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On component commonality for periodic review assemble-to-order systems

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Abstract Akçay and Xu (Manag Sci 50(1):99–116, 2004) studied a periodic review assemble-to-order (ATO) system with an independent base stock policy and a first-come-first-served allocation rule, where the base stock levels and the component allocation are optimized jointly. The formulation is non-convex and, thus theoretically and computationally challenging. In their computational experiments, Akçay and Xu (Manag Sci 50(1):99–116, 2004) modified the right hand side of the inventory availability constraints by substituting linear functions for piece-wise linear ones. This modification may have a significant impact for low budget levels. The optimal solutions obtained via the original formulation; that is, without the modification, include zero base stock levels for some components and indicate consequently a bias against component commonality. We study the impact of component commonality on ATO systems. We show that lowering component commonality may yield a higher type-II service level. The lower degree of component commonality is achieved via separating inventories of the same component for different products. We substantiate this property via computational and theoretical approaches. We show that for low budget levels the use of separate inventories of the same component for different products could achieve a higher reward than with shared inventories. Finally, considering a simple ATO system consisting of one component shared by two products, we characterize the budget ranges such that

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the use of separate inventories is beneficial, as well as the budget ranges such that component commonality is beneficial.

Keywords Assemble-to-order · Component commonality · Stochastic programming · Inventory replenishment · Component allocation

1 Introduction

Given the pressure of high capital costs and the competitive environment in manufacturing, more and more manufacturers have adopted assemble-to-order (ATO) systems to increase product customization and reduce response time. The main difference between ATO and make-to-stock (MTS) approaches is that ATO eliminates the necessity for final product inventories. When a customer order arrives, an ATO system satisfies the order by assembling the products from component inventories. Manufacturers will benefit from an ATO system if their product assembly times are negligible compared with their component replenishment lead times. While ATO systems provide numerous benefits, efficiently matching the demand and the supply for ATO systems is a challenging task. In particular, if the matching problem is not efficiently solved, those benefits may be offset, see Song and Zipkin (2003). Our approach focuses on a periodic review ATO system with an independent base stock policy and a first-come-first-served (FCFS) allocation rule. We analyze the theoretical and computational aspects of the formulation of Akçay and Xu (2004) which jointly optimizes the base stock levels and the component allocation. In particular, we discuss the impact of substituting linear inventory availability constraints for piece-wise linear ones in the Akçay and Xu formulation and the efficiency of component commonality for ATO systems.

1.1 Literature review

Component commonality is widely adopted and often preferred in ATO systems in order to offset the reduction of economies of scale when providing customized products. The economic impact of component commonality for single period models has been extensively studied. Eynan and Rosenblatt (1996) presented three models to compare and analyze the effects of increasing component commonality, and demonstrated that some forms of commonality might not always be beneficial. They also provided conditions for which commonality should be either employed or avoided. Mirchandani and Mishra (2002) compared a non-commonality model with two different commonality models—based on whether or not the products are prioritized—for a system with two products and independent uniform demand distributions. They derived theoretical conditions when component commonality is beneficial for this specific system. Both Eynan and Rosenblatt (1996) and Mirchandani and Mishra (2002) allowed the common component to be more expensive than those it replaces. However, in our formulation, we apply component commonality to the inventory management rather than to the design process. We assume, like Baker et al. (1986) and Gerchak et al. (1988), that the costs of the dedicated component and the common component are identical. Baker et al. (1986) studied the effect of component commonality on optimal safety stock levels for an ATO system with two end-products and two components. They considered the problem of minimizing safety stock levels while satisfying a service level constraint under independent uniform demand distributions and showed that component commonality induced a reduction in the optimal safety stock levels. Gerchak et al. (1988) extended this work by investigating whether the results hold for a system with an arbitrary number of products and a general joint demand distribution.

In contrast to the above works where a single period model is assumed, our commonality study focuses on a multi-period model. Considering a simple multi-period ATO model, Hillier (1999) observed that component commonality is not always beneficial. Hillier (1999) studied a periodic review ATO system with zero lead times and uniformly distributed demands, and derived a closed-form solution for a cost minimization model with service level constraints. The results demonstrated that, for a multi-period model, the use of a common component is always beneficial if its price does not exceed the price of the replaced components. If the common component is more expensive than the replaced ones, then in contrast to the single period case, it is almost never beneficial to use it. Hillier (2000) further extended these results to systems with an arbitrary number of final products and components. Song and Zhao (2009) considered a continuous-review ATO system with one common component, two end products, and Poisson distributed demands, and showed that, while component commonality is generally beneficial, its added value depends strongly on the component costs, lead times, and allocation rules. Based on the general setting proposed by Huang and Kok (2015), our approach aims at further analyzing complicated ATO systems while taking into account component commonality. Huang and Kok (2015) considered a periodic-review ATO system with component base stock policy and correlated demands, and presented a FCFS formulation for a cost minimization model involving the inventory holding cost, remnant stock holding cost and backlogging cost.

In the literature reviewed above, minimizing inventory level or inventory cost subject to some service level constraints is commonly used to model ATO systems. However, the problem we consider follows another line of research: component commonality for systems with a given budget for all the components. Jönsson and Silver (1989) analyzed the impact of component commonality for an ATO system with two end products and two components, with one being common to both products. Fong et al. (2004) pursued the approach of Baker et al. (1986) and provided analytical formulations for a commonality problem minimizing the expected shortage subject to a fixed budget constraint and assuming independent Erlang demand distributions. They observed that the relative reduction in the expected shortage can be substantial when the budget level is high relative to the demand requirements for the end products—even if the component is much more expensive. Note that all these models assume a single period.

Another relevant work is Nonås (2009) who formulated a two-stage stochastic program for an ATO system with three products and an arbitrary number of components, and introduced a gradient-based search method to find the optimal inventory levels for a profit maximization problem. The key difference is that we consider a budget constraint.

1.2 Akçay and Xu formulation

Following the model proposed by Akçay and Xu (2004), we assume:

(1) a periodic review system,
(2) an independent base stock policy is used for each component,
(3) the product demands are satisfied by a FCFS rule,
(4) the product demands are correlated within each period, while the demands over different periods are independent,
(5) the replenishment lead time for each component is constant,
(6) a product reward is collected if the assembly is completed within the given time window.

In addition, the following sequence of events is assumed for each period:

(i) inventory position reviewed,
(ii) new replenishment order of components placed,
(iii) earlier component replenishment order arrive,
(iv) demand realized,
(v) component allocated and product assembled,
(vi) associated rewards accounted for.

