arc length* $t$, *such that* $\Psi([0,T]) \subset \{(x,y) : 0 \leq x \leq a+b, 0 \leq y \leq b\} \cup \{(x,y) : a \leq x \leq a+b, -a \leq y \leq b\}$ *and* $\Psi(0) \in \{0\} \times [0,b]$, $\Psi(T) \in [a, a+b] \times \{-a\}$. *Then the total curvature* $K$ *of* $\Psi$ *satisfies* $K \geq \arcsin\left(1 - 2b^2/a^2\right)$.

*Proof.* By the mean-value theorem, for any $\tau$ such that $\Psi_1(\tau) = a$ we have $\tau \geq a$; recall that $\|\dot{\Psi}\| = 1$. Thus, by the same theorem, $\exists t_1$ such that $|\Psi_2(t_1)| \leq b/a$. Similarly, $\exists t_2$ such that $|\Psi_1(t_2)| \leq b/a$. Now map the values of the derivative of $\Psi$ at $t_1$ and $t_2$ onto a sphere and recall that the total curvature $K$ between these two points corresponds to the length of a connecting curve on the sphere, thus bounded below by the length of the geodesic (which in this case is the same as the angular distance).

A simple calculation completes the proof. □

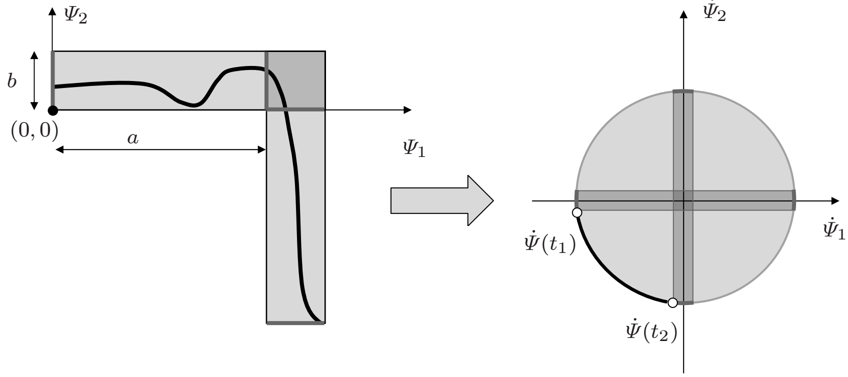


**Fig. 7.1** Total curvature and geodesics.

**Remark 7.3.** Note that if $b/a \to 0$, then the corresponding lower bound on the total curvature $K$ approaches $\pi/2$.

Next we construct a simple bound on the total curvature of $\mathcal{P}$ by picking suitable $d, \varepsilon$, and finally $\delta$ small enough, together with $h$, that results in a "narrow" polyhedral tube $\mathcal{T}$.

For $X \subseteq \mathbb{R}^n$ denote its orthogonal projection onto a linear subspace spanned by a subset $S \subseteq \{1, \dots, n\}$ of coordinates, with coordinates corresponding to $S^c$ suppressed, by $X_S$. For $x, z \in \mathbb{R}^n$ we denote $(x, z)$ the straight line segment connecting the point $x$ and $z$.

**Corollary 7.2.** *Fix $n \geq 2$. If $d_i = (n-1)2^{n-i+2}$, $i = 1, \dots, n$,*

$$\varepsilon = \frac{n-1}{2n},$$

$$\delta = \frac{1}{32n^2}\left(\frac{4}{5}\right)^{n-2},$$

*and $h$ satisfies*

$$h = \left\lfloor \left(1 + \frac{\delta}{3} \max_i \sum_{j=1}^{n} |a_{ij}|\right) \frac{3}{\delta} \widetilde{h} \right\rfloor,$$

*where $A\widetilde{h} = \mathbf{1}$, then the total curvature of the central path $\mathcal{P}$ satisfies*

$$K \geq \frac{1}{2n}\left(\frac{8}{5}\right)^{n-2}.$$

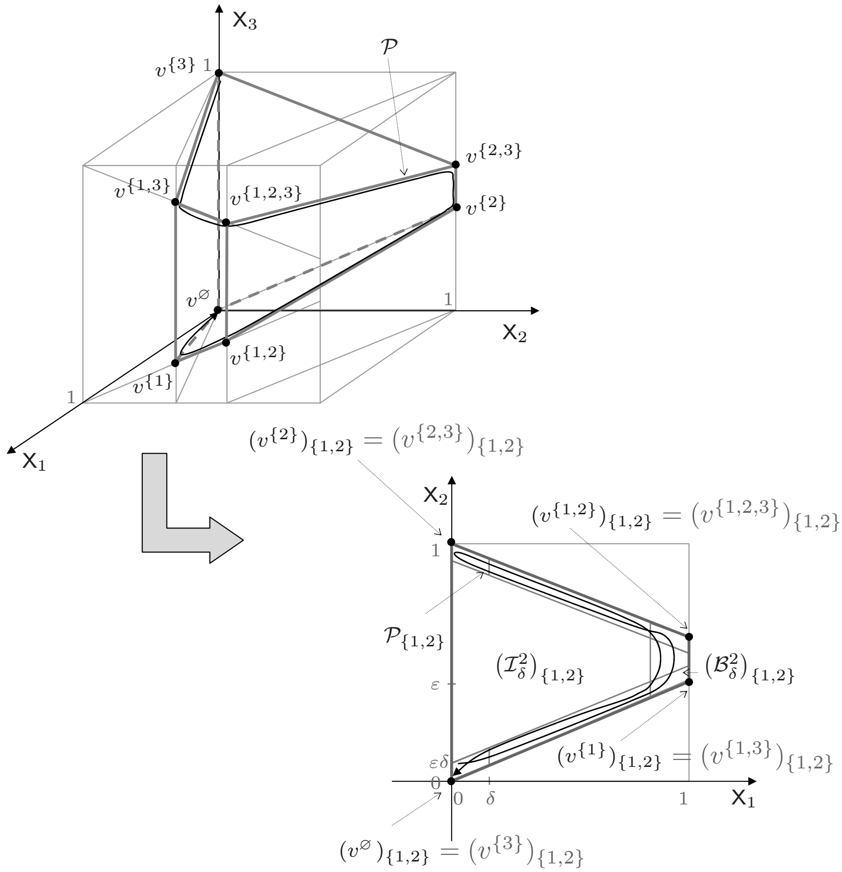


**Fig. 7.2** Planar projection of the central path for $n = 3$.

*Proof.* That $\varepsilon, \delta, h = h(d, \varepsilon, \delta)$ above satisfies the conditions of Corollary 7.1 and thus $\mathcal{P}$ is confined to the polyhedral tube $\mathcal{T}$ is established in Section 7.5.

Instead of analyzing $\mathcal{P} \in \mathbb{R}^n$ directly we derive the lower bound on the total curvature of $\mathcal{P}$ based on its planar projection $\mathcal{P}_{\{1,2\}}$.

From $\mathcal{I}_\delta^k \cap \mathcal{P} \subset \mathcal{B}_\delta^k, \quad k = 2, \ldots, n$, it follows that $\mathcal{P}_{\{1,2\}}$ will traverse the two-dimensional Klee–Minty cube $\mathcal{C}_{\{1,2\}}$ at least $2^{n-2}$ times, every time originating in either $\mathcal{N}_\delta(v^{\varnothing})_{\{1,2\}}$ or $\mathcal{N}_\delta(v^{\{2\}})_{\{1,2\}}$ and terminating in the other neighborhood, while confined to the polyhedral tube $\mathcal{T}_{\{1,2\}} = (\{s_2 \le \varepsilon\delta\} \cup \{\bar{s}_1 \le \delta\} \cup \{\bar{s}_2 \le \varepsilon\delta\}) \cap \mathcal{C}_{\{1,2\}}$. Thus, $\mathcal{P}_{\{1,2\}}$ will make at least $2^{n-1}$ "sharp turns", each corresponding to a turn in $\mathcal{N}_\delta(v^{\{1,2\}})_{\{1,2\}}$ or $\mathcal{N}_\delta(v^{\{1\}})_{\{1,2\}}$.

In order to understand how the turns of $\mathcal{P}_{\{1,2\}}$ contribute to the total curvature of $\mathcal{P}$ we need the following lemma.

**Lemma 7.1.** *Let $\widehat{u}, \widehat{v} \in \mathbb{R}^3$ and $u = (\widehat{u}_{\{1,2\}}, 0), v = (\widehat{v}_{\{1,2\}}, 0)$. If the angle*

$$\epsilon := \pi - \arccos \arg \min_{\substack{\widehat{w} \in \mathrm{span}\{\widehat{u}, \widehat{v}\}, \\ w \in \mathrm{span}\{u, v\}, \\ \|\widehat{w}\| = \|w\| = 1}} \widehat{w}^T w$$

*between the hyperplane spanned by $\widehat{u}, \widehat{v}$ and the hyperplane spanned by $u, v$ does not exceed* $\arcsin \varepsilon$, *then the angle $\widehat{\alpha}$ between $\widehat{u}$ and $\widehat{v}$ satisfies*

$$\frac{\cos\alpha - \varepsilon^2 \left(\frac{1-\cos\alpha}{2}\right)}{1 + \varepsilon^2 \left(\frac{1-\cos\alpha}{2}\right)} \leq \cos\widehat{\alpha} \leq \frac{\cos\alpha + \varepsilon^2 \left(\frac{1+\cos\alpha}{2}\right)}{1 + \varepsilon^2 \left(\frac{1+\cos\alpha}{2}\right)}$$

*where $\alpha$ is the angle between $u$ and $v$.*

*Proof.* Without loss of generality we may assume $\|u\| = \|v\| = 1$ with

$$u_1 = \sin\tfrac{\alpha}{2} = \sqrt{\tfrac{1-\cos\alpha}{2}},\ v_1 = -\sin\tfrac{\alpha}{2} = -\sqrt{\tfrac{1-\cos\alpha}{2}},$$
$$u_2 = \cos\tfrac{\alpha}{2} = \sqrt{\tfrac{1+\cos\alpha}{2}},\ v_2 = \cos\tfrac{\alpha}{2} = \sqrt{\tfrac{1+\cos\alpha}{2}},$$

and, assuming that the angle $\epsilon$ is precisely $\arcsin\varepsilon$, parameterize $\mathrm{span}\{\widehat{u}, \widehat{v}\}$ by $\mathrm{span}\{u, v\}$ and $z = (z_1, z_2, 0)$ such that $\|z\| = 1$, writing $x \in \mathrm{span}\{\widehat{u}, \widehat{v}\}$ as $x = (x_1, x_2, x_{\{1,2\}}^T z_{\{1,2\}} \varepsilon)$.

