Hyperplane Arrangements with Large Average Diameter: a Computational Approach

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Abstract.

We consider the average diameter of a bounded cell of a simple arrangement defined by n hyperplanes in dimension d. In particular, we investigate the conjecture stating that the average diameter is no more than the dimension d. Previous results in dimensions 2 and 3 suggested that specific extensions of the cyclic arrangement might achieve the largest average diameter. We show that the suggested arrangements do not always achieve the largest diameter and disprove a related conjecture dealing with the minimum number of facets belonging to exactly one bounded cell. In addition, we computationally determine the largest possible average diameter in dimensions 3 and 4 for arrangements defined by no more than 8 hyperplanes via the associated uniform oriented matroids. These new entries substantiate the hypothesis that the largest average diameter is achieved by an arrangement minimizing the number of facets belonging to exactly one bounded cell. The computational framework to generate specific arrangements, and to compute the average diameter and the number of facets belonging to exactly one bounded cell is presented.

§1. Introduction

Let $\mathcal{A}_{d,n}$ be a simple arrangement formed by n hyperplanes in dimension d. We recall that an arrangement is called simple if $n \geq d+1$ and any d hyperplanes intersect at a unique distinct point. The number of bounded cells (closures of the bounded connected components of the complement) of $\mathcal{A}_{d,n}$ is $I = \binom{n-1}{d}$. Let $\delta(\mathcal{A}_{d,n})$ denote the average diameter of a bounded cell P_i of $\mathcal{A}_{d,n}$; that is,

$$\delta(\mathcal{A}_{d,n}) = \frac{\sum_{i=1}^{i=I} \delta(P_i)}{I}$$

Key words and phrases. simple hyperplane arrangements, average diameter, Hirsch conjecture, oriented matroids.

where $\delta(P_i)$ denotes the diameter of P_i, i.e., the smallest number such that any two vertices of P_i can be connected by a path with at most $\delta(P_i)$ edges. Let $\Delta_{\mathcal{A}}(d,n)$ denote the largest possible average diameter of a bounded cell of a simple arrangement defined by n inequalities in dimension d. We consider the following conjecture stating that the average diameter is no more than the dimension d.

Conjecture 1. [**7**] *The average diameter of a bounded cell of a simple arrangement defined by n inequalities in dimension d is not greater than d, i.e., $\Delta_{\mathcal{A}}(d,n) \leq d$.*

1.1. Conjecture 1 as a discrete analogue of a result of Dedieu, Malajovich, and Shub

Conjecture 1 can be regarded as a discrete analogue of a result of Dedieu, Malajovich, and Shub [**6**] on the average total curvature of the central path associated to a bounded cell of a simple arrangement. We first recall the definitions of the central path and of the total curvature. For a polytope, i.e. bounded polyhedron, $P = \{x : Ax \geq b\}$ with $A \in \Re^{n\times d}$, the central path corresponding to $\min\{c^T x : x \in P\}$ is a set of minimizers of $\min\{c^T x + \mu f(x) : x \in P\}$ for $\mu \in (0,\infty)$ where $f(x) = -\sum_{i=1}^{n} \ln(A_i x - b_i)$ – the standard logarithmic barrier function [**19**]. Intuitively, the total curvature [**21**] is a measure of how far off a certain curve is from being a straight line. Let $\psi : [\alpha,\beta] \to \Re^d$ be a $C^2((\alpha - \varepsilon, \beta+\varepsilon))$ map for some $\varepsilon > 0$ with a non-zero derivative in $[\alpha,\beta]$. Denote its arc length by $l(t) = \int_\alpha^t \|\dot\psi(\tau)\| d\tau$, its parametrization by the arc length by $\psi_{\text{arc}} = \psi \circ l^{-1} : [0, l(\beta)] \to \Re^d$, and its curvature at the point t by $\kappa(t) = \ddot\psi_{\text{arc}}(t)$. The total curvature is defined as $\int_0^{l(\beta)} \|\kappa(t)\| dt$. The requirement $\dot\psi \neq 0$ insures that any given segment of the curve is traversed only once and allows to define a curvature at any point on the curve. Let $\lambda^c(\mathcal{A}_{d,n})$ denote the average associated total curvature of a bounded cell P_i of a simple arrangement $\mathcal{A}_{d,n}$; that is,