In this model, assembly takes zero time while component lead times are greater than zero. The model is based on a multi-matching approach proposed by Huang and Kok (2015) and Huang (2014) where multiple components are matched with multiple products to satisfy demands. In each period within the time window, rewards are collected by satisfying product demands. We recall that the time window is the number of periods between the order receiving period and the order fulfillment period. In particular, a time window equal to 0 means that the demand must be fulfilled within the period the order is received; that is, we must have enough components to satisfy the demand within that period in order to collect rewards. The base stocks of the ATO system are constrained by a pre-set overall budget. The approach is based on a two-stage decision model. The first stage consists of determining a base stock level for each component, and the second stage consists of determining products that need to be assembled in each period with respect to some constraints reflecting the inventory availability. The first stage decisions are made before the second stage decisions following a two-stage stochastic programming framework, see Birge and Louveaux (2011). The objective of the approach is to maximize the expected total reward collected from the products assembled within given time windows. Note that while all products are eventually assembled within $L+1$ periods, the rewards are collected only within the pre-set time windows. The notations are summarized in Table 1.

The second stage corresponds to the allocation problem $\big(Alloc(S,\xi)\big)$, where $S=(S_i)$ is the vector representing base stock levels, $\xi=\{P_{j,k}|j=1,\ldots,m;\ k=0,-1,\ldots,-L\}$ is the vector representing random demands, and $O_{i,k}$ is the number of component i available at period k. Note that $O_{i,k}=(S_i-D_i^{L_i-k})^+$ for $0\le k\le L_i$ where $D_i^{L_i-k}=\sum_{s=0}^{L_i-k}D_{i,-s}$, and $O_{i,k}=D_{i,0}$ for $L_i+1\le k\le L+1$ are inferred from the base stock policy and a FCFS rule, see Huang and Kok (2015) and Huang (2014).

$$\max\sum_{j=1}^{m}\sum_{k=0}^{w_j}(r_{j,k}\times x_{j,k}) \qquad (Alloc(S,\xi))$$

$$\sum_{k=0}^{L+1}x_{j,k}=P_j \quad j=1,\ldots,m$$

$$\sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j}\times x_{j,\mu})\le O_{i,k} \quad i=1,\ldots,n,\quad k=0,\ldots,L+1$$

$$x_{j,k}\in\mathbb{Z}_+ \quad j=1,\ldots,m,\quad k=0,\ldots,L+1$$

The first set of constraints guarantees that assembly will satisfy customer demand. The second set of constraints—called inventory availability constraints—guarantees that assembly could only happen when there are enough component inventories. While an optimal allocation can be computed for a given base stock level S and demand ξ, we still need to determine the optimal base stock levels. Thus, we use the two-stage stochastic integer program $\big(Joint(B)\big)$

Table 1 Notations

n	Number of components
m	Number of products
i	Index of component $i = 1, \ldots, n$
j	Index of product $j = 1, \ldots, m$
S_i	Base stock level of component $i = 1, \ldots, n$
c_i	Unit base stock level cost of component $i = 1, \ldots, n$
L_i	Lead time of component $i = 1, \ldots, n$
L	Maximum lead time among all components; that is, $L = \max_{i=1}^{n} L_i$
w_j	Time window of product j
k	Index of period k corresponding to the duration $[k, k+1)$; $k = 0$ implies the current period; negative values of k imply previous periods
$x_{j,k}$	Number of product j assembled in period k
$r_{j,k}$	Reward for satisfying the demand for product j in period k
$a_{i,j}$	Number of component i used to assemble one unit of product j; that is, the bill of materials (BOM)
B	The budget, i.e., $\sum_{i=1}^{n} c_i \times S_i \leq B$
$P_{j,k}$	Demand of product j at period k
P_j	Demand of product j at the current period; that is, $P_{j,0}$
$D_{i,k}$	Demand of component i at period k; that is, $D_{i,k} = \sum_{j=1}^{n} a_{i,j} P_{j,k}$
M	Number of independent samples
N	Number of realizations in one sample
l	Index of sample $l = 1, \ldots, M$
h	Index of realization $h = 1, \ldots, N$
x^+	The positive part of x; that is, $x^+ = (\lvert x \rvert + x)/2$

where the first stage determines the base stock levels and the the second stage maximizes the expectation of the component allocations:

$$\begin{aligned} \max \quad & \mathbf{E}[Alloc(S, \xi)] \\ & \sum_{i=1}^{n} (c_i \times S_i) \leq B \\ & S_i \in \mathbb{Z}_+ \quad i = 1, \ldots, n \end{aligned} \qquad (Joint(B))$$

We recall in Sect. 1.3 the sample average approximation method used to solve $\big(Joint(B)\big)$.

1.3 Sample average approximation method

The sample average approximation (SAA) method, see Kleywegt et al. (2002), consists of the following steps:

(i) generate M independent samples for $l = 1, \ldots, M$ with N realizations for each sample. The vector $\xi_l^N = (\xi(\omega_l^1), \xi(\omega_l^2), \ldots, \xi(\omega_l^N))$ represents the N realizations of the l-th sample,

(ii) solve the optimization problem $(INLP)$ for each sample, which is the associated deterministic version of $\big(Joint(B)\big)$. where the objective function is set to $\frac{1}{N}\sum_{h=1}^{N} Alloc(S, \xi(\omega_l^h))$ as described below. Note that $(INLP)$ is non-linear not only due to the integrality constraints but also due to the right hand side of the inventory availability constraints. Let $\hat{S}_l$ denote the optimal base stock levels for $(INLP)$ and $\hat{G}(\hat{S}_l)$ denote its optimal objective value.

$$\max \quad \frac{1}{N}\sum_{h=1}^{N}\sum_{j=1}^{m}\sum_{k=0}^{w_j}(r_{j,k} \times x_{j,k}^h) \tag{INLP}$$

$$\sum_{k=0}^{L+1} x_{j,k}^h = P_j^h \quad j = 1, \ldots, m, \quad h = 1, \ldots, N$$

$$\sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j} \times x_{j,\mu}^h) \le O_{i,k}^h \quad i = 1, \ldots, n, \quad k = 0, \ldots, L+1, \quad h = 1, \ldots, N$$

$$\sum_{i=1}^{n}(c_i \times S_i) \le B$$

$$S_i \in \mathbb{Z}_+ \quad i = 1, \ldots, n$$

$$x_{j,k}^h \in \mathbb{Z}_+ \quad j = 1, \ldots, m, \quad k = 0, \ldots, L+1, \quad h = 1, \ldots, N$$

(iii) generate a different sample $\xi^{N'}$ with $N' \gg N$ realizations and compare the performance among all the base stock vectors $\hat{S}_l$ solved in (ii) by solving $\big(Alloc(S, \xi^{N'})\big)$ with $S = \hat{S}_l$. Let $\bar{G}(\hat{S}_l)$ be the new optimal objective value.