Introducing $\beta$ such that $z_1 = \cos\beta$ and $z_2 = \sin\beta$ we have

$$\widehat{u} = \left(\sqrt{\frac{1-\cos\alpha}{2}}, \sqrt{\frac{1+\cos\alpha}{2}}, \varepsilon\cos\left(\beta - \frac{\pi}{2} + \frac{\alpha}{2}\right)\right),$$

$$\widehat{v} = \left(-\sqrt{\frac{1-\cos\alpha}{2}}, \sqrt{\frac{1+\cos\alpha}{2}}, \varepsilon\cos\left(\beta - \frac{\pi}{2} + \frac{\alpha}{2}\right)\right),$$

and, therefore,

$$\cos\widehat{\alpha} = \frac{\widehat{u}^T\widehat{v}}{\|\widehat{u}\|\|\widehat{v}\|} = \frac{\cos\alpha + \varepsilon^2\cos\left(\beta - \frac{\pi}{2} + \frac{\alpha}{2}\right)\cos\left(\beta - \frac{\pi}{2} - \frac{\alpha}{2}\right)}{\sqrt{1 + \varepsilon^2\cos^2\left(\beta - \frac{\pi}{2} + \frac{\alpha}{2}\right)}\sqrt{1 + \varepsilon^2\cos^2\left(\beta - \frac{\pi}{2} - \frac{\alpha}{2}\right)}}.$$

Denoting $\gamma := \beta - (\pi/2)$ and differentiating the above with respect to $\gamma$ we get

$$(\cos\widehat{\alpha})'_\gamma = \frac{(1+\varepsilon^2)(-32\varepsilon^2\sin 2\gamma + 16\varepsilon^2\sin(2\gamma + 2\alpha) + 16\varepsilon^2\sin(2\gamma - 2\alpha))}{D},$$

where

$$D = \big(16 + 8\varepsilon^2\cos(2\gamma-\alpha) + 16\varepsilon^2 + 8\varepsilon^2\cos(2\gamma+\alpha) + 2\varepsilon^4\cos 2\alpha \\ + 2\varepsilon^4\cos 4\gamma + 4\varepsilon^4\cos(2\gamma+\alpha) + 4\varepsilon^4\cos(2\gamma-\alpha) + 4\varepsilon^4\big)^{3/2}.$$

Setting the derivative to 0 and simplifying the numerator we obtain the necessary condition for the extremum of $\cos\widehat{\alpha}$,

$$32\varepsilon^2(1+\varepsilon^2)\sin 2\gamma(\cos 2\alpha - 1) = 0.$$

That is, $\gamma = k\,(\pi/2)$ for $k = 0, \pm 1, \pm 2$, and so on. In particular, it follows that the minimum of $\cos\widehat{\alpha}$ is attained at $\beta_{\min} = 0$ and the maximum is attained at $\beta_{\max} = \pi/2$. The bounds are obtained by further substituting the critical values of $\beta$ into the expression for $\cos\widehat{\alpha}$ and observing the monotonicity with respect to $\varepsilon$. □

Although the full-dimensional tube $\mathcal{T}$ might make quite wide turns, the projected tube $\mathcal{T}_{\{1,2\}}$ is bound to make the same sharp turn equal to $\big((\pi/2) + \arcsin\big(\varepsilon/\sqrt{1+\varepsilon^2}\big)\big)$ each time $\mathcal{T}$ passes through the $\delta$-neighborhood of a vertex $v^S$, $1 \in S$ (e.g., consider the turn adjacent to $v^{\{1,3\}}$ for $n = 3$).

For a moment, equip $\mathcal{C}$ and $\mathcal{T}$ with a superscript $\bar{n}$ to indicate the dimension of the cube, that is, the largest number of linearly independent vectors in $\operatorname{span}(\{v^S : v^S \in \mathcal{C}^{\bar{n}}\})$. Recalling the $\mathcal{C}^n$ defining constraints, namely $\varepsilon x_{n-1} \le x_n \le 1 - \varepsilon x_{n-1}$, we note that by construction of the Klee–Minty cube, whenever we increase the dimension from $\bar{n}$ to $\bar{n}+1$, $\mathcal{C}^{\bar{n}}$ is affinely transformed into "top" and "bottom" $\bar{n}$-dimensional faces $F^{\bar{n}+1}_{\text{top}}$ and $F^{\bar{n}+1}_{\text{bottom}}$ of $\mathcal{C}^{\bar{n}+1}$; that is,

$$F^{\bar{n}+1}_{\text{top}} = \begin{pmatrix} I \\ (0,\dots,0,-\varepsilon) \end{pmatrix} \mathcal{C}^{\bar{n}} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

$$F^{\bar{n}+1}_{\text{bottom}} = \begin{pmatrix} I \\ (0,\dots,0,\varepsilon) \end{pmatrix} \mathcal{C}^{\bar{n}},$$

where $I$ is the identity $\bar{n}\times\bar{n}$ matrix, and $\mathcal{C}^{\bar{n}+1}$ is the convex hull of $F^{\bar{n}+1}_{\text{top}}$ and $F^{\bar{n}+1}_{\text{bottom}}$. Consequently, any two-dimensional space spanned by two connected edges of $\mathcal{C}^{\bar{n}+1}$ from $\mathcal{T}^{\bar{n}+1}_0 \cap F^{\bar{n}+1}_{\text{top}}$ or $\mathcal{T}^{\bar{n}+1}_0 \cap F^{\bar{n}+1}_{\text{bottom}}$ is obtained by tilting the two-dimensional space spanned by the two corresponding edges of $C^{\bar{n}}$ from $\mathcal{T}^{\bar{n}}_0$, lifted to $\mathbb{R}^{\bar{n}+1}$ by setting the $(\bar{n}+1)$st coordinate to zero, by an angle not exceeding $\arcsin\big(\varepsilon/\sqrt{1+\varepsilon^2}\big)$, and moreover, not exceeding $\arcsin\varepsilon$. Therefore, we are in position to apply Lemma 7.1 to bound how fast the cosine of a turn $\alpha^S$ of $\mathcal{T}^n$ adjacent to any $v^S \in \mathcal{C}^n$ with $1 \in S$ may approach its two boundary values of 1 or $-1$ by induction on the dimension $n$.

Fixing $n = 3$, $S \subseteq \{1,2,3\}$ such that $1 \in S$, adding and subtracting 1 to $\cos\alpha^S$ we get

$$1 + \cos \alpha^S \geq \frac{1 + \cos \alpha^{S_{\{1,2\}}}}{1 + \varepsilon^2 \left( \frac{1 - \cos \alpha^{S_{\{1,2\}}}}{2} \right)}$$

and

$$1 - \cos \alpha^S \geq \frac{1 - \cos \alpha^{S_{\{1,2\}}}}{1 + \varepsilon^2 \left( \frac{1 + \cos \alpha^{S_{\{1,2\}}}}{2} \right)}.$$

Furthermore, for any $n \geq 3$ and $v^S$ with $1 \in S$ we can write

$$1 + \cos \alpha^S \geq \frac{1 + \cos \alpha^{S_{\{1,2\}}}}{(1 + \varepsilon^2)^{n-2}} \geq \frac{1 - \varepsilon}{(1 + \varepsilon^2)^{n-2}},$$
$$1 - \cos \alpha^S \geq \frac{1 - \cos \alpha^{S_{\{1,2\}}}}{(1 + \varepsilon^2)^{n-2}} \geq \frac{1 + \left(2\varepsilon/\sqrt{5}\right)}{(1 + \varepsilon^2)^{n-2}},$$

recalling $-2\varepsilon/\sqrt{5} \geq \cos \alpha^{S_{\{1,2\}}} = -\varepsilon/\sqrt{1 + \varepsilon^2} \geq -\varepsilon$ because $\varepsilon \geq 2$.

Observe that by construction of a polyhedral tube $\mathcal{T}$, a single linearly connected component of $\mathcal{T} \setminus \left( \bigcup_{S \subseteq \{1,\ldots,n\}} N_\delta(v^S) \right)$ may be uniquely identified with an edge $(v^R, v^S)$, $R, S \subseteq \{1, \ldots, n\}$, of $\mathcal{C}$ from $\mathcal{T}_0$ by having a nonempty intersection with this component and thus we denote such a component by $L_{(v^R,v^S)}$ and refer to it as a section of $\mathcal{T}$ corresponding to $(v^R, v^S)$. Moreover, recalling the definition of $N_\delta(v^S)$ and $\mathcal{T}$, and noting that $\sqrt{\delta^2 + (\varepsilon\delta)^2 + (\varepsilon\delta)^3 + \cdots} \leq \delta + \varepsilon\delta + \cdots \leq 2\delta$ because $\varepsilon \leq 1/2$, we get that within a given section of a tube $L_{(v^R,v^S)}$ the Euclidean distance from $\forall x \in L_{(v^R,v^S)}$ to the compact closure of $(v^R, v^S) \cap L_{(v^R,v^S)}$ is bounded from above by $2\delta$.

Let us consider what happens to the central path in the proximity of a vertex $v^S \in \mathcal{C}$ such that $1 \in S$. We do so by manufacturing a surrogate for a part of $\mathcal{T}$ that is easier to analyze.

Fix $v^S \in \mathcal{C}$ with $1 \in S$ and denote the two adjacent vertices to which $v^S$ is connected by the two edges from $\mathcal{T}_0$ by $v^R$ and $v^Q$. Without loss of generality we may assume that

$$\begin{aligned} v^R_{\{1,2\}} &= (0, 1), \\ v^S_{\{1,2\}} &= (1, 1 - \varepsilon), \\ v^Q_{\{1,2\}} &= (1, \varepsilon), \end{aligned}$$

and $v^R_n > v^S_n > v^Q_n$, so that the central path $\mathcal{P}$ enters the part of the polyhedral tube $\mathcal{T}$ sectioned between these three vertices via $N_\delta(v^R)$ and exits via $N_\delta(v^Q)$.