$$\lambda^c(\mathcal{A}_{d,n}) = \sum_{i=1}^{i=I} \frac{\lambda^c(P_i)}{I}$$

where $\lambda^c(P)$ denotes the total curvature of the central path corresponding to the linear optimization problem $\min\{c^T x : x \in P\}$. Dedieu, Malajovich and Shub [**6**] demonstrated that $\lambda^c(\mathcal{A}_{d,n}) \leq 2\pi d$ for any fixed c. Keeping the linear optimization approach but replacing central path following interior point methods by simplex methods, Haimovich's probabilistic analysis of the shadow-vertex simplex algorithm, see [**3**, Section 0.7], showed that the expected number of pivots is bounded by d. Note that while Dedieu, Malajovich and Shub consider only the bounded cells (the central path may not be defined over some unbounded ones), Haimovich considers the average over bounded and unbounded cells. The result of Haimovich and Conjecture 1 are similar in nature but differ in some aspects as, while the diameter is a lower bound for the number of pivots in the worst case, the number of pivots could be smaller than the diameter for some cells.

Considering the largest $\lambda^c(P)$ over all possible c, we obtain the quantity $\lambda(P)$, referred to as the curvature of a polytope. Following the approach regarding the curvature $\lambda(P)$ as a continuous analogue of the diameter $\delta(P)$, analogues of the results of Holt and Klee [**14**], and Klee and Walkup [**15**] were given in [**7**, **8**]. Namely, a family of polytopes which attain the conjectured order of the largest curvature was given, and it was proved that if the order of the curvature is less than the dimension d for all polytopes defined by $2d$ inequalities and for all d, then the order of the curvature is less than the number of inequalities for all polytopes.

As pointed out by an anonymous referee, it would be interesting to consider the average diameter over all cells, bounded and unbounded. The following property, see [**12**], illustrates the mathematical appeal of this approach. For an oriented matroid of rank $d+1$, $f_k \leq \binom{d}{k} f_d$ where f_k denotes the number of k-faces. Setting $k = d-1$ yields $f_{d-1} \leq d\, f_d$; that is, since any facet belongs to exactly 2 cells, the average number of facets of a cell $\frac{2f_{d-1}}{f_d}$ is no more than $2d$. Note that the inequality holds without the simplicity assumption.

1.2. Conjecture 1 and the conjecture of Hirsch

The conjecture of Hirsch formulated in 1957 and reported in [**5**] states that the diameter of a polyhedron defined by n inequalities in dimension d is not greater than $n-d$. The conjecture holds for $d \leq 3$ and for $n-d \leq 6$, but it had been speculated to be false for large enough n and d, and was recently disproved by Santos [**20**]. It was noticed in [**7**] that Conjecture 1 is nearly implied by the conjecture of Hirsch using the following straightforward argument. If we assume that the diameter of a bounded cell P_i defined by n_i hyperplanes of $\mathcal{A}_{d,n}$ is bounded by $n_i - d$, then:

$$\delta(\mathcal{A}_{d,n}) \leq \frac{\sum_{i=1}^{i=I}(n_i - d)}{I} = \frac{\sum_{i=1}^{i=I} n_i}{I} - d = \frac{2n\binom{n-2}{d-1} - \phi(\mathcal{A}_{d,n})}{I} - d$$

where $\phi(\mathcal{A}_{d,n})$ denotes the number of *external* facets of $\mathcal{A}_{d,n}$; that is, the number of facets belonging to exactly one bounded cell. This inequality yields $\Delta_{\mathcal{A}}(d,n) \leq d + \frac{2d}{n-1}$ as $\phi(\mathcal{A}_{d,n}) \geq 0$, and suggests that arrangements minimizing $\phi(\mathcal{A}_{d,n})$ might be good candidates for achieving a large $\delta(\mathcal{A}_{d,n})$. Let $\Phi_{\mathcal{A}}(d,n)$ denote the minimum number of external facets for any simple arrangement defined by n hyperplanes in dimension d. It was hypothesized in [**7**] that: $\Phi_{\mathcal{A}}(d,n) \geq d\binom{n-2}{d-1}$. While the conjecture of Hirsch turned out to be false, we speculate that the weaker Conjecture 1 could still hold.