(iv) select the optimal base stock vector $\hat{S}^*$ achieving the best performance among all the base stock vectors; that is, $\hat{S}^* = \text{argmax}\{\bar{G}(\hat{S}_l) : l = 1, \ldots, M\}$.

Let $\hat{G}_M = \frac{1}{M}\sum_{l=1}^{M} \hat{G}(\hat{S}_l)$, $\bar{G}_{N'} = \bar{G}(\hat{S}^*)$, and G^* be the optimal objective value of $\big(Joint(B)\big)$. Since $\bar{G}_{N'} \le G^* \le \hat{G}_M$ under certain conditions for N, M, N', see Birge and Louveaux (2011), $\bar{G}_{N'}$ and $\hat{G}_M$ are, respectively, a lower and an upper bound for G^*. For more details concerning the statistical testing of optimality for the SAA method, and the selection of N, M, and N', see Kleywegt et al. (2002). Note that $O_{i,k} = (S_i - D_i^{L_i-k})^+$ is a non-convex function of S_i; and we use the standard Big-M method to check whether $(S_i - D_i^{L_i-k})$ is positive.

2 Impact of modifying the inventory availability constraints

While the $(INLP)$ formulation uses a plus sign in the right hand side of the inventory availability constraints, $(S_i - D_i^{L_i-k})^+$ is substituted by $(S_i - D_i^{L_i-k})$ in the computational experiments performed by Akçay (2002). The obtained new formulation (ILP) allows faster computation. Note that the feasible region of (ILP) is a subset of the feasible region of $(INLP)$. In addition, while relaxing the integrality constraints on the variables would make (ILP) convex, $(INLP)$ would remain non-convex due to the $(S_i - D_i^{L_i-k})^+$ term in the right hand side of the inventory availability constraints. Note that substituting $(S_i - D_i^{L_i-k})$ for $(S_i - D_i^{L_i-k})^+$ may lead to infeasibility. This issue can be addressed by filtering out samples leading to infeasibility and by assuming sufficiently large budget level; that is, by allowing large base stock levels. We argue that substituting $(S_i - D_i^{L_i-k})$ for $(S_i - D_i^{L_i-k})^+$ might yield an intractable sample generation process for the SAA approach for low budget levels.

2.1 Impact of modifying the inventory availability constraints on the sample generation

Generating enough samples such that the associated (ILP) formulation is feasible could be highly challenging for low budget levels. Note that under the extreme case of setting the budget to zero, the only sample yielding a feasible formulation is the trivial zero sample. Disregarding infeasible ones, we generate samples for (ILP) until the required number of samples, or a pre-set number of feasibility tests, is reached. For a given budget, the feasibility check is done by comparing with a computed minimum budget for a sample having a feasible solution. The computed minimum budget is determined from the (ILP) minimum base stock levels using Algorithm 1 described below. The non-negativity of the left hand side of the inventory availability constrains implies $(S_i - D_i^{L_i-k}) \geq 0$. Note that while we can generate enough feasible samples for (ILP), the mean and variance—i.e., the distribution—of generated sample are impacted and, thus, the SAA method.

Algorithm 1 Computing minimum feasible budget

Initialize $maxS \leftarrow zeros(n)$
for any realization h **do**
  for for any component i **do**
    if $D_i^{L_i} > maxS(i)$ **then**
      $maxS(i) \leftarrow D_i^{L_i}$
    end if
  end for
end for
$B = \sum_{i=1}^{n} c_i \times maxS(i)$

2.2 Impact of modifying the inventory availability constraints on the SAA method

Following the notation and discussion of Sect. 1.3, let $\bar{G}^{\bullet}_{N'}$, $G^*_{\bullet}$, and $\hat{G}^{\bullet}_M$ denote respectively the (ILP) lower bound, optimal value, and upper bound. Since $x \leq x^+$, any feasible solution of (ILP) is a feasible solution of $(INLP)$. In addition, this inclusion is typically strict as one can set some base stocks to zero to build a solution feasible for $(INLP)$ but infeasible for (ILP). To ensure a fair comparison, we only consider samples yielding feasible (ILP) and $(INLP)$ formulations. Since, for a given sample, the optimal objective value for $(INLP)$ is at least the one for (ILP), we have $G^{\bullet}_M \leq \hat{G}_M$.

2.3 Computational results for the Zhang system

We consider an ATO system proposed in 1997 by Zhang (1997) with four products and five components as described in Table 2. The computational results are presented in Table 3 where *N/A* corresponds to budgets for which not enough sample yielding a feasible formulation can be generated, and *LB* and *UB* denote, respectively, the lower and upper bounds for the (ILP) and $(INLP)$ formulations. The parameters for the SAA method are set to: $N = 25$, $N' = 100$ and $M = 5000$. If a million samples are not enough to yield 100 feasible $(INLP)$ samples, the process stops and outputs *N/A*.

Table 2 Settings for the Zhang system

						Component				
					i	1	2	3	4	5
					c_i	2	3	6	4	1
Product					L_i	3	1	2	4	4
j	Mean	Std Dev	r_j	w_j		Bill of materials				
1	100	25	1	0		1	2	1	0	0
2	150	30	1	0		1	1	1	0	0
3	50	15	1	0		0	1	1	1	0
4	30	11	1	0		0	0	0	1	1

Table 3 Type-II service levels for (*ILP*) and (*INLP*) for the Zhang system

Budget	(*ILP*)-LB	(*ILP*)-UB	(*INLP*)-LB	(*INLP*)-UB
2000	N/A	N/A	9.08	9.11
3000	N/A	N/A	9.08	9.12
4000	N/A	N/A	9.46	9.88
5000	N/A	N/A	21.59	22.98
6000	N/A	N/A	46.47	47.83
7000	N/A	N/A	65.78	66.49
7500	57.74	60.03	71.73	71.99
8000	68.58	70.54	74.88	75.01
8500	79.78	80.22	81.13	82.40
9000	87.85	88.85	89.07	90.02
9500	92.68	93.60	94.77	95.35
10000	97.76	98.12	98.20	98.34
10500	98.69	98.89	99.88	99.59
11000	99.62	99.66	100.50	99.86

Remark that systems with low service level occur, for example in the IT industry. In particular, instances of low service level occur under so-called *hunger marketing*, or *guerrilla marketing*, strategies which are driven not only by targeting more sales, but also by restricted capital cost; that is, low budget. Consider for example a new iPhone released by Apple. The service level is actually low given the fact that the time window is small. Similarly, Xiaomi grew exponentially in China over the last few years by reducing significantly the required capital cost. Xiaomi service level is quite low but, despite many resulting complaints, the activity flourished as Xiaomi could cut competitors' price in half. In other words, reducing the service level could be strategic in some situations.