Define four auxiliary points $\overline{x}, \overline{z} \in (v^R, v^S)$ and $\underline{x}, \underline{z} \in (v^S, v^Q)$ satisfying

$$\begin{aligned} \overline{x}_{\{1,2\}} &= (1 - 3\delta, 1 - \varepsilon + 3\varepsilon\delta) + \tfrac{1/2 - \varepsilon - 3\delta}{\sqrt{1+\varepsilon^2}}(-1, \varepsilon), \\ \overline{z}_{\{1,2\}} &= (1 - 3\delta, 1 - \varepsilon + 3\varepsilon\delta), \\ \underline{x}_{\{1,2\}} &= (1, 1 - \varepsilon - 3\delta), \\ \underline{z}_{\{1,2\}} &= (1, 1/2). \end{aligned}$$

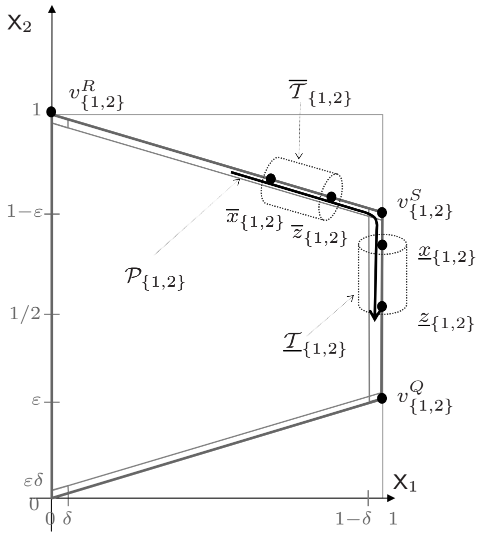


**Fig. 7.3** Schematic drawing for the cylindrical tube segments.

Because the distance from any point to the (part of the) identifying edge of $L_{(v^R,v^S)}$ or $L_{(v^S,v^Q)}$ is no greater than $2\delta$ and because $(\cdot)_{\{1,2\}}$ corresponds to the orthogonal projection from $\mathbb{R}^n$ onto its first two coordinates, we can define two cylindrical tube segments:

$$\begin{aligned}\overline{\mathcal{T}} := &\{x \in \mathbb{R}^n : \min_{z\in(\overline{x},\overline{z})} \|x - z\| \le 2\delta\}\\ &\cap \{x \in \mathbb{R}^n : (\overline{x} - \overline{z})^T x \le (\overline{x} - \overline{z})^T \overline{x}\}\\ &\cap \{x \in \mathbb{R}^n : (\overline{z} - \overline{x})^T x \le (\overline{z} - \overline{x})^T \overline{z}\}\end{aligned}$$

and

$$\begin{aligned}\underline{\mathcal{T}} := &\{x \in \mathbb{R}^n : \min_{z\in(\underline{x},\underline{z})} \|x - z\| \le 2\delta\}\\ &\cap \{x \in \mathbb{R}^n : (\underline{x} - \underline{z})^T x \le (\underline{x} - \underline{z})^T \underline{x}\}\\ &\cap \{x \in \mathbb{R}^n : (\underline{z} - \underline{x})^T x \le (\underline{z} - \underline{x})^T \underline{z}\}\end{aligned}$$

such that

$$\overline{\mathcal{T}} \supset L_{(v^R,v^S)} \cap \{x \in \mathbb{R}^n : (\overline{x} - \overline{z})^T x \le (\overline{x} - \overline{z})^T \overline{x}, (\overline{z} - \overline{x})^T x \le (\overline{z} - \overline{x})^T \overline{z}\},$$

$$\underline{\mathcal{T}} \supset L_{(v^S,v^Q)} \cap \{x \in \mathbb{R}^n : (\underline{x} - \underline{z})^T x \le (\underline{x} - \underline{z})^T \underline{x}, (\underline{z} - \underline{x})^T x \le (\underline{z} - \underline{x})^T \underline{z}\},$$

and

$$\overline{\mathcal{T}} \cap \big(N_\delta(v^R) \cup N_\delta(v^S)\big) = \underline{\mathcal{T}} \cap \big(N_\delta(v^S) \cup N_\delta(v^Q)\big) = \varnothing.$$

Therefore, $\mathcal{P}$ will traverse $\overline{\mathcal{T}}$ and $\underline{\mathcal{T}}$, first entering $\overline{\mathcal{T}}$ through its face corresponding to $(\overline{x} - \overline{z})^T x = (\overline{x} - \overline{z})^T \overline{x}$ and exiting through the face corresponding to $(\overline{z} - \overline{x})^T x = (\overline{z} - \overline{z})^T \overline{z}$, and then entering $\underline{\mathcal{T}}$ at a point with $(\underline{x} - \underline{z})^T x = (\underline{x} - \underline{z})^T \underline{x}$ and exiting through a point with $(\underline{z} - \underline{x})^T x = (\underline{z} - \underline{x})^T \underline{z}$.

Now we choose a new system of orthogonal coordinates in $\mathbb{R}^n$ that allows us apply the argument similar to that of Proposition 7.3 as follows. Let the first two coordinates correspond to the linear subspace spanned by $(\overline{x}, \overline{z})$ and $(\underline{x}, \underline{z})$; align the second coordinate axis with the vector $(\underline{z}, \underline{x})$, so that the vector $(\overline{x}, \overline{z})$ forms the same angle equal to $\alpha^S$ with the second coordinate axis as with $(\underline{x}, \underline{z})$. Choose the rest $(n-2)$ coordinates so that they form an orthogonal basis for $\mathbb{R}^n$.

Consider parameterization of $\mathcal{P}$ by its arc length, $\mathcal{P}_{\text{arc}}$. Because the shortest distance between the two parallel faces of $\overline{\mathcal{T}}$ that correspond to $\{x \in \mathbb{R}^n : (\overline{x} - \overline{z})^T x = (\overline{x} - \overline{z})^T \overline{x}\}$ and $\{x \in \mathbb{R}^n : (\overline{z} - \overline{x})^T x = (\overline{z} - \overline{x})^T \overline{z}\}$ is equal to $\|(\overline{x}, \overline{z})\| = 1/2 - \varepsilon - 3\delta$, by the mean-value theorem it takes at least $(1/2 - \varepsilon - 3\delta)$ change of the arc length parameter for $\mathcal{P}_{\text{arc}}$ to traverse $\overline{\mathcal{T}}$. Noting that while traversing the tube $\overline{\mathcal{T}}$ the second coordinate of $\mathcal{P}_{\text{arc}}$ might change at most by $2 \cdot |2\delta \sin \alpha^S| + |(1/2 - \varepsilon - 3\delta) \cos \alpha^S|$, by the same theorem we deduce that $\exists t_1$ such that

$$\left| \left( \dot{\mathcal{P}}_{\text{arc}}(t_1) \right)_2 \right| \leq \frac{2|2\delta \sin \alpha^S| + |(1/2 - \varepsilon - 3\delta) \cos \alpha^S|}{1/2 - \varepsilon - 3\delta} \leq |\cos \alpha^S| + \frac{4\delta}{1/2 - \varepsilon - 3\delta}.$$

Analogously, considering $\underline{\mathcal{T}}$ along the $i$th coordinate with $i \neq 2$ we conclude that $\forall i \neq 2$, $\exists t_i$ such that

$$\left| \left( \dot{\mathcal{P}}_{\text{arc}}(t_i) \right)_i \right| \leq \frac{4\delta}{1/2 - \varepsilon - 3\delta}.$$

We use the points $t_1, t_2, \ldots, t_n$ to compute a lower bound on the total curvature contribution of a turn of $\mathcal{P}$ next to $v^S$: recalling $\|\dot{\mathcal{P}}_{\text{arc}}\| = 1$, the total curvature of the part of $\mathcal{P}$ that passes through $\overline{\mathcal{T}}$ and $\underline{\mathcal{T}}$ (i.e., resulting from a turn of $\mathcal{T}$ adjacent to $v^S$) may be bounded below by the length of the shortest curve on a unit $n$-sphere that connects points $\dot{\mathcal{P}}_{\text{arc}}(t_1), \dot{\mathcal{P}}_{\text{arc}}(t_2), \ldots, \dot{\mathcal{P}}_{\text{arc}}(t_n)$ in any order. For simplicity, the latter length may be further bounded below by

$$\mathcal{K}^S := \min_{\substack{x^i \in \mathbb{R}^n, \quad i=1,\ldots,n: \\ \|x^i\|=1, \quad \forall i, \\ |x_1^1| \leq |\cos \alpha^S| + \frac{4\delta}{1/2-\varepsilon-3\delta}, \\ |x_j^j| \leq \frac{4\delta}{1/2-\varepsilon-3\delta}, \quad j \geq 2}} \max_{j \geq 2} \operatorname{dist}(x^1, x^j)$$

$$\geq \min_{\substack{x^i \in \mathbb{R}^n, \quad i=1,\ldots,n: \\ \|x^1\|=1, \\ |x_1^1| \leq |\cos \alpha^S| + \frac{4\delta}{1/2-\varepsilon-3\delta}, \\ |x_j^j| \leq \frac{4\delta}{1/2-\varepsilon-3\delta}, \quad j \geq 2}} \max_{j \geq 2} \|x^1 - x^j\|,$$

where $\operatorname{dist}(x,z)$ is the length of the shortest curve on a unit sphere between points $x$ and $z$, that is, the geodesic. Clearly, the critical value for the last expression is attained, in particular, at $x^i \in \mathbb{R}^n_+$, $\forall i$, when $\|x^1\| = 1$, $\|x^1 - x^j\| = \|x^1 - x^i\|$, $i, j \geq 2$, and

$$x_1^1 = |\cos\alpha^S| + \frac{4\delta}{1/2 - \varepsilon - 3\delta}, \qquad x_j^j = \frac{4\delta}{1/2 - \varepsilon - 3\delta}, \qquad j \geq 2.$$