We recall previous results concerning Conjecture 1 in Section 2. In Section 3 we consider specific single element extensions of the cyclic arrangements. The computational framework to generate such arrangements, as well as to compute the average diameter and the number of external facets, is presented in Section 4. Finally, in Section 5 we show that, while providing a promising lower bound, the specific single element extensions of the cyclic arrangements do not always achieve the largest diameter and disprove the hypothesized lower bound for the number of external facets. We also substantiate Conjecture 1 by determining new entries for the largest possible average diameter in dimensions 3 and 4 for arrangements defined by no more than 8 hyperplanes. We provide computational evidence that maximizing the average diameter and minimizing the number external facets might be achieved simultaneously.

We refer to the books of Grünbaum [**13**] and Ziegler [**25**] for polytopes and arrangements, to the books of Renegar [**18**] and Roos et al. [**19**] for linear programming and the central path, and to the books of Bokowski [**1**] and Bjöner et al. [**4**] for oriented matroids.

§2. Previous Results

Let $\mathcal{A}^*_{d,n}$ denote a simple arrangement combinatorially equivalent to the cyclic hyperplane arrangement which is dual to the cyclic polytope. Proposition 2 recalls that, since the bounded cells of $\mathcal{A}^*_{d,n}$ are mainly combinatorial cubes, the dimension d is an asymptotic lower bound for $\Delta_{\mathcal{A}}(d,n)$ for fixed d. Additional results for small d and $n-d$ are recalled in Proposition 3. See Figure 1 and Figure 2 for an illustration of arrangements maximizing the average diameter for $(d,n) = (2,7)$ and $(3,6)$.

Proposition 2. [**9**] *For $n \geq 2d$, we have*
$\Delta_{\mathcal{A}}(d,n) \geq \frac{(d-1)\binom{n-d}{d}+\binom{n-1}{d}+(n-d)(n-d-1)}{\binom{n-1}{d}}$.

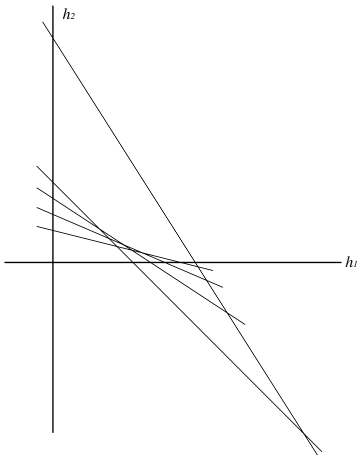

Fig. 1. An arrangement formed by 7 lines maximizing the average diameter

Proposition 3. [**9**] *We have*

(*a*) $\Delta_{\mathcal{A}}(d, d+2) = \frac{2d}{d+1}$

(*b*) $\Delta_{\mathcal{A}}(2, n) = 2 - \frac{2\lceil \frac{n}{2} \rceil}{(n-1)(n-2)}$ *for* $n \geq 3$

(*c*) $\Delta_{\mathcal{A}}(3, 6) = 2$

(*d*) $3 - \frac{6}{n-1} + \frac{6(\lfloor \frac{n}{2} \rfloor - 2)}{(n-1)(n-2)(n-3)} \leq \Delta_{\mathcal{A}}(3, n) \leq 3 + \frac{4(2n^2-16n+21)}{3(n-1)(n-2)(n-3)}$.

One can easily check that the arrangement resulting from the addition of one hyperplane to $\mathcal{A}^*_{2,n-1}$ such that all the vertices are on one side of the added hyperplane, simultaneously maximizes the average diameter and minimizes the number of external facets. Noticing that the arrangement given in Figure 2 maximizing the average diameter for $(d, n) = (3, 6)$ is obtained by adding a hyperplane to $\mathcal{A}^*_{3,5}$ such that all the vertices of $\mathcal{A}^*_{3,5}$ are on one side of the added hyperplane, we investigate extensions of $\mathcal{A}^*_{d,n-1}$ as potential candidates to achieve the largest average diameter $\Delta_{\mathcal{A}}(d, n)$ and minimize the number of external facets $\Phi_{\mathcal{A}}(d, n)$.