2.4 Computational results for the IBM system

We consider an ATO system proposed in 2002 by Cheng et al. (2002) with six products and seventeen components as described in Table 4. The computational results are presented in Fig. 1 where *LB* and *UB* denote, respectively, the lower and upper bounds for the (*ILP*) and (*INLP*) formulations. The parameters for the SAA method are set to $N = 25$, $N' = 100$, and $M = 1000$. If a million samples are not enough to yield 100 feasible (*INLP*) samples,

Table 4 Settings for the IBM system

			Component						
			j	1	2	3	4	5	6
			c_i	1363	1595	1765	1494	1494	1628
i	c_i	L_i		Bill of materials					
1	42	5		1	1	1	1	1	1
2	114	5		1	1	1	1	1	1
3	114	5		1	1	1	1	1	1
4	307	5		1	0	0	0	0	0
5	538	5		0	1	0	0	0	0
6	395	5		0	0	1	1	1	0
7	790	5		0	0	0	0	0	1
8	290	5		1	1	1	1	1	1
9	155	5		1	1	0	0	0	1
10	198	5		0	0	1	1	1	0
11	114	5		1	1	1	1	1	1
12	114	5		1	1	1	0	1	0
13	114	5		0	0	0	1	0	0
14	43	5		0	0	1	0	0	0
15	114	5		0	0	1	0	0	0
16	114	5		1	1	1	1	1	0
17	114	5		0	0	1	0	0	0

the process stops and the reward is set to 0. In particular, a budget of at least 11 million is needed to yield 100 feasible (*INLP*) samples for the (*ILP*) model, see Fig. 1.

3 Component commonality for specific ATO systems

3.1 Component commonality for the Zhang system

The computational experiments performed for the Zhang system with (*INLP*) formulation show that, for some low level budgets, the optimal base stock levels of some components are set to zero, see Table 5. This computation indicates a bias against component commonality and suggests that dedicating the components to different products may yield a higher objective value. For example, for a budget of 2000, the inventory levels for C_1, C_2 and C_3 are set to zero implying that an optimal solution only considers assembling product 4. Similarly, for a budget between 5000 and 8000, the optimal base stock levels for components C_4 and C_5 are set to zero and thus products 3 and 4 are ignored.

We propose a model separating component inventories with respect to the different products; that is, each product is served by dedicated components. We consider a modified BOM for the Zhang system as described in Table 6. In the first row, the subscript is the component index in the original BOM, and the superscript is the index of the product served by the component. The components with the same subscript have the same cost and lead time. Computational experiments, presented in Table 7, are performed to compare the Zhang system with maximum component commonality, denoted as (*INLP*), and the Zhang system with no component commonality, denoted as ($INLP_\Delta$). Table 7 indicates that the ($INLP_\Delta$)

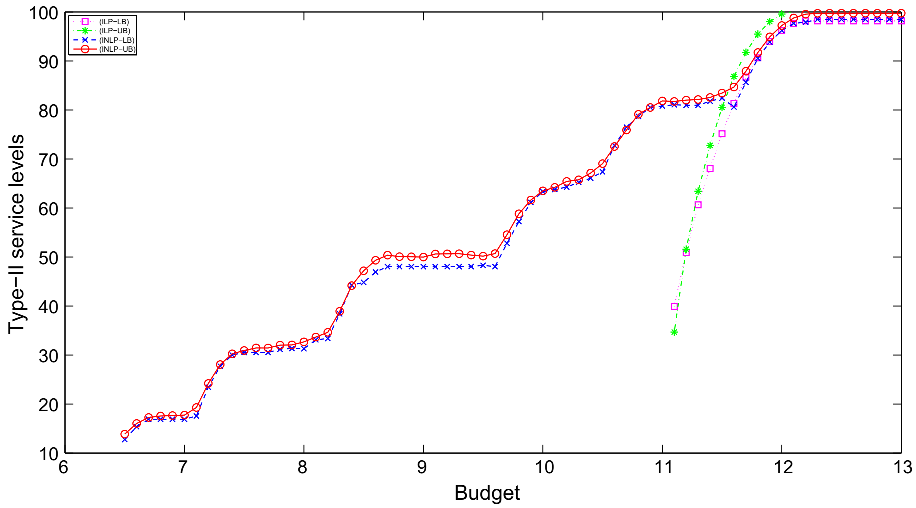

Fig. 1 Type II service levels for (*ILP*) and (*INLP*) for the IBM system

Table 5 Optimal base stock levels and Type-II service levels for the Zhang system

Budget	C_1	C_2	C_3	C_4	C_5	**LB**	**UB**
2000	0	0	0	428	199	9.08	9.11
3000	0	0	162	413	376	9.08	9.12
4000	0	325	249	339	175	9.46	9.88
5000	613	492	383	0	0	21.59	22.98
6000	699	598	468	0	0	46.47	47.83
7000	782	722	545	0	0	65.78	66.49
7500	819	786	584	0	0	71.73	71.99
8000	865	846	622	0	0	74.88	75.01
8500	766	727	562	316	151	81.13	82.40
9000	793	779	595	339	151	89.07	90.02
9500	823	835	632	350	157	94.77	95.35
10000	855	876	665	377	163	98.20	98.34
10500	883	932	696	400	162	99.88	99.59
11000	899	981	744	402	187	100.50	99.86

model outperforms the original (*INLP*) model for a budget no greater than 8000. While the gap decreases with the increase of the budget, it is significant for a low to medium budget. Since the model uses a FCFS policy, we need to satisfy all the demands that are ahead of time t to gain reward for a product for current period, i.e., at time t. Then, there could be insufficient inventories to meet the high reward demand generated at time t. Intuitively, our results highlight that tweaking the domaine of the FCFS policy can be beneficial.