It follows that

$$\sum_{j=2}^{n}(x_j^1)^2 = 1 - \left(|\cos\alpha^S| + \frac{4\delta}{1/2 - \varepsilon - 3\delta}\right)^2 \geq 1 - |\cos\alpha^S| - \frac{4\delta}{1/2 - \varepsilon - 3\delta}$$

and, because $|\cos\alpha^S| \leq 1 - \left((1-\varepsilon) \,/\, \left(1+\varepsilon^2\right)^{n-2}\right)$,

$$\sum_{j=2}^{n}(x_j^1)^2 \geq \frac{1-\varepsilon}{(1+\varepsilon^2)^{n-2}} - \frac{4\delta}{1/2 - \varepsilon - 3\delta},$$

resulting in

$$\begin{aligned} x_j^1 \geq (x_j^1)^2 &\geq \frac{1}{n-1}\left(\frac{1-\varepsilon}{(1+\varepsilon^2)^{n-2}} - \frac{4\delta}{1/2 - \varepsilon - 3\delta}\right) \\ &\geq \frac{1}{n-1} \cdot \frac{1}{2(1+1/4)^{n-2}} - \frac{1}{n-1}\left(\frac{4\delta}{1/2 - \varepsilon - 3\delta}\right), \qquad j \geq 2. \end{aligned}$$

Therefore, recalling $\varepsilon = (n-1)/2n$ and $\delta = \left(1/32n^2\right)(4/5)^{n-2}$, we can write

$$\begin{aligned} \mathcal{K}^{\mathcal{S}} &\geq \|x^1 - x^2\| \\ &\geq x_2^1 - x_2^2 \\ &\geq \frac{1}{2(n-1)}\left(\frac{4}{5}\right)^{n-2} - \left(1 + \frac{1}{n-1}\right)\frac{4\delta}{1/2 - \varepsilon - 3\delta} \\ &\geq \frac{1}{2(n-1)}\left(\frac{4}{5}\right)^{n-2} - \frac{n}{8n^2(n-1)}\left(\frac{4}{5}\right)^{n-2}\frac{1}{\frac{1}{2n} - \frac{3}{32n^2}\left(\frac{4}{5}\right)^{n-2}} \\ &\geq \frac{1}{2(n-1)}\left(\frac{4}{5}\right)^{n-2} - \frac{n}{8n^2(n-1)}\left(\frac{4}{5}\right)^{n-2} 2n \\ &\geq \frac{1}{4n}\left(\frac{4}{5}\right)^{n-2}. \end{aligned}$$

Finally, recalling that the polyhedral tube $\mathcal{T}$ makes $2^{n-1}$ such turns, we conclude that the total curvature of $\mathcal{P}$ indeed satisfies $K \geq (1/2n)\,(8/5)^{n-2}$. □

The bound on the total curvature $K$ of $\mathcal{P}$ established above is obviously not tight. We expect the true order of $K$ to be $2^n$ up to a multiplier, rational in $n$.

**Remark 7.4.** In $\mathbb{R}^2$, by combining the optimality conditions (7.1) and (7.2) for the analytic center $\chi$ with that of the central path $\mathcal{P}$ visiting the $\delta$-neighborhoods of the vertices $v^{\{1\}}$ and $v^{\{1,2\}}$ one can show that for $\delta$ below a certain threshold both $d_1$ and $d_2$ are bounded away from 0 by a constant. In turn, this implies that for fixed feasible $d_1, d_2$, the necessary conditions (7.1) and (7.2) for $h$ chosen such that the central path visits the $\delta$-neighborhoods of all the vertices of $\mathcal{C}$ are "asymptotically equivalent" as $\delta \downarrow 0$ to the sufficient conditions (7.3) and (7.4), up to a constant multiplier. Here the term asymptotic equivalence refers to the convergence of the normalized extreme rays and the vertices of the unbounded polyhedra given by the set of necessary conditions for a fixed $d$ to those of the polyhedra given by the set of sufficient conditions (7.3) and (7.4).

This suggests that the following might be true. In $\mathbb{R}^n$

$$\min_{i=1,\ldots,n} d_i \geq \hat{d} > 0,$$

where $\hat{d}$ is independent of $n, \delta, \varepsilon$. Moreover, the necessary conditions for $\mathcal{P}$ to visit the $\delta$-neighborhoods of all the vertices of $\mathcal{C}$ for a fixed $d$ are asymptotically equivalent as $\delta \downarrow 0$ to the sufficient conditions (7.3) and (7.4). If, furthermore, we confine ourselves to only bounded subsets of all such feasible $(d, h)$ corresponding to, say,

$$\sum_{i=1}^{n} h_i \leq H_\delta^* := 2 \min_{d,h} \sum_{i=1}^{n} h_i$$

then the conditions (7.3) and (7.4) are tight, in a sense that if we denote the set of all $(d, h)$ satisfying the necessary conditions for $\mathcal{P}$ to visit all the $\delta$-neighborhoods intersected with $\{h : \sum_i h \leq H_\delta^*\}$ as $\mathrm{Necc}_\delta$, the set of all $(d, h)$ satisfying (7.3) and (7.4) intersected with $\{h : \sum_i h \leq H_\delta^*\}$ as $\mathrm{Suff}_\delta$, then for some small enough $\hat{\delta}$ there exists $M, m > 0$ independent of $\varepsilon$ such that

$$\mathrm{Necc}_{m\delta} \subseteq \mathrm{Suff}_\delta \subseteq \mathrm{Necc}_{M\delta}, \qquad 0 < \delta \leq \hat{\delta}.$$

## 7.3 Finding $h \in \mathbb{N}^n$ Satisfying Conditions (7.3) and (7.4)

We write $f(n) \approx g(n)$ for $f(n), g(n) : \mathbb{N} \to \mathbb{R}$ if $\exists c, C > 0$ such that $cf(n) \leq g(n) \leq Cf(n)$, $\forall n$; the argument $n$ is usually omitted from the notation.

Denote $b := \mathbf{1}\,(3/\delta)$. Let us first concentrate on finding $h \in \mathbb{N}^n$ such that (7.3) holds. If the integrality condition on $h$ is relaxed, a solution to (7.3) can be found by simply solving $Ah = b$. Note that

$$\|A\|_{1,\infty} = \max_i \sum_{j=1}^{n} |a_{ij}|$$

is, in fact, small for $d$ – large componentwise and $\varepsilon < 1/2$. So to find an integral $h$ we can

- Solve $A\widehat{h} = (1+\gamma)b$ for some small $\gamma > 0$.
- Set $h = \lfloor \widehat{h} \rfloor$.

Observe that for $h$ to satisfy (7.3), it is enough to require $\max_i (A(\widehat{h} - h) - \gamma b)_i \leq 0$. In turn, this can be satisfied by choosing $\gamma > 0$ such that

$$\gamma \frac{3}{\delta} \geq \max_i \sum_{j=1}^{n} |a_{ij}|.$$

In Section 7.3.1 we show how to solve this system of linear equations. In Section 7.3.2 we demonstrate that under some assumption on $d$, (7.4) is already implied by (7.3), and consequently the rounding of $\widehat{h}$ will not cause a problem for (7.4) either.

**Remark 7.5.** The choice of rounding down instead of rounding up is arbitrary.

## *7.3.1 Solving the Linear System*

Because $(1+\gamma)b = (1+\gamma)\,(3/\delta)\,\mathbf{1}$, we can first solve $A\widetilde{h} = \mathbf{1}$ and then scale $\widetilde{h}$ by $(1+\gamma)\,(3/\delta)$. Our current goal is to find the solution to $A\widetilde{h} = \mathbf{1}$.

For an arbitrary invertible $B \in \mathbb{R}^{n\times n}$ and $y, z \in \mathbb{R}^n$ such that $1 + z^T B^{-1} y \neq 0$, the solution to

$$(B + yz^T)x = b$$

can be written as

$$x = B^{-1}(b - \alpha y),$$

where

$$\alpha = \frac{x^T B^{-1} b}{1 + z^T B^{-1} y}$$

(for writing $(B + yz^T)x = Bx + y(z^T x) = b$, denoting $\alpha := z^T x$, we can express $x = B^{-1}(b - \alpha y)$ and substitute this $x$ into $(B + yz^T)x = b$ again to compute $\alpha$).

Denoting

$$B := \begin{pmatrix} \frac{1}{d_1+1} & \frac{-\varepsilon}{d_2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{2\varepsilon}{d_2+1} & \frac{-\varepsilon^2}{d_3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{2\varepsilon^{k-1}}{d_k+1} & \frac{-\varepsilon^k}{d_{k+1}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2\varepsilon^{n-2}}{d_{n-1}+1} & \frac{-\varepsilon^{n-1}}{d_n} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{2\varepsilon^{n-1}}{d_n+1} \end{pmatrix},$$

$$y^T := \frac{-1}{d_1}(0, 1, 1, \dots, 1),$$

$$z^T := (1, 0, 0, \dots, 0),$$

we can compute the solution to $(B + yz^T)\widetilde{h} \equiv A\widetilde{h} = \mathbf{1}$ as

$$\widetilde{h} = B^{-1}\mathbf{1} + \frac{\alpha}{d_1}B^{-1}\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{7.5}$$

where

$$\alpha = \frac{\sum_{j=1}^{n} B^{-1}_{1j}}{1 - \frac{1}{d_1}\sum_{j=2}^{n} B^{-1}_{1j}}. \tag{7.6}$$

So to get the explicit formula for $\widetilde{h}$ we need to compute $B^{-1}$ and show that $d$ can be chosen such that $\alpha$ is well defined; that is, $1 - (1/d_1)\sum_{j=2}^{n} B^{-1}_{1j} \neq 0$.