§3. Covering extensions of cyclic arrangements

3.1. Covering extensions of cyclic arrangements

The alternating oriented matroid $\mathcal{M}^*_{d+1,n}$ of rank $d+1$ and with n elements can be realized as a cyclic arrangement $\mathcal{A}^*_{d,n-1}$. Let $(\mathcal{M}^*_{d+1,n}, k)$

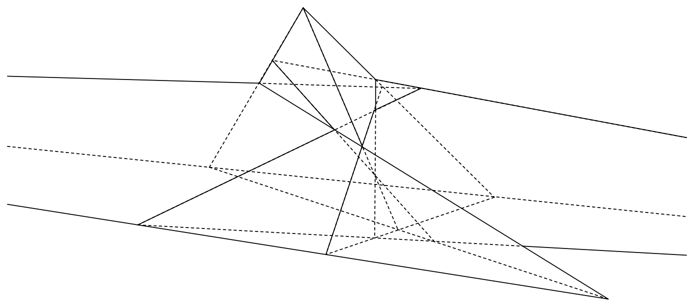

Fig. 2. An arrangement formed by 6 planes maximizing the average diameter

be the corresponding affine alternating oriented matroid where k is the infinity element. The combinatorics of the addition of a pseudo-hyperplane to the cyclic hyperplane arrangement, and its relationship with higher Bruhat orders, is studied in detail in [**24**]. Note that since the combinatorial type of an arrangement defined by $d+2$ hyperplanes is unique, all arrangements defined by $d+3$ hyperplanes are extensions of $\mathcal{A}^*_{d,d+2}$. In order to avoid the NP-hard realizability problem [**23**], we focus on single element extensions of $(\mathcal{M}^*_{d+1,n}, k)$ for which all the cocircuits are on one side of the added pseudo-hyperplane. These affine oriented matroids are, by definition, realizable. Let us call *covering extensions* of $\mathcal{A}^*_{d,n-1}$ arrangements whose underlying affine oriented matroid is a single element extension of $(\mathcal{M}^*_{d+1,n}, k)$ for which all the cocircuits are on one side of the added pseudo-hyperplane. In other words, covering extensions of cyclic arrangements are obtained by adding one hyperplane to $\mathcal{A}^*_{d,n-1}$ such that all the vertices are on one side of the added hyperplane. For example, the cyclic arrangement $\mathcal{A}^*_{d,n}$ and the arrangements given in Figures 1 and 2 are covering extensions of a cyclic arrangement.

3.2. Generating covering extensions of cyclic arrangements

One can generate covering extensions of cyclic arrangements via single element extensions of the alternating oriented matroid $\mathcal{M}^*_{d+1,n}$, see [**1**, **4**], and selecting one element as the infinity element.

We recall that the alternating oriented matroid $\mathcal{M}^*_{d+1,n}$ can be represented as an arrangement of n hemispheres on the d-sphere. We generate the covering extensions by adding one hemisphere resulting from a proper perturbation of the infinity hemisphere. Let $\mathcal{C}^*$ be the set of cocircuits of $\mathcal{M}^*_{d+1,n}$. Each cocircuit $X \in \mathcal{C}^*$ corresponds to a vertex on the hemisphere arrangement representing $\mathcal{M}^*_{d+1,n}$. A single element extension of $\mathcal{M}^*_{d+1,n}$ is determined by the cocircuit signature $\sigma : \mathcal{C}^* \to \{+,-,0\}$, see [**4**]. We obtain the cocircuit signature σ_c corresponding to the covering extension via a series of sphere rotations. Let H_k with $1 \leq k \leq n$ be the infinity hemisphere to be perturbed, and H_{n+1} denote the resulting new hemisphere. Regardless of the directions of the rotations, we can keep it small enough so that the vertices which are on the positive, respectively negative, side of H_k are also on the positive, respectively negative, side of H_{n+1}, i.e., we have $\sigma_c(X) = X_k$ if $X_k \neq 0$, where X_k is the k^{th} sign of X. The signature of the remaining cocircuits $\{X \in \mathcal{C}^* : X_k = 0\}$ is determined by the rotations in the following way. Let us choose a pair of antipodal vertices Y and $-Y$ both on the hemisphere H_k, i.e. which have, besides the k^{th} sign, $(d-1)$ zero-signs. Let the ordered index set of zeros in Y, except the one at index k, be $(i_1, i_2, \ldots, i_{d-1})$. We use a sign vector $O \in \{+,-\}^{d-1}$ to represent the rotations, where O_j records the direction of rotation of the d-sphere around the axis defined by the intersection of the hemispheres H_k and H_{i_j} for $j = 1, \ldots, d-1$. The signature of any cocircuit X, except for the antipodal pair Y and $-Y$, is therefore given by:

$$\sigma_c(X) = \begin{cases} X_k, & \text{if } X_k \neq 0; \\ X_{i_j} O_j, & \text{if } X_k = 0,\ j \ \text{: smallest index such that } X_{i_j} \neq 0. \end{cases}$$

Finally, for computational purposes, we set the signature of the antipodal pair Y and $-Y$ by extending the length of the orientation vector O by 1 and use O_d to record the chosen orientation of the antipodal pair. Thus, the signature of any cocircuit X is given by:

$$\sigma_c(X) = \begin{cases} O_d, & \text{if } X = Y; \\ -O_d, & \text{if } X = -Y; \\ X_k, & \text{if } X_k \neq 0; \\ X_{i_j} O_j, & \text{if } X_k = 0,\ j \ \text{: smallest index such that } X_{i_j} \neq 0. \end{cases}$$

See Figure 3 for an illustration of a covering extension of $\mathcal{M}^*_{3,4}$. The covering hemisphere, colored in red, is obtained as a perturbation of hemisphere H_4 corresponding to the choice $Y =$`0++0` and $-Y =$`0--0` as antipodal pair, rotating the sphere around the intersection of hemispheres H_4 and H_1 towards the reader, and then perturbing the antipodal pairs clockwise, that is, $O_d = \{-\}$.

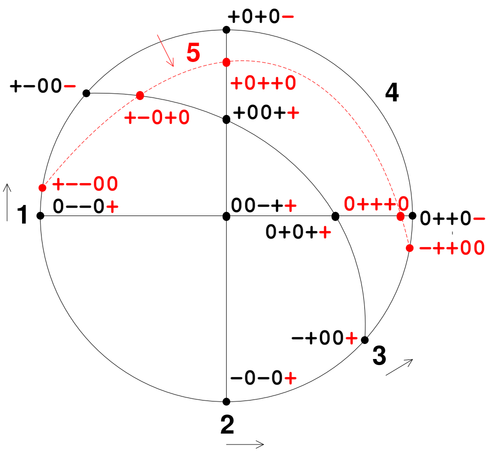

Fig. 3. A covering extension of $\mathcal{M}^*_{3,4}$, the alternating oriented matroid of rank 3 with 4 elements

We have n choices for the infinity hemisphere, $\binom{n-1}{d-1}$ possible pairs of antipodal vertices, $(d-1)!$ different ways of ordering the $(d-1)$ rotations, 2 directions per rotation, and finally 2 choices for the signature of the antipodal pair. Therefore, the total number of potential perturbations is $n\binom{n-1}{d-1}2^{d-1}(d-1)!\;2 = n\binom{n-1}{d-1}2^{d}(d-1)!$. Note that some affine oriented matroids obtained by the operation described above might belong to the same dissection type [**10**], i.e., the equivalence class of a realizable oriented matroid defined by combinations of relabeling and reorientation that map infinity element to infinity element.

§4. Computational framework

The developed C++ package for computing the average diameter and the number of external facets of an affine oriented matroid can be found at [**22**]. All computations are performed in a combinatorial way which allows us to avoid numerical errors. The code was run on a 32-core server with each core running at a clock rate of 2.3 GHz. Realizability is checked only for the few affine oriented matroids which either maximize the average diameter or minimize the number of external facets. The realizability checking was done on a 64-core server with each core running at a clock rate 2.2 GHz.