3.2 Component commonality for the IBM system

We compare our formulation and the original one for the IBM system in Fig. 2. Our formulation is at least as good as the original one for a service level around 80%, and a budget

Table 6 Bill of materials for the Zhang system without component commonality

	C_1^1	C_2^1	C_3^1	C_1^2	C_2^2	C_3^2	C_2^3	C_3^3	C_4^3	C_4^4	C_5^4
P_1	1	2	1	0	0	0	0	0	0	0	0
P_2	0	0	0	1	1	1	0	0	0	0	0
P_3	0	0	0	0	0	0	1	1	1	0	0
P_4	0	0	0	0	0	0	0	0	0	1	1

Table 7 Type II service levels for $(INLP)$ and $(INLP_\Delta)$ for the Zhang system

Budget	$(INLP)$-**LB**	$(INLP)$-**UB**	$(INLP_\Delta)$-**LB**	$(INLP_\Delta)$-**UB**
2000	9.08	9.11	14.26	14.40
3000	9.08	9.12	25.62	26.60
4000	9.46	9.88	43.53	43.70
5000	21.59	22.98	53.72	53.67
6000	46.47	47.83	58.44	59.40
7000	65.78	66.49	68.73	69.78
7500	71.73	71.99	74.85	75.67
8000	74.88	75.01	79.74	80.24
8500	81.13	82.40	82.83	82.92
9000	89.07	90.02	84.39	85.29
9500	94.77	95.35	89.54	90.59
10000	98.20	98.34	94.05	94.68
10500	99.88	99.59	97.12	97.35
11000	100.50	99.86	98.94	98.89

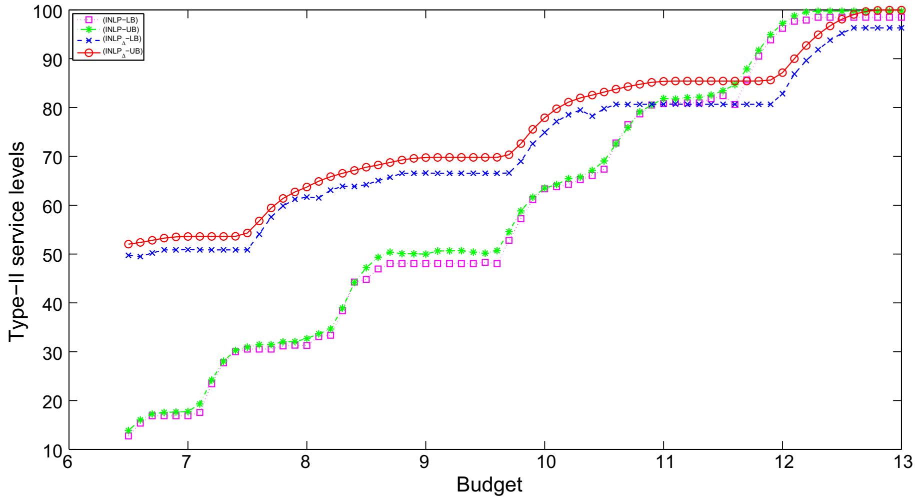

Fig. 2 Type II service levels for $(INLP)$ and $(INLP_\Delta)$ for the IBM system

up to around 11.5 million. For a budget above 12.5 million, both formulations yield similar service level as the budget is large enough to satisfy all the demands immediately. For a budget between 11.5 and 12.5 million, the benefit of common component is clear.

Table 8 Bill of materials for Λ- and Λ_Δ-systems

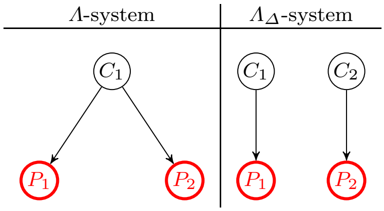

Table 9 Parameters for Λ- and Λ_Δ-systems

	Mean	Std Dev	$r_{j,0}$
P_1	100	25	1
P_2	150	30	1

3.3 Component commonality for Λ-system

While in Sect. 3.1 the gap between the $(INLP_\Delta)$ and $(INLP)$ models is substantiated computationally, we provide a theoretical analysis for a simpler system, denoted Λ-system, consisting of one component shared by two products. See Table 8 for a description of the original Λ-system and of our modified model, denoted Λ_Δ-system, that removes component commonality (Table 9).

To simplify the analysis, the component costs and product rewards are set to 1, and the product time windows are set to 0. The corresponding SAA formulations $(INLP^N)$ and $(INLP^N_\Delta)$ are as follows:

$$\max \quad \frac{1}{N}\sum_{h=1}^{N}(x_1^h + x_2^h) \tag{$INLP^N$}$$
$$x_1^h + x_2^h \le (B - D_1^h - D_2^h)^+ \quad h = 1, \dots, N$$
$$x_1^h \le P_1^h, \quad x_2^h \le P_2^h \quad h = 1, \dots, N$$
$$x_1^h, x_2^h \in \mathbb{Z}_+ \quad h = 1, \dots, N$$

$$\max \quad \frac{1}{N}\sum_{h=1}^{N}(x_1^h + x_2^h) \tag{$INLP^N_\Delta$}$$
$$x_1^h \le (B_1 - D_1^h)^+ \quad h = 1, \dots, N$$
$$x_2^h \le (B_2 - D_2^h)^+ \quad h = 1, \dots, N$$
$$x_1^h \le P_1^h, \quad x_2^h \le P_2^h \quad h = 1, \dots, N$$
$$B_1 + B_2 = B$$
$$x_1^h, x_2^h \in \mathbb{Z}_+ \quad h = 1, \dots, N$$
$$B_1, B_2 \in \mathbb{R}_+.$$

Theorem 1 characterizes the budget ranges such that component commonality is beneficial for Λ-system over Λ_Δ-system. For example, the "$<$" sign means that common commonality is non-beneficial for a budget ranging from B_{min} to B^+_{min} as specified in Theorem 1. The proof of Theorem 1 is given in Sect. 4.