In order to invert $B$ first note that it satisfies

$$B = \mathrm{Diag}\left(\frac{1}{d_1+1}, \frac{2\varepsilon}{d_2+1}, \frac{2\varepsilon^2}{d_3+1} \dots, \frac{2\varepsilon^{n-1}}{d_n+1}\right)(I + S),$$

where a superdiagonal matrix $S \in \mathbb{R}^{n\times n}$ is such that

$$S_{ij} = \begin{cases} \frac{-\varepsilon(d_i+1)}{d_{i+1}} & j = i+1, i = 1, \dots, n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $(I + Z)^{-1} = I - Z + Z^2 - Z^3 + \cdots$ for any $Z \in \mathbb{R}^{n\times n}$ such that these matrix-power series converge. In our case, the powers of $S$ are easy to compute for $1 \le k \le n-1$,

$$S^k_{ij} = \begin{cases} \prod_{l=i}^{i+k-1} S_{l,l+1} & j = i+k, i = 1, \ldots, n-k \\ 0 & \text{otherwise,} \end{cases}$$

and $S^m = 0$ for all $m \geq n$, so the inverse of $(I + S)$ can be computed as above and the inverse of $B$ can be further computed by post-multiplying by the inverse of

$$\text{Diag}\left(\frac{1}{d_1+1}, \frac{2\varepsilon}{d_2+1}, \frac{2\varepsilon^2}{d_3+1} \ldots, \frac{2\varepsilon^{n-1}}{d_n+1}\right).$$

Therefore, $B^{-1}$ is equal to

$$\begin{pmatrix} d_1+1 & \frac{(d_1+1)(d_2+1)}{2d_2} & \frac{(d_1+1)(d_2+1)(d_3+1)}{4d_2d_3} & \frac{(d_1+1)\cdots(d_4+1)}{8d_2d_3d_4} & \cdots & \frac{\prod_{j=1}^n(d_j+1)}{2^{n-1}\prod_{j=2}^n d_j} \\ 0 & \frac{d_2+1}{2\varepsilon} & \frac{(d_2+1)(d_3+1)}{4d_3\varepsilon} & \frac{(d_2+1)\cdots(d_4+1)}{8d_3d_4\varepsilon} & \cdots & \frac{\prod_{j=2}^n(d_j+1)}{2^{n-1}\varepsilon\prod_{j=3}^n d_j} \\ 0 & 0 & \frac{d_3+1}{2\varepsilon^2} & \frac{(d_3+1)(d_4+1)}{4d_4\varepsilon^2} & \cdots & \frac{\prod_{j=3}^n(d_j+1)}{2^{n-2}\varepsilon^2\prod_{j=4}^n d_j} \\ 0 & 0 & 0 & \frac{d_4+1}{2\varepsilon^3} & \cdots & \frac{\prod_{j=4}^n(d_j+1)}{2^{n-3}\varepsilon^3\prod_{j=5}^n d_j} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{d_n+1}{2\varepsilon^{n-1}} \end{pmatrix}.$$

## *7.3.2 Partial Implication for Sufficient Conditions*

Observe that in order for $\widetilde{h} \in \mathbb{R}^n_+$, we must have $\alpha > 0$ as in (7.6). For if not (i.e., if $\alpha < 0$), then denoting

$$\beta_1 := z^T B^{-1} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \sum_{j=2}^n B^{-1}_{1j}$$

and writing

$$\alpha = \frac{\beta_1 + d_1 + 1}{1 - \frac{\beta_1}{d_1}} \tag{7.7}$$

we must have $\beta_1 > d_1 > 0$. So

$$\frac{-\alpha}{d_1} = \frac{\beta_1 + d_1 + 1}{\beta_1 - d_1} > 1$$

and from (7.5) it follows that if $(\alpha/d_1) < -1$, then $\widetilde{h}_2, \widetilde{h}_3, \ldots, \widetilde{h}_n < 0$.

From now on we assume $\alpha > 0$ (in Sections 7.4.1, 7.5.1 we show how to achieve this by choosing $d$ appropriately). Note that in this case $(\alpha/d_1) > 1$.

Suppose $h \in \mathbb{N}^n$ is such that (7.3) holds. If, furthermore, $\left(h_i \varepsilon^{i-1}\right)/d_i$ is dominated by $h_1/d_1$ for $i = 1, \ldots, n$, then (7.3) already implies (7.4). Therefore, it is left to show that $d$ can be chosen such that $h = \lfloor \widehat{h} \rfloor$ satisfies the domination condition above.

For this to hold it suffices

$$\frac{\frac{3}{\delta}(1+\gamma)\widetilde{h}_1 - 1}{d_1} \geq \frac{\frac{3}{\delta}(1+\gamma)\widetilde{h}_i + 1}{d_i} \varepsilon^{i-1}, \qquad i = 2, \ldots, n,$$

where $\widetilde{h}$ solves $A\widetilde{h} = \mathbf{1}$. The above is implied by

$$\frac{d_1 + 1 + \beta_1 + \frac{\alpha}{d_1}\beta_1 - \frac{1}{6}}{d_1} \geq \frac{(1 + \frac{\alpha}{d_1})\frac{\beta_i}{\varepsilon^{i-1}} + \frac{1}{6}}{d_i} \varepsilon^{i-1}, \qquad i = 2, \ldots, n$$

because $\gamma > 0$, $\delta < 1/2$, where

$$\beta_i := \varepsilon^{i-1}(B^{-1}\mathbf{1})_i.$$

This can be written as

$$\left(1 + \frac{\alpha}{d_1}\right)\frac{\beta_1}{d_1} + 1 + \frac{5}{6d_1} \geq \left(1 + \frac{\alpha}{d_1}\right)\frac{\beta_i}{d_i} + \frac{\varepsilon^{i-1}}{6d_i}, \qquad i = 2, \ldots, n.$$

In particular, if we have

$$\frac{\beta_1}{d_1} > \frac{\beta_i}{d_i}, \qquad i = 2, \ldots, n \tag{7.8}$$

then the above inequality holds true if

$$1 \geq \frac{\varepsilon}{6d_1} - \frac{5}{6d_1},$$

that is, because $\varepsilon < 1/2$, for $d_i > 0$ for $i = 1, \ldots, n$.

Finally, observe that if $d_1 \geq d_i$, $i \geq 2$, and $d_1 = O(2^n)$, then the magnitude of $\widetilde{h}$ is primarily determined by $\alpha$: recalling (7.5), (7.7), we write

$$\widetilde{h} = \alpha \left( \frac{B^{-1}}{d_1} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \frac{B^{-1}}{d_1}\mathbf{1} \right)$$

$$\leq \alpha \left( \frac{B^{-1}}{d_1} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \frac{\left(1 - \frac{\beta_1}{d_1}\right) B^{-1}}{d_1 + 1} \mathbf{1} \right)$$

$$= \alpha \left( \frac{B^{-1}}{d_1} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \left(1 - \frac{\beta_1}{d_1}\right) \frac{B^{-1}}{d_1 + 1} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \left(1 - \frac{\beta_1}{d_1}\right) \begin{pmatrix} d_1 + 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right).$$

Because $d_i > 0$ for $i = 1, \ldots, n$, we have $(d_i + 1)/2d_i > \frac{1}{2}$ and so $1 - (\beta_1/d_1) < 1/2^{n-1}$, implying

$$\widetilde{h} < \alpha \left( \frac{B^{-1}}{d_1} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \frac{B^{-1}}{2^{n-1}(d_1 + 1)} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \frac{1}{2^{n-1}} \begin{pmatrix} d_1 + 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) \tag{7.9}$$

$$\approx \alpha \frac{B^{-1}}{d_1} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{for large } n.$$

## 7.4 Uniform Distances to the Redundant Hyperplanes

Clearly, many different choices for $d$ are possible. In this section we explore the uniform model for $d$; that is $d_1 = d_2 = \cdots = d_n$.

### *7.4.1 Bounding* $\alpha$

For $\alpha > 0$ we need

$$\beta_1 < d_1.$$

Denoting

$$\xi := \frac{d_1 + 1}{2d_1}$$

the above can be written as

$$(d_1 + 1)\xi + (d_1 + 1)\xi^2 + \cdots + (d_1 + 1)\xi^{n-1} < d_1$$

or, equivalently,

$$\xi^2(1+\xi+\xi^2+\cdots+\xi^{n-2}) < \frac{1}{2},$$

same as

$$\xi^2\left(\frac{1-\xi^{n-1}}{1-\xi}\right) < \frac{1}{2}.$$

In other words, we want

$$p(\xi) := 1-\xi-2\xi^2+2\xi^{n+1} > 0.$$

Note that as $d_1 \to \infty$, $\xi \to \frac{1}{2}$ and $p(\xi) \to 2\xi^{n+1} > 0$, so $p(\xi) > 0$ for $d_1$ large enough.

Observe that $\xi > \frac{1}{2}$ for any $d_1 > 0$, and

$$\begin{array}{rcll} p'\left(\frac{1}{2}\right) &=& -3+\frac{2(n+1)}{2^n} < 0 & \text{for } n=1,2,\ldots \\ p''\left(\frac{1}{2}\right) &=& -4+\frac{2(n+1)n}{2^{n-1}} \begin{cases} =0 \\ >0 \\ <0 \end{cases} & \begin{array}{l} \text{for } n=1 \\ \text{for } n=2,3,4 \\ \text{otherwise} \end{array} \\ p^{(k)}\left(\frac{1}{2}\right) &=& \frac{1}{2^{n-k}}\frac{(n+1)!}{(n-k+1)!} > 0 & \text{for } k\geq 3, \end{array}$$

so

$$p\left(\frac{1}{2}+\Delta\xi\right) \geq \widetilde{p}\left(\frac{1}{2}+\Delta\xi\right) := p\left(\frac{1}{2}\right)+p'\left(\frac{1}{2}\right)\Delta\xi+p''\left(\frac{1}{2}\right)\frac{\Delta\xi^2}{2}.$$

Thus, to guarantee $p(\xi)>0$, it is enough to require

$$\widetilde{p}\left(\frac{1}{2}+\Delta\xi\right) \geq 0$$

letting $\xi = \frac{1}{2}+\Delta\xi$.