4.1. Uniform oriented matroids enumeration

The list of all (uniform) oriented matroids $\mathcal{M}_{d+1,n+1}$ for $d+1 \leq 5$ and $n+1 \leq 8$ as well as for $(d+1, n+1) = (3, 9)$ can be found on the online database [**11**]. The enumeration of all uniform oriented matroids for $(d+1, n+1) = (4, 9)$ and $(5, 9)$ was recently performed by Finschi, Fukuda, and Moriyama and will be uploaded on the online database [**11**] soon.

4.2. Computing the average diameter of a simple arrangement

An affine uniform oriented can be represented as a simple affine pseudo-sphere arrangement. One can easily check if a cell is bounded by verifying that any vertex on the infinity pseudo-sphere is not conformal to the sign vector of the cell. Recall that the sign vector of a cell does not contain any 0. Given the sign vector of a bounded cell, we compute its diameter by constructing its skeleton graph. Two vertices X and Y are adjacent if X can be pivoted to Y, or vice versa, i.e. if there exists a pair of indices (i, j) with $1 \leq i, j \leq n$, $i \neq j$ such that: (i) $X_k = Y_k$ for all $k \neq i, j$, (ii) $X_i = 0$, $Y_i \neq 0$, and (iii) $X_j \neq 0$, $Y_i = 0$. The diameter of the cell is obtained by computing the diameter of the resulting skeleton graph.

4.3. Computing the number of external facets of a simple arrangement

One can check if a facet is bounded by verifying that the sign vector of any vertex at infinity is not conformal to the sign vector of the facet. Recall that the sign vector of a facet contains exactly one 0. A bounded facet is external facet if its sign vector is conformal to the sign vector of some unbounded cell.

4.4. Realizability of uniform oriented matroids

Deciding the realizability of an oriented matroid is known to be NP-hard [**16**] in general but could be tractable for small instances. In particular, following the approach used in [**17**] for $\mathcal{M}_{4,8}$, $\mathcal{M}_{3,9}$, and $\mathcal{M}_{6,9}$, a software was developed to check the realizability for $\mathcal{M}_{4,9}$ and $\mathcal{M}_{5,9}$.

Let $(\{1, \ldots, n+1\}, \chi)$ be a uniform oriented matroid where χ is the associated chirotope, and $(v_1, \ldots, v_{n+1}) \in \mathbb{R}^{(d+1)\times(n+1)}$ the corresponding vector configuration if realizable. The realizability of $(\{1, \ldots, n+1\}, \chi)$ is equivalent to the feasibility of the following polynomial system:

$$\mathrm{sign}(\det(v_{i_1}, \ldots, v_{i_{d+1}})) = \chi(i_1, \ldots, i_{d+1}) \quad \text{for } 1 \leq i_1 < \cdots < i_{d+1} \leq n+1.$$

This system contains many redundancies which can be exploited to solve it efficiently for small instances. First, since the feasibility of this system is invariant under linear transformations A, with $\det(A) > 0$, of the configuration $(v_1, \ldots, v_{n+1})$ and positive scalar multiplications of $v_1, \ldots v_{n+1}$, we can assume the submatrix $(v_{i_1}, \ldots, v_{i_{d+1}})$ to be an identity matrix for some $(d+1)$-tuple $(i_1, \ldots, i_{d+1})$, and the 1st coordinate of $v_1, \ldots, v_{n+1}$ to be 1 or -1 according to the sign constraints. In addition, square-free variables can be regarded as redundant in the following case:

$$\begin{cases} y < R_i(x_1, \ldots, x_{n+1}) \text{ for } i = 1, \ldots, l \\ y > L_j(x_1, \ldots, x_{n+1}) \text{ for } j = 1, \ldots, m \end{cases}$$

where R_i, L_j are rational functions for $i = 1, \ldots, l, j = 1, \ldots, m$. In this case, we can eliminate the variable y:

$$L_j(x_1, \ldots, x_n) < R_i(x_1, \ldots, x_n) \text{ for } i = 1, \ldots, l, j = 1, \ldots, m$$

The solvability sequence method applies this rule to special polynomial systems consisting of determinants under the *bipartiteness condition* [**2**]. This elimination rule can be combined with the following *branching rule*.