Table 10 Λ_{Δ}-system where a positive sign corresponds to common commonality being beneficial

N / B	$[0, B_{min}]$	$(B_{min}, B_{min}^{+}]$	$(B_{min}^{+}, B_{max}^{+}]$	$(B_{max}^{+}, B_{max}^{\Sigma}]$	$(B_{max}^{\Sigma}, +\infty)$
1	$=$	$<$	$<$	$\leq$	$=$
2	$=$	$<$	$\leq$	$\leq$ or $>$	$=$
N_0	$=$	$<$	$\leq$ or $>$	$\leq$ or $>$	$=$

Theorem 1 *Given a budget B, let $f^*(B)$ and $f_{\Delta}^*(B)$ be the optimal objective values of $(INLP^N)$ and $(INLP_{\Delta}^N)$. Considering the cases $N = 1, 2$ and N_0, the sign of $f^*(B)$ - $f_{\Delta}^*(B)$ is given in Table 10 where*

$$\begin{aligned} B_{min} &= \min_{i=1}^{2}\{\min_{h=1}^{N}\{D_i^h\}\} \\ B_{min}^{+} &= \min_{h=1}^{N}\{D_1^h + D_2^h\} \\ B_{max}^{+} &= \max_{h=1}^{N}\{D_1^h + D_2^h\} \\ B_{max}^{\Sigma} &= \sum_{i=1}^{2} \max_{h=1}^{N}\{D_i^h + P_i^h\}. \end{aligned}$$

Some intuition behind Theorem 1 arises from the non-convexity of the formulation resulting from the right hand side of the inventory availability constraints, i.e., $O_{i,k} = (S_i - D_i^{L_i-k})^+$. If the budget level is high enough, the inventory availability constraints become classic linear constraints and, thus, the non-convexity decreases. Using a management science formulation; if the base stock is large enough to meet the demand, the inventory availability become less important and the number of back orders decrease. If the budget level, which can de derived from the sample, can meet the demand, the base stock is not an issue.

4 Proof of Theorem 1

4.1 Case $N = 1$

We first consider the case $N = 1$; that is for one realization in the SAA method. The associated formulations $(INLP^1)$ and $(INLP_{\Delta}^1)$ correspond to a deterministic demand where P_1^1 and P_2^1 represent the demands in the current period for, respectively, product 1 and 2, and D_1^1 and D_2^1 represent the overall demands from all previous periods. The budget level B is given and since the cost of the component is set to one, the budget level is equivalent to the base stock level.

$$\begin{aligned} \max \quad & x_1^1 + x_2^1 \qquad\qquad (INLP^1) \\ & x_1^1 + x_2^1 \leq (B - D_1^1 - D_2^1)^+ \\ & x_1^1 \leq P_1^1, \quad x_2^1 \leq P_2^1 \\ & x_1^1, x_2^1 \in \mathbb{Z}_+ \end{aligned}$$

$$\max \quad x_1^1 + x_2^1 \qquad (INLP_\Delta^1)$$
$$x_1^1 \le (B_1 - D_1^1)^+$$
$$x_2^1 \le (B_2 - D_2^1)^+$$
$$x_1^1 \le P_1^1, \quad x_2^1 \le P_2^1$$
$$B_1 + B_2 = B$$
$$x_1^1, x_2^1 \in \mathbb{Z}_+$$
$$B_1, B_2 \in \mathbb{R}_+$$

Property 1 Given a budget B, let $f^*(B)$ and $f_\Delta^*(B)$ be the optimal objective values of ($INLP^1$) and ($INLP_\Delta^1$). Both $f^*(B)$ and $f_\Delta^*(B)$ are monotonically non-decreasing with B and $f^*(B) \le f_\Delta^*(B)$. □

Proof Since the feasible region of ($INLP^1$) for a given B is a subset of the feasible region of ($INLP^1$) for $B' \ge B$, $f^*(B)$ is non-decreasing with B increasing. The same holds for $f_\Delta^*(B)$. We then prove that $f^*(B) \le f_\Delta^*(B)$ by showing that an optimal solution for ($INLP^1$) yields a feasible solution for ($INLP_\Delta^1$). Assume first that an optimal solution for ($INLP^1$) satisfies $(x_1^1)^* = 0$. Then, the solution $\hat{x}_1^1 = (x_1^1)^* = 0$, $\hat{x}_2^1 = (x_2^1)^*$, $B_1 = 0$, $B_2 = B$ is feasible for ($INLP_\Delta^1$) as $\hat{x}_2^1 \le (B - D_2^1)^+$ holds since $\hat{x}_2^1 = (x_2^1)^* \le (B - D_1^1 - D_2^1)^+ \le (B - D_2^1)^+$. Assume then that an optimal solution for ($INLP^1$) satisfies $(x_1^1)^* > 0$. Then, the solution $\hat{x}_1^1 = (x_1^1)^*$, $\hat{x}_2^1 = (x_2^1)^*$, $B_1 = (x_1^1)^* + D_1^1$, $B_2 = B - (x_1^1)^* - D_1^1$ is feasible for ($INLP_\Delta^1$) as $\hat{x}_2^1 = (x_2^1)^* \le (B - (x_1^1)^* - D_1^1 - D_2^1)^+$ holds since $(x_1^1)^* > 0$ implies $B > D_1^1 + D_2^1$; that is : $B - D_1^1 - D_2^1 \ge (x_1^1)^* + (x_2^1)^*$ by the first constraint of ($INLP^1$).

Property 2 refines the inequality $f^*(B) \le f_\Delta^*(B)$ for $N = 1$ by providing budget ranges for which the inequality is strict or holds with equality.

Property 2 Given a budget B, let $f^*(B)$ and $f_\Delta^*(B)$ be the optimal objective values of ($INLP^1$) and ($INLP_\Delta^1$). We have: $f^*(B) = f_\Delta^*(B)$ if $B \le B_{min}$ or $B \ge D_1^1 + D_2^1 + \max\{P_1^1, P_2^1\}$, and $f^*(B) < f_\Delta^*(B)$ if $B_{min} < B < D_1^1 + D_2^1 + \max\{P_1^1, P_2^1\}$.