Denoting

$$\begin{aligned} \widetilde{a} &:= -2+\frac{n(n+1)}{2^{n-1}} \\ \widetilde{b} &:= -3+\frac{2(n+1)}{2^n} \\ \widetilde{c} &:= \frac{1}{2^n} \end{aligned}$$

and

$$\Delta\xi^* := \begin{cases} \frac{-\widetilde{b}+\sqrt{\widetilde{b}^2-4\widetilde{a}\widetilde{c}}}{2\widetilde{a}}, & 2\leq n\leq 4 \\ \frac{-\widetilde{b}-\sqrt{\widetilde{b}^2-4\widetilde{a}\widetilde{c}}}{2\widetilde{a}}, & n>4 \end{cases} \tag{7.10}$$

(the smallest positive root of $\widetilde{p}(\frac{1}{2} + \Delta\xi) = 0$), we conclude that $\beta_1 < d_1$ as long as

$$\xi = \frac{d_1 + 1}{2d_1} \leq \frac{1}{2} + \Delta\xi^*.$$

That is,

$$d_1 = \frac{1}{2\Delta\xi} \geq \frac{1}{2\Delta\xi^*}. \tag{7.11}$$

Note that for $d_1 = d_2 = \cdots = d_n$, (7.8) holds, so (7.4) is readily implied by (7.3).

It is left to demonstrate how to choose $d_1$ satisfying the above to guarantee a moderate growth in $\widetilde{h}$ as $n \to \infty$.

### *7.4.2 Picking a "Good" $d_1$*

Note that as $n \to \infty$, we have

$$\Delta\xi^* \to \frac{1}{3 \cdot 2^n}$$

by expanding the square root in (7.10) as a first-order Taylor series. Also,

$$1 - \frac{\beta_1}{d_1} = p(\xi) \geq \widetilde{p}(\xi)$$

and hence

$$\frac{1}{1 - \frac{\beta_1}{d_1}} \leq \frac{1}{\widetilde{p}(\xi)}.$$

In fact, for large $n$, $p(\xi) \approx \widetilde{p}(\xi)$, for $\frac{1}{2} \leq \xi \leq \frac{1}{2} + \Delta\xi^*$. In turn, $\widetilde{p}(\xi)$ is almost linear on this interval because, as $n \to \infty$,

$$\widetilde{b} \to -3 \leftarrow \frac{\widetilde{c}}{\Delta\xi^*}$$

(we compare the slope of $\widetilde{p}(\frac{1}{2} + \Delta\xi)$ at $\Delta\xi = 0$ with its decrement as a function of $\Delta\xi$ over $[0, \Delta\xi^*]$).

Recalling that the growth of $\widetilde{h}$ is primarily determined by $\alpha$ for large $n$ (see (7.9)), our goal becomes to minimize $\alpha$. From (7.7), (7.11), noting that $\beta_1 \approx d_1$ for large $n$ (also recall $\beta_1 < d_1$), we get

$$\alpha \leq \frac{\beta_1 + (d_1 + 1)}{\widetilde{p}(\xi)} \approx \frac{2d_1}{\widetilde{p}(\xi)} \approx \frac{2}{2\Delta\xi} \cdot \frac{1}{\widetilde{c} - \widetilde{c}\frac{\Delta\xi}{\Delta\xi^*}}$$

and, moreover the right-hand side of this expression approximates $\alpha$ fairly well. So, to approximately minimize $\widetilde{h}$, we maximize

$$\Delta\xi \cdot \left(\widetilde{c} - \widetilde{c}\frac{\Delta\xi}{\Delta\xi^*}\right)$$

for $0 \leq \Delta\xi \leq \Delta\xi^*$, which corresponds to setting

$$\Delta\xi = \frac{\Delta\xi^*}{2}$$

thus resulting in

$$\alpha \to 6 \cdot 2^{2n} \quad \text{as} \quad n \to \infty$$

and

$$\widetilde{h}_i = O\left(\frac{2^{2n}}{\varepsilon^{i-1}}\right), \qquad i = 1, \ldots, n.$$

### 7.4.3 An Explicit Formula

For a given $n, \varepsilon, \delta$, compute $\Delta\xi^*$ according to (7.10), set $\Delta\xi = \Delta\xi^*/2$, $d_1$ as in (7.11). Compute the solution to $A\widetilde{h} = \mathbf{1}$ using (7.5), where $\alpha$ is computed according to (7.6). Set $\gamma = (\delta/3)\max_i \sum_{j=1}^n |a_{ij}|$ and, finally,

$$h = \left\lfloor (1+\gamma)\frac{3}{\delta}\widetilde{h} \right\rfloor.$$

From (7.9) it follows that (for large $n$)

$$\widetilde{h}_i \approx \frac{\alpha}{d_1} B^{-1} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \leq \frac{\alpha}{\varepsilon^{i-1}}, \qquad i = 1, \ldots, n.$$

We are interested in picking $\varepsilon, \delta$, to minimize the total number of the redundant constraints, $\sum_{i=1}^n h_i$. Recalling $\varepsilon + \delta < \frac{1}{2}$, denoting

$$g(\varepsilon) := \frac{\varepsilon(\varepsilon^{-n} - 1)}{1 - 3\varepsilon + 2\varepsilon^2}$$

for large $n$ we can write

$$\begin{aligned} \textstyle\sum_{i=1}^n h_i &\approx \tfrac{3}{\delta} \textstyle\sum_{i=1}^n \widetilde{h}_i \approx \tfrac{3}{\delta} \textstyle\sum_{i=1}^n \alpha \left(\tfrac{1}{\varepsilon}\right)^{i-1} \\ &\approx 6 \cdot 2^{2n} \tfrac{6}{1-2\varepsilon} \tfrac{1-\left(\frac{1}{\varepsilon}\right)^n}{1-\frac{1}{\varepsilon}} \\ &= 9 \cdot 2^{2n+2} g(\varepsilon) \end{aligned}$$

(it is natural to pick $\delta$ as close to $\frac{1}{2} - \varepsilon$ as possible, say $\delta = .999\left(\frac{1}{2} - \varepsilon\right)$).

In order to bound

$$g^* := \min_{0<\varepsilon<\frac{1}{2}} g(\varepsilon)$$

we first bound the minimizer $\varepsilon^*$ of the above, next we bound the derivative of $g(\varepsilon)$, and finally using the mean-value theorem we bound $g^*$ itself.

Observing

$$\begin{aligned} g'(\varepsilon) &= \frac{(1-3\varepsilon+2\varepsilon^2)((1-n)\varepsilon^{-n}-1)-(4\varepsilon-3)\varepsilon(\varepsilon^n-1)}{(1-3\varepsilon+2\varepsilon^2)^2} \\ &= \frac{(1-2\varepsilon)(1-n-\varepsilon^n)+\varepsilon(3-4\varepsilon)(\varepsilon^{n-1}+\varepsilon^{n-2}+\cdots+1)}{\varepsilon^n(1-\varepsilon)(1-2\varepsilon)^2} \\ &= \frac{3-4\varepsilon}{\varepsilon^n(1-\varepsilon)(1-2\varepsilon)^2}\left(\frac{1-n}{2}+\frac{n-1+\varepsilon^n}{2(3-4\varepsilon)}+\left(\frac{1}{2}\varepsilon^n+\varepsilon^{n-1}+\cdots+\varepsilon\right)\right) \end{aligned}$$

and noting that in the expression above the second and the third summands are monotone-increasing functions of $\varepsilon$ (for $\varepsilon \in (0, 1/2), n \geq 1$), for $n \geq 3$ we can write $\varepsilon^* \in (\varepsilon^L, \varepsilon^U)$ with

$$\varepsilon^L := \frac{n-5/4}{2n}$$

because for $n \geq 2$ we have

$$\begin{aligned} &(1-2\varepsilon)(1-n-\varepsilon^n)+\varepsilon(3-4\varepsilon)(\varepsilon^{n-1}+\varepsilon^{n-2}+\cdots+1)\Big|_{\varepsilon=\varepsilon^L} \\ &\quad = \frac{(-4n^2+15n-25)+(-16n^2-40n+25)\left(\frac{n-5/4}{2n}\right)^n}{4n(4n+5)} < 0, \end{aligned}$$

where the first summand in brackets has no real roots with respect to $n$ and the second summand is negative for $n \geq 2$, so $g'(\varepsilon^L) < 0$ and thus $g'(\varepsilon) < 0$ for $0 < \varepsilon < \varepsilon^L$, and

$$\varepsilon^U := \frac{n-1}{2n}$$

because for $n \geq 3$,

$$\begin{aligned} &(1-2\varepsilon)(1-n-\varepsilon^n)+\varepsilon(3-4\varepsilon)(\varepsilon^{n-1}+\varepsilon^{n-2}+\cdots+1)\Big|_{\varepsilon=\varepsilon^U} \\ &\quad = \frac{n-1-n^2-2n+2\left(\frac{n-1}{2n}\right)^n}{n(n+1)} \\ &\quad = \frac{n-1-\left((n-1)(n+3)\left(\frac{n-1}{2n}\right)^n+\left(\frac{n-1}{2n}\right)^n\right)}{n(n+1)} \\ &\quad > \frac{n-1}{n(n+1)}\left(1-\frac{n+4}{2^n}\right) > 0, \end{aligned}$$

so $g'(\varepsilon^U) > 0$ and thus $g'(\varepsilon) > 0$ for $\varepsilon^U < \varepsilon < 1/2$. Consequently, for $\varepsilon \in (\varepsilon^L, \varepsilon^U)$ we have

$$g'(\varepsilon) \geq \min_{\varepsilon \in [\varepsilon^L, \varepsilon^U]} \frac{-n(1-2\varepsilon)}{\varepsilon^n (1-\varepsilon)(1-2\varepsilon)^2} = \frac{-2n^3}{n+1} \left( \frac{8n}{4n-5} \right)^n$$

and, therefore, by the mean-value theorem,

$$g(\varepsilon^L) + \min_{\varepsilon \in [\varepsilon^L, \varepsilon^U]} g'(\varepsilon)(\varepsilon^U - \varepsilon^L) \leq g^* \leq g(\varepsilon^L);$$

that is,

$$0 < \frac{4n}{5} \left( \frac{4n-5}{4n+5} \right) \left( \left( \frac{8n}{4n-5} \right)^n - 1 \right) - \frac{n^2}{4(n+1)} \left( \frac{8n}{4n-5} \right)^n$$
$$\leq g^* \leq -\frac{4n}{5} \left( \frac{4n-5}{4n+5} \right) \left( \left( \frac{8n}{4n-5} \right)^n - 1 \right)$$

for $n \geq 3$.