Proposition 4.
Let $A_1(x_1, \ldots, x_n), \ldots, A_p(x_1, \ldots, x_n), B_1(x_1, \ldots, x_n), \ldots, B_p(x_1, \ldots, x_n)$ *be real polynomials. Then the following system*

$$A_k(x_1, \ldots, x_n) y < B_k(x_1, \ldots, x_n) \text{ for } k = 1, \ldots, p$$

is feasible if and only if, for at least one $s : \{1, \ldots, p\} \to \{+, -\}$*, the following system is feasible*

$$\begin{cases} \mathrm{sign}(A_i(x_1, \ldots, x_n)) = s(i) \text{ for } i = 1, \ldots, p \\ y < \frac{B_i(x_1, \ldots, x_n)}{A_i(x_1, \ldots, x_n)} \text{ for } s(i) = + \\ y > \frac{B_i(x_1, \ldots, x_n)}{A_i(x_1, \ldots, x_n)} \text{ for } s(i) = - \end{cases}$$

We search a square-free variable y, and then apply the branching and elimination rules. If we can eliminate all variables by successive applications of the branching and elimination rules, and obtain consistent inequalities of rationals at some branch of the tree, the oriented matroid is realizable. If we have a polynomial system without a square-free variable at some branch, we try random assignments to the remaining variables. If the random assignments at some branch are successful, the oriented matroid is realizable. Thus, we transform the realizability problem into a tree search problem for which an iterative deepening depth-first search can be used.

§5. Computational results

5.1. Maximal average diameter

For $d \leq 4$ and $n \leq 8$, in order to determine the entries for $\Delta_{\mathcal{A}}(d,n)$, we consider the set $\mathcal{M}_{d+1,n+1}$ of uniform oriented matroids. For each uniform oriented matroid, we consider the $n+1$ choices of setting one element as the infinity element, and compute the average diameter of the resulting affine oriented matroid. Finally, we check the realizability of the oriented matroids maximizing the average diameter. Note that all oriented matroids maximizing the average diameter turned out to be realizable which leads to the following question: *Can an affine non-realizable oriented matroid achieve the maximal average diameter?* The entries for $\Delta_{\mathcal{A}}(d,n)$, including the four new entries for $(d,n) = (3,7), (3,8), (4,7)$ and $(4,8)$, are listed in Table 1. The list of hyperplane arrangements satisfying $\delta(\mathcal{A}_{d,n}) = \Delta_{\mathcal{A}}(d,n)$ can be found in [**22**] where arrangements are represented by the chirotope of its corresponding affine oriented matroid. The signs of the chirotope are ordered reverse lexicographically and the infinity element is always the last one. To avoid redundancy, affine oriented matroids with equivalent dissection types were removed.

(d,n)	$\|\|\mathcal{M}_{d+1,n+1}\|\|$	$\Delta_{\mathcal{A}}(d,n)$
(2,5)	4	1.5
(2,6)	11	1.7
(2,7)	135	1.73...
(2,8)	4,382	1.80...
(3,6)	11	2
(3,7)	2,628	2.25
(3,8)	9,276,595	2.42...
(4,7)	135	2.2
(4,8)	9,276,595	2.71...

Table 1. Entries for $\Delta_{\mathcal{A}}(d,n)$ for $d \leq 4$ and $n \leq 8$

One can easily check, by removing a hyperplane and checking if the associated oriented matroid is alternating, if an arrangement satisfying $\delta(\mathcal{A}_{d,n}) = \Delta_{\mathcal{A}}(d,n)$ corresponds to a single element extension of $\mathcal{A}^*_{d,n-1}$. Similarly, one can easily check if an arrangement satisfying $\delta(\mathcal{A}_{d,n}) = \Delta_{\mathcal{A}}(d,n)$ corresponds to a covering extension. As stated in Proposition 5, the computational results disprove the hypothesis that $\Delta_{\mathcal{A}}(d,n)$ is always achieved by an (covering) extension of $\mathcal{A}^*_{d,n-1}$.