Proof Consider first the case $B \le B_{min} = \min(D_1^1, D_2^1)$, then $(x_1^1)^* = (x_2^1)^* = (\hat{x}_1^1)^* = (\hat{x}_2^1)^* = 0$, and thus $f^*(B) = f_\Delta^*(B) = 0$. Consider then the case $B \ge D_1^1 + D_2^1 + \max\{P_1^1, P_2^1\}$. Adding the last two constraints of ($INLP_\Delta^1$) yields that $P_1^1 + P_2^1$ is an upper bound; that is, $f^*(B) \le f_\Delta^*(B) \le P_1^1 + P_2^1$. Without loss of generality, we assume $P_1^1 > P_2^1$ and consider two sub-cases. Sub-case $B \ge D_1^1 + D_2^1 + P_1^1 + P_2^1$: then the solution $(x_1^1)^* = P_1^1$ and $(x_2^1)^* = P_2^1$ is feasible for ($INLP^1$) and, thus, $P_1^1 + P_2^1 \le f^*(B) \le f_\Delta^*(B) \le P_1^1 + P_2^1$ which implies $f^*(B) = f_\Delta^*(B)$. Sub-case $D_1^1 + D_2^1 + \max\{P_1^1, P_2^1\} \le B < D_1^1 + D_2^1 + P_1^1 + P_2^1$: then an optimal solution for ($INLP^1$) satisfies $(x_1^1)^* = P_1^1$ and $(x_2^1)^* = B - D_1^1 - D_2^1 - P_1^1$. Furthermore, for ($INLP_\Delta^1$), if $B_1 - D_1^1 < 0$ then $x_1^1 = 0$ and $x_2^1 \le P_2^1 < P_1^1 < f^*(B)$ which is not an optimal solution, therefore we can assume that $B_1 - D_1^1 \ge 0$. In addition, if $B_2 - D_2^1 < 0$ then $x_2^1 = 0$ and $x_1^1 \le P_1^1$ which can not yield a strictly larger objective value. Thus we can assume $B_1 - D_1^1 \ge 0$ and $B_2 - D_2^1 \ge 0$. Adding the first two constraints shows that $f_\Delta^*(B) \le B - D_1^1 - D_2^1$, and thus a strictly larger objective value can not be achieve; that is $f^*(B) = f_\Delta^*(B)$. Finally, consider the case $\min(D_1^1, D_2^1) < B < D_1^1 + D_2^1 + \max\{P_1^1, P_2^1\}$. We consider 2 sub-cases. Sub-case $\min(D_1^1, D_2^1) < B \le D_1^1 + D_2^1$: then $f^*(B) = 0$ while $B_1^* = B$ and $B_2^* = 0$ yields a feasible solution for ($INLP^1_\Delta$) which a strictly positive objective value and, thus, $f^*_\Delta(B) > f^*(B)$. Sub-case $D^1_1 + D^1_2 < B < D^1_1 + D^1_2 + \max\{P^1_1, P^1_2\}$ and, without loss of generality, $P^1_1 > P^1_2$: then $f^*(B) \leq B - D^1_1 - D^1_2 < P^1_1$ by the first constraint of ($INLP^1$). On the other hand, setting $B^*_1 = B$, $B^*_2 = 0$ and $\hat{x}^1_1 = \min\{B - D^1_1, P^1_1\}$ yields a feasible solution for ($INLP^1_\Delta$) with an objective value of at least P^1_1; that is, $f^*_\Delta(B) \geq P^1_1 > f^*(B)$. □

4.2 Case $N = 2$

We consider the case $N = 2$; that is the simplest random demand with only two realizations. We assume that both realizations have probability 0.5 and omit this constant term in the objectives for clarity. In the associated formulations ($INLP^2$) and ($INLP^2_\Delta$) below, superscripts are use to distinguish different realizations. For example, x^2_1, D^2_1, and P^2_1 refer to the second realization.

$$\begin{aligned} \max \quad & x^1_1 + x^1_2 + x^2_1 + x^2_2 \\ & x^1_1 + x^1_2 \leq (B - D^1_1 - D^1_2)^+ \\ & x^2_1 + x^2_2 \leq (B - D^2_1 - D^2_2)^+ \\ & x^1_1 \leq P^1_1, \quad x^1_2 \leq P^1_2 \\ & x^2_1 \leq P^2_1, \quad x^2_2 \leq P^2_2 \\ & x^1_1, x^1_2, x^2_1, \quad x^2_2 \in \mathbb{Z}_+ \end{aligned} \tag{$INLP^2$}$$

$$\begin{aligned} \max \quad & x^1_1 + x^1_2 + x^2_1 + x^2_2 \\ & x^1_1 \leq (B_1 - D^1_1)^+ \\ & x^1_2 \leq (B_2 - D^1_2)^+ \\ & x^2_1 \leq (B_1 - D^2_1)^+ \\ & x^2_2 \leq (B_2 - D^2_2)^+ \\ & x^1_1 \leq P^1_1, \quad x^1_2 \leq P^1_2 \\ & x^2_1 \leq P^2_1, \quad x^2_2 \leq P^2_2 \\ & B_1 + B_2 = B \\ & x^1_1, x^1_2, x^2_1, x^2_2 \in \mathbb{Z}_+ \\ & B_1, B_2 \in \mathbb{R}_+ \end{aligned} \tag{$INLP^2_\Delta$}$$

As the number of cases to consider in order to provide an analogue of Property 2 essentially increases exponentially with the number of realizations, comparing ($INLP^2$) and ($INLP^2_\Delta$) can be quite tedious. Thus, Property 3. focuses on the following 3 scenarios : (i) the demands are large for both realizations, (ii) the demands are large for one realization and small for the other, and (iii) the demands are small for both realizations.

Property 3 Given a budget B, let $f^*(B)$ and $f^*_\Delta(B)$ be the optimal objective values of ($INLP^2$) and ($INLP^2_\Delta$) We have: $f^*(B) < f^*_\Delta(B)$ if $B_{min} < B \leq B^+_{min}$,

$f^*(B) \leq f^*_\Delta(B)$ if $B^+_{min} < B \leq B^+_{max}$, and

$f^*(B)= f^*_\Delta(B)$ if $0 \leq B \leq B_{min}$ or $B \geq \max\{D^1_1 + P^1_1, D^2_1 + P^2_1\} + \max\{D^1_2 + P^1_2, D^2_2 + P^2_2\}$.