In turn, this results in $\sum_{i=1}^n h_i = O(n2^{3n})$, and, in fact, $\sum_{i=1}^n h_i$ has the same order lower bound as well by the above, noting that

$$\left( \frac{8n}{4n-5} \right)^n = 2^n \left( 1 + \frac{5}{4n-5} \right)^n \to e^{5/4} \cdot 2^n$$

as $n \to \infty$, for a suitably chosen small $\varepsilon > 0$.

## 7.5 Geometrically Decaying Distances to the Redundant Hyperplanes

Next we explore the geometric model for $d$: $d_i = \omega \left(1/\tilde{\varepsilon}\right)^{n-i+1}$, $i = 1, \ldots, n$.

### *7.5.1 Bounding α*

As in Section 7.4.1, we need to guarantee $\beta_1 < d_1$.

Firstly, we give a lower bound on $(\Delta_k)_{k=1,\ldots,n}$ recursively defined by

$$1 - \Delta_{k+1} = \frac{\tilde{d}_{k+1} + 1}{2\tilde{d}_{k+1}} (2 - \Delta_k), \qquad k = 0, \ldots, n-1 \tag{7.12}$$

with $\Delta_0 = 1$, and where

$$\tilde{d}_k = \omega \left(\frac{1}{\tilde{\varepsilon}}\right)^k, \qquad k = 1, \ldots, n$$

with some constant $\omega$.

We have $d_i = \tilde{d}_{n-i+1}$ for $i = 1, \ldots, n$, and

$$\frac{\beta_1}{d_1} = \frac{\tilde{d}_n + 1}{\tilde{d}_n}(1 - \Delta_{n-1}), \qquad \frac{\beta_i}{d_i} = 1 - \Delta_{n-i+1}, \qquad i = 2, \ldots, n.$$

Note that to satisfy $\beta_1 < d_1$ we necessarily must have $1 - \Delta_k < 1$ for $k = 1, \ldots, n-1$.

From (7.12) it follows that

$$\Delta_{k+1} = \frac{\Delta_k}{2} - \frac{\tilde{\varepsilon}^{k+1}}{\omega} + \frac{\Delta_k}{2}\frac{\tilde{\varepsilon}^{k+1}}{\omega} \geq \frac{\Delta_k}{2} - \frac{\tilde{\varepsilon}^{k+1}}{\omega}, \qquad k = 1, \ldots, n-1$$

and hence

$$\Delta_k \geq \frac{\Delta_1}{2^{k-1}} - \frac{\tilde{\varepsilon}^2}{\omega 2^{k-2}}(1 + (2\tilde{\varepsilon}) + (2\tilde{\varepsilon})^2 + \cdots + (2\tilde{\varepsilon})^{k-2}), \qquad k = 2, \ldots, n.$$

Observing $\Delta_1 = \frac{1}{2}(1 - (\tilde{\varepsilon}/\omega))$ we can write the above inequality as

$$\begin{aligned} \Delta_k &\geq \frac{1}{2^k}\left(1 - \frac{\tilde{\varepsilon}}{\omega}\right) - \frac{\tilde{\varepsilon}^2}{\omega 2^{k-2}} \sum_{i=0}^{k-2} (2\tilde{\varepsilon})^i \\ &= \frac{1}{2^k}\left(1 - \frac{\tilde{\varepsilon}}{\omega} - \frac{4\tilde{\varepsilon}^2}{\omega} \sum_{i=0}^{k-2} (2\tilde{\varepsilon})^i\right), \qquad k = 2, \ldots, n. \end{aligned}$$

Now, for $\alpha$ to be positive, that is, for

$$1 - \frac{\tilde{d}_n + 1}{\tilde{d}_n}(1 - \Delta_{n-1}) = 1 - \left(1 - \Delta_{n-1} + \frac{1}{\tilde{d}_n} - \frac{\Delta_{n-1}}{\tilde{d}_n}\right) > 0,$$

it suffices

$$\Delta_{n-1} - \frac{1}{\tilde{d}_n} > 0$$

which is implied by

$$\frac{1}{2^{n-1}}\left(1 - \frac{\tilde{\varepsilon}}{\omega} - \frac{4\tilde{\varepsilon}^2}{\omega} \sum_{i=0}^{n-3} (2\tilde{\varepsilon})^i\right) - \frac{\tilde{\varepsilon}^n}{\omega} > 0.$$

If $\tilde{\varepsilon} = \frac{1}{2}$, the above translates into

$$\frac{1}{2^{n-1}} - \frac{n-1}{\omega \cdot 2^{n-1}} > 0;$$

that is

$$\omega > n - 1$$

resulting in

$$d_i = \omega 2^{n-i+1} > (n-1)2^{n-i+1}, \qquad i = 1, \dots, n.$$

It is left to verify that $h_i \varepsilon^{i-1}/d_i$ is indeed dominated by $h_1/d_1$ for $i = 1, \dots, n$, to ensure (7.4) as in Section 7.3.2. In particular, we demonstrate (7.8), as in the case of uniform $d$. Recalling

$$\frac{\beta_1}{d_1} = \frac{\tilde{d}_n + 1}{\tilde{d}_n}(1 - \Delta_{n-1})$$

and

$$\frac{\beta_i}{d_i} = 1 - \Delta_{n-i+1} \qquad \text{for } i = 2, \dots, n,$$

it immediately follows that

$$\frac{\beta_1}{d_1} > \frac{\beta_2}{d_2}.$$

Also observe

$$\frac{\beta_1}{d_1} > \frac{\beta_2}{d_2} > \frac{\beta_3}{d_3} > \cdots > \frac{\beta_n}{d_n}$$

because, recalling (7.12) and $0 < \Delta_k < 1, k = 1, \dots, n-1$,

$$\begin{aligned}(1 - \Delta_{k+1}) - (1 - \Delta_k) &= \frac{1}{2}\left(2 - \Delta_k + \frac{2 - \Delta_k}{\tilde{d}_{k+1}}\right) - 1 + \Delta_k \\ &= \frac{\Delta_k}{2} + \frac{2 - \Delta_k}{2\tilde{d}_{k+1}} > 0.\end{aligned}$$

### 7.5.2 Picking a "Good" $\omega$

As in Section 7.4.2, we would like to minimize $\alpha$ with respect to $\omega$, which, in the case of $\tilde{\varepsilon} = \frac{1}{2}$, can be well approximated from above by

$$\left(2\tilde{d}_n + 1\right)\left(\Delta_{n-1} - \frac{1}{\tilde{d}_n}\right)^{-1} \approx 2\tilde{d}_n \left(\frac{1}{2^{n-1}} - \frac{n-1}{\omega \cdot 2^{n-1}}\right)^{-1}.$$

We look for

$$\min_{\omega > n-1} \omega \cdot 2^n \left( \frac{1}{2^{n-1}} - \frac{n-1}{\omega \cdot 2^{n-1}} \right)^{-1};$$

that is,

$$\min_{\omega > n-1} \frac{\omega^2}{\omega - n + 1}$$

or equivalently

$$\min_{\omega > n-1} (2 \ln \omega - \ln(\omega - n + 1)).$$

Setting the gradient to 0, we obtain

$$\frac{2}{\omega} - \frac{1}{\omega - n + 1} = \frac{2(\omega - n + 1) - \omega}{\omega(\omega - n + 1)} = 0$$

which gives us the minimizer

$$\omega = 2(n-1)$$

with the corresponding value of $\alpha \approx (n-1)2^{2n+1}$. This results in

$$d_i = (n-1)2^{n-i+2}, \qquad i = 1, \ldots, n$$

and

$$\widetilde{h}_i = O\left( \frac{n 2^{2n}}{(2\varepsilon)^{i-1}} \right), \qquad i = 1, \ldots, n.$$

## 7.5.3 An Explicit Formula

For a given $n, \varepsilon, \delta$, set $d_i = (n-1)2^{n-i+2}$ for $i = 1, \ldots, n$ and compute the solution to $A\widetilde{h} = \mathbf{1}$ using (7.5). Set

$$h = \left\lfloor \left( 1 + \frac{\delta}{3} \max_i \sum_{j=1}^{n} |a_{ij}| \right) \frac{3}{\delta} \widetilde{h} \right\rfloor.$$

From (7.9) it follows that for large $n$

$$\widetilde{h}_i \approx \frac{\alpha}{d_1} B^{-1} \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \leq \alpha \left( \frac{1}{2\varepsilon} \right)^{i-1}, \qquad i = 1, \ldots, n.$$

We choose $\varepsilon, \delta$, to minimize the total number of the redundant constraints, $\sum_{i=1}^{n} h_i$. Recalling $\varepsilon + \delta < 1/2$, for large $n$ we can write

$$\begin{aligned}
\sum_{i=1}^{n} h_i &\approx \frac{3}{\delta}\sum_{i=1}^{n}\widetilde{h}_i \approx \frac{3}{\delta}\sum_{i=1}^{n}\alpha\left(\frac{1}{2\varepsilon}\right)^{i-1} \\
&\approx 3(n-1)2^{2n+1}\,\frac{2}{1-2\varepsilon}\,\frac{1-\left(\frac{1}{2\varepsilon}\right)^n}{1-\frac{1}{2\varepsilon}} \\
&= 3(n-1)2^{2n+2}\,2\varepsilon\frac{\left(\frac{1}{2\varepsilon}\right)^n-1}{(2\varepsilon-1)^2} \\
&\leq 3(n-1)2^{2n+2}\,\frac{\left(\frac{1}{2\varepsilon}\right)^n-1}{(2\varepsilon-1)^2}.
\end{aligned} \tag{7.13}$$

In fact, we would expect $\varepsilon$ to be close to 1/2 in order for $\sum_{i=1}^{n} h_i$ to be minimized, so the last inequality also gives us a good approximation, namely

$$\sum_{i=1}^{n} h_i \approx 3(n-1)2^{2n+2}\,\frac{\left(\frac{1}{2\varepsilon}\right)^n-1}{(2\varepsilon-1)^2}. \tag{7.14}$$

Indeed, denoting

$$\zeta := 2\varepsilon$$

and introducing

$$f(\zeta) := \frac{\left(\frac{1}{\zeta}\right)^n-1}{(\zeta-1)^2}$$

we can write the last two lines in (7.13) as

$$3(n-1)2^{2n+2}\zeta f(\zeta) \leq 3(n-1)2^{2n+2} f(\zeta).$$

Differentiating $\zeta f(\zeta)$ we get

$$(\zeta f(\zeta))' = \frac{(n+1-\zeta^n)(\zeta+1)-2n}{\zeta^n(1-\zeta)^3} < 0 \qquad \text{for } 0 < \zeta < \frac{n-1}{n+1}$$

because $(n+1-\zeta^n)(\zeta+1)-2n < (n+1)(\zeta+1)-2n < 0$ for such $\zeta$, and therefore, the function is decreasing on this interval. So to maximize

$$\zeta f(\zeta) \approx \frac{\sum_{i=1}^{n} h_i}{3(n-1)2^{2n+2}}$$

we must take $\zeta > (n-1)\,/\,(n+1)$, thus justifying (7.14).

Next we demonstrate how to minimize $\sum_{i=1}^{n} h_i$ with respect to $0 < \varepsilon < 1/2$. We note that the approximate minimum of $\sum_{i=1}^{n} h_i$ corresponds to

$$f^* := \min_{0<\zeta<1} f(\zeta).$$

To analyze $f^*$ we proceed as follows: first we demonstrate that for $n$ large enough $f(\zeta)$ is convex on $(0,1)$; then we produce lower and upper bounds on the root of $f'(\zeta) = 0$ in this interval, thus bounding the minimizer; and finally we compute upper and lower bounds on $f^*$ using Taylor expansions of $f(\zeta)$.

Observe

$$f'(\zeta) = \frac{(n+2)\zeta^{-n} - n\zeta^{-n-1} - 2}{(1-\zeta)^3}$$

and

$$f''(\zeta) = \frac{(n^2+6n+6)\zeta^2 - (2n^2+6n)\zeta + n^2 + n - 6\zeta^{n+2}}{\zeta^{n+2}(1-\zeta)^4}.$$

Denoting

$$\hat{a} := n^2 + 6n + 6$$
$$\hat{b} := -(2n^2 + 6n)$$
$$\hat{c} := n^2 + n$$

for $f''(\zeta)$ to be positive on $(0,1)$ it suffices to show that the minimum of

$$\hat{a}\zeta^2 + \hat{b}\zeta + \hat{c}$$

is greater or equal than 6 (recall $0 < \zeta < 1$, so that $6\zeta^{n+2} < 6$). In turn,

$$\min_{\zeta}\left(\hat{a}\zeta^2 + \hat{b}\zeta + \hat{c}\right) = \left(\hat{a}\zeta^2 + \hat{b}\zeta + \hat{c}\right)\Big|_{\zeta=-\hat{b}/2\hat{a}} = \frac{-\hat{b}^2}{4\hat{a}} + \hat{c}$$

so for $f''(\zeta) > 0$ on $(0,1)$ it is enough to have

$$-\frac{(2n^2+6n)^2}{4(n^2+6n+6)} + n^2 + n \geq 6.$$

The above can be rewritten as

$$\frac{4n^3 + 12n^2 + 24n}{4(n^2+6n+6)} = n - \frac{3n^2}{n^2+6n+6} \geq 6$$

and is clearly implied by

$$n - 3 \geq 6.$$

Therefore, $f(\zeta)$ is convex on $(0,1)$ as long as $n \geq 9$.

Furthermore, at

$$\zeta^L := \frac{n-1}{n+1}$$

we have

$$f'(\zeta^L) = \frac{\left(\frac{n+1}{n-1}\right)^n \frac{2}{n-1} - 2}{\left(1 - \frac{n-1}{n+1}\right)^3} < 0, \qquad \text{for } n \geq 2.$$

On the other hand, at

$$\zeta^U := \frac{n-1}{n}$$

we have

$$f'(\zeta^U) = \frac{\left(\frac{n}{n-1}\right)^n \left(\frac{n-2}{n-1}\right) - 2}{\left(1 - \frac{n-1}{n}\right)^3} < 0$$

and relying on all the derivatives of $(1+z)^n$ for $n \in \mathbb{N}$ being positive at $z = 0$, expanding

$$\left(\frac{n}{n-1}\right)^n = \left(1 + \frac{1}{n-1}\right)^n$$

into second-order Taylor series, we get

$$f'(\zeta^U) \geq \frac{\left(1 + \frac{n}{n-1} + \frac{n(n-1)}{2(n-1)^2}\right)\left(\frac{n-2}{n-1}\right) - 2}{\left(1 - \frac{n-1}{n}\right)^3} = \frac{n^2 - 4n}{2(n-1)\left(1 - \frac{n-1}{n}\right)^3} \geq 0$$

for $n \geq 4$.

Now we are in position to give bounds on $f^*$ for $n \geq 9$: by convexity it follows that

$$f(\zeta^L) + f'(\zeta^L)(\zeta^U - \zeta^L) \leq f^* \leq f(\zeta^L);$$

that is,

$$\begin{aligned} 0 &< \left(\frac{n+1}{n-1}\right)^n \left(\frac{(n+1)^2}{4} - \frac{(n+1)^2}{4n}\right) - \frac{(n+1)^2}{4n} \\ &\leq f^* \leq \left(\left(\frac{n+1}{n-1}\right)^n - 1\right) \frac{(n+1)^2}{4}. \end{aligned}$$

Consequently $\sum_{i=1}^n h_i = O(n^3 2^{2n})$ with the same order lower bound in the best case for $\varepsilon \approx (n-1)\,/\,(2(n+1))$.

**Remark 7.6.** Similar analysis can be easily carried out for $\tilde{d}_k = \omega\,(1/\tilde{\varepsilon})^k$ for $k = 1, \ldots, n$, with $\tilde{\varepsilon} = \varepsilon$. Because

$$\begin{aligned} \alpha &\approx \frac{2\tilde{d}_n + 1}{1 - \frac{\tilde{d}_n + 1}{\tilde{d}_n}(1 - \Delta_{n-1})} \\ &\approx \frac{2\omega}{\varepsilon^n} \left( \frac{1}{2^{n-1}} \left( 1 - \frac{\varepsilon}{\omega} - \frac{4\varepsilon^2}{\omega} \sum_{i=0}^{n-3} (2\varepsilon)^i \right) - \frac{\varepsilon^n}{\omega} \right)^{-1} \\ &= \frac{2^n}{\varepsilon^n} \frac{\omega^2}{\omega - \left(\varepsilon + (2\varepsilon)^2 \sum_{i=0}^{n-3} (2\varepsilon)^i + (2\varepsilon)^{n-1}\varepsilon\right)} \end{aligned}$$

and is approximately minimized at $\omega = 2\left(\varepsilon + (2\varepsilon)^2 \sum_{i=0}^{n-3} (2\varepsilon)^i + (2\varepsilon)^{n-1}\varepsilon\right)$, resulting in $\alpha \approx 4\,(2/\varepsilon)^n \left(\varepsilon + (2\varepsilon)^2 \sum_{i=0}^{n-3} (2\varepsilon)^i + (2\varepsilon)^{n-1}\varepsilon\right)$, we have

$$\sum_{i=1}^{n} h_i \approx \frac{3}{\delta} \sum_{i=1}^{n} \widetilde{h}_i \approx 3\frac{2}{1-2\varepsilon} 4 \left(\frac{2}{\varepsilon}\right)^n \left(\varepsilon + (2\varepsilon)^2 \sum_{i=0}^{n-3} (2\varepsilon)^i + (2\varepsilon)^{n-1}\varepsilon\right) n$$

$$= 12n2^{2n} \frac{1}{(1-2\varepsilon)(2\varepsilon)^n} \left(2\varepsilon + (2\varepsilon)^n + 2(2\varepsilon)^2 \frac{1-(2\varepsilon)^{n-2}}{1-2\varepsilon}\right).$$

Noting that

$$\frac{1}{(1-2\varepsilon)(2\varepsilon)^n} \left(2\varepsilon + (2\varepsilon)^n + 2(2\varepsilon)^2 \frac{1-(2\varepsilon)^{n-2}}{1-2\varepsilon}\right) \le \frac{2(n-1)}{(1-2\varepsilon)(2\varepsilon)^n}$$

for $\varepsilon \in (0, 1/2)$ and thus the latter may be bounded from above by the value of the right-hand side at $\varepsilon = n/\left(2(n+1)\right)$, that is, $2e(n+1)(n-1)$, we get

$$\sum_{i=1}^{n} h_i \le 24e(n+1)(n-1)n2^{2n} = O(n^3 2^{2n}).$$

The resulting estimate for the number of the redundant constraints is not much different from the case of $\tilde{\varepsilon} = 1/2$, so this model for $d$ is not discussed here in any more details.

## 7.6 Conclusions

We provide sufficient conditions for the central path to intersect small neighborhoods of all the vertices of the Klee–Minty $n$-cube; see Propositions 7.1, 7.2, and Corollary 7.1. More precisely, we derive explicit formulae for the number of redundant constraints for the Klee–Minty $n$-cube example given in [4, 5]. We give a smaller number of redundant constraints of order $n2^{3n}$ when the distances to those are chosen uniformly, as opposed to the previously established $O(n^2 2^{6n})$. When these distances are chosen to decay geometrically, we give a slightly tighter bound of the same order $n^3 2^{2n}$ as in [5], that results in a provably smaller number of constraints in practice.

We argue that in $\mathbb{R}^2$ the sufficient conditions presented are tight and indicate that the same is likely true in higher dimensions.

Our construction potentially gives rise to linear programming instances that exhibit the worst case iteration-complexity for path-following interior-point methods, which almost matches its theoretical counterpart.

Considering the $n$-dimensional simplex, Megiddo and Shub [8] demonstrated that the total curvature of the central path can be as large as order

$n$. Combined with Corollary 7.2, it follows that the worst-case lower bound on the total curvature of the central path is at least exponential in $n$ up to a rational multiplier. We conjecture that the total curvature of the central path is $O(m)$; see [3].

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