Proof Consider first the case $B \leq B_{min}$, then $(x_1^1, x_1^2, x_2^1, x_2^2)$ must be set to (0, 0, 0, 0) to obtain a feasible solution for ($INLP^2$) and ($INLP^2_\Delta$) and, thus we have $f^*(B) = f^*_\Delta(B) = 0$. Consider the case $B \geq \max\{D_1^1 + P_1^1, D_1^2 + P_1^2\} + \max\{D_2^1 + P_2^1, D_2^2 + P_2^2\}$. First note that $P_1^1 + P_2^1 + P_1^2 + P_2^2$ is an upper bound both $f^*(B)$ and $f^*_\Delta(B)$ as implied by adding the last 4 constraints. Then, as $x_i^h = P_i^h$ is a feasible solution for ($INLP^2$), $f^*(B) = P_1^1 + P_2^1 + P_1^2 + P_2^2$. Similarly, $x_i^h = P_i^h$, $B_1 = \max\{D_1^1 + P_1^1, D_1^2 + P_1^2\}$ and $B_2 = B - B_1$ a feasible solution for ($INLP^2_\Delta$) and the corresponding objective is also $P_1^1 + P_2^1 + P_1^2 + P_2^2$; that is, $f^*_\Delta(B) = f^*(B)$. Consider the case $B \leq B^+_{min} = \min\{D_1^1 + D_2^1, D_1^2 + D_2^2\}$, then while $f^*(B) = 0$, setting $B_1^* = B$ and $B_2^* = 0$ yields a feasible solution for ($INLP^2_\Delta$) with a strictly positive objective value; that is, $f^*(B) < f^*_\Delta(B)$. Consider the case $B^+_{min} < B \leq B^+_{max}$, and assume without loss of generality that $D_1^2 + D_2^2 > D_1^1 + D_2^1$. Since $B \leq D_1^2 + D_2^2$, the second constraints of ($INLP^2$) is $x_1^2 + x_2^2 \leq 0$; that is $x_1^2 = x_2^2 = 0$. In other words, we can restrict to $(x_1^1, x_2^1, 0, 0)$ feasible solutions and use Property 2 to derive $f^*(B) \leq f^*_\Delta(B)$. □

4.3 Case $N = N_0$

Similarly to Sect. 4.2, we assume for $N = N_0$, that the N_0 realizations have probability $1/N_0$ and omit this constant term in the objectives for clarity. In the associated formulations ($INLP^{N_0}$) and ($INLP^{N_0}_\Delta$) below, superscripts are use to distinguish different realizations. For example, x_1^h, x_2^h, D_1^h, D_2^h, P_1^h, and P_1^h refer to the h-th realization.

$$\max \sum_{h=1}^{N_0} (x_1^h + x_2^h) \qquad (INLP^{N_0})$$
$$x_1^h + x_2^h \leq (B - D_1^h - D_2^h)^+ \quad h = 1, \ldots, N_0$$
$$x_1^h \leq P_1^h, \quad x_2^h \leq P_2^h \quad h = 1, \ldots, N_0$$
$$x_1^h, x_2^h \in \mathbb{Z}_+ \quad h = 1, \ldots, N_0$$

$$\max \sum_{h=1}^{N_0} (x_1^h + x_2^h) \qquad (INLP^{N_0}_\Delta)$$
$$x_1^h \leq (B_1 - D_1^h)^+ \quad h = 1, \ldots, N_0$$
$$x_2^h \leq (B_2 - D_2^h)^+ \quad h = 1, \ldots, N_0$$
$$x_1^h \leq P_1^h, \quad x_2^h \leq P_2^h \quad h = 1, \ldots, N_0$$
$$B_1 + B_2 = B$$
$$x_1^h, x_2^h \in \mathbb{Z}_+ \quad h = 1, \ldots, N_0$$
$$B_1, B_2 \in \mathbb{R}_+.$$

Similarly to Sect. 4.2, the number of cases being essentially intractable, Property 4. focuses on the following 2 scenarios : (i) the demands are large for all N_0 realizations, and (ii) the demands are small for all N_0 realizations.

Property 4 Given a budget B, let $f^*(B)$ and $f^*_\Delta(B)$ be the optimal objective values of ($INLP^{N_0}$) and ($INLP^{N_0}_\Delta$) We have: $f^*(B) = f^*_\Delta(B)$ if $0 \leq B \leq B_{min}$ or $B \geq B^\Sigma_{max}$, and $f^*(B) < f^*_\Delta(B)$ if $B_{min} < B \leq B^+_{min}$.

Proof Similarly to Property 3, for $B \leq B_{min}$, x_i^h must be set to 0 for $h = 1, \ldots, N_0$ and $i = 1$ and 2 to obtain a feasible solution for ($INLP^{N_0}$) and ($INLP_{\Delta}^{N_0}$) and, thus, $f^*(B) = f_{\Delta}^*(B) = 0$.

Consider then, Since $B \geq B_{max}^{\Sigma} = \sum_{i=1}^{2} \max_{h=1}^{N_0}\{D_i^h + P_i^h\}$. First note that $\sum_{i=1}^{2}\sum_{h=1}^{N_0}\{P_i^h\}$ is an upper bound both $f^*(B)$ and $f_{\Delta}^*(B)$ as implied by adding the last $2N_0$ constraints. Then, as $x_i^h = P_i^h$ is a feasible solution for ($INLP^{N_0}$), $f^*(B) = \sum_{i=1}^{2}\sum_{h=1}^{N_0}\{P_i^h\}$. Similarly, $x_i^h = P_i^h$, $B_1 = \max_{h=1}^{N_0}\{D_1^h + P_1^h\}$ and $B_2 = B - B_1$ a feasible solution for ($INLP_{\Delta}^{N_0}$) and the corresponding objective is also $\sum_{i=1}^{2}\sum_{h=1}^{N_0}\{P_i^h\}$; that is, $f_{\Delta}^*(B) = f^*(B)$.

Consider the case $B \leq B_{min}^{+} = \min_{h=1}^{N_0}\{D_1^h + D_2^h\}$, then while $f^*(B) = 0$, setting $B_1^* = B$ and $B_2^* = 0$ yields a feasible solution for ($INLP_{\Delta}^{N_0}$) with a strictly positive objective value; that is, $f^*(B) < f_{\Delta}^*(B)$. □

5 Conclusions and future work

We highlighted the critical role played by the piece-wise linear inventory availability constraints and the associated feasibility issue and challenges for sample generation. The computational results estimate the impact resulting from substituting linear functions for piece-wise linear ones: While the impact decreases when the budget increases, it remains significant for low to medium level budgets. In addition, the benefits of component commonality are analyzed from theoretical and computational aspects and illustrated for specific ATO systems. We introduce a simple inventory control method applicable in practice where a more flexible design of products and components allows us to exploit the different degrees of component commonality according to the budget. Future work includes an enhanced analysis of the sample generation process for (*ILP*) and a tighter estimate of the gap between the optimal objective values of (*ILP*) and (*INLP*). Further flexibility for the proposed inventory control method might be achieved via component commonality for subset of components and products.

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