Proposition 5. *While a single element extension of $\mathcal{A}^*_{3,6}$ achieves $\Delta_{\mathcal{A}}(3,7)$, none of the arrangements achieving $\Delta_{\mathcal{A}}(3,8)$ and $\Delta_{\mathcal{A}}(4,8)$ is a single element extension of the cyclic arrangement. In addition, the single element extension of $\mathcal{A}^*_{3,6}$ achieving $\Delta_{\mathcal{A}}(3,7)$ is not a covering extension.*

Remark 6. *For line arrangements, $\Delta_{\mathcal{A}}(2,n)$ is always achieved by covering extensions of $\mathcal{A}^*_{2,n-1}$. The largest average diameter $\Delta_{\mathcal{A}}(2,n)$ is achieved only by covering extensions for $n \leq 6$, by covering and non-covering extensions as well as non-extensions of $\mathcal{A}^*_{2,6}$ for $n = 7$, and by covering extensions as well as non-extensions of $\mathcal{A}^*_{2,7}$ for $n = 8$.*

While the extensions of $\mathcal{A}^*_{d,n-1}$ fail to always reach $\Delta_{\mathcal{A}}(d,n)$, the covering extensions of $\mathcal{A}^*_{d,n-1}$ could provide a good lower bound. See Table 2 where the value of $\Delta_{\mathcal{A}}(d,n)$ is compared with the maximal average diameter $\Delta^c_{\mathcal{A}}(d,n)$ over all covering extensions of $\mathcal{A}^*_{d,n-1}$ as well as with $\delta(\mathcal{A}^*_{d,n})$, the average diameter of the cyclic arrangement $\mathcal{A}^*_{d,n}$. In particular, we obtain a new lower bound for $\Delta_{\mathcal{A}}(3,9)$. It was showed in [**9**] that $\Delta_{\mathcal{A}}(2,n) = \Delta^c_{\mathcal{A}}(2,n)$ and $\Delta_{\mathcal{A}}(3,6) = \Delta^c_{\mathcal{A}}(3,6)$. The computation shows that $\Delta_{\mathcal{A}}(4,7) = \Delta^c_{\mathcal{A}}(4,7)$, and all arrangements achieving $\Delta_{\mathcal{A}}(3,6)$ and $\Delta_{\mathcal{A}}(4,7)$ are covering arrangements.

(d,n)	$\Delta_{\mathcal{A}}(d,n)$	$\Delta^c_{\mathcal{A}}(d,n)$	$\delta(\mathcal{A}^*_{d,n})$
(3,6)	2	2	1.8
(3,7)	2.25	2.1	2
(3,8)	2.42	2.34...	2.14...
(3,9)	?	2.39...	2.25
(4,7)	2.2	2.2	2
(4,8)	2.71...	2.45...	2.28...

Table 2. Covering extensions of as a lower bound for $\Delta_{\mathcal{A}}(d,n)$

5.2. Minimal number of external facets

While computing entries for $\Delta_{\mathcal{A}}(d,n)$, one can also obtain $\Phi_{\mathcal{A}}(d,n)$, the minimum number of external facets, i.e. facets belonging to exactly one bounced cell, over all arrangements formed by n hyperplanes in dimension d. It was showed in [**9**] that $\Phi_{\mathcal{A}}(2,n) = 2(n-1)$. The computed entries for $\Phi_{\mathcal{A}}(d,n)$ are listed in Table 3. In particular, as stated in Proposition 7, the entry for $(d,n) = (3,8)$ is a counterexample for the hypothesized lower bound. Similarly to Section 5.1, all affine oriented matroids minimizing the number of external facets turned out to be realizable. The list of hyperplane arrangements satisfying $\phi(\mathcal{A}_{d,n}) = \Phi_{\mathcal{A}}(d,n)$ can be found in [**22**].

(d,n)	$\|\|\mathcal{M}_{d+1,n+1}\|\|$	$\Phi_{\mathcal{A}}(d,n)$
(2,5)	4	8
(2,6)	11	10
(2,7)	135	12
(2,8)	4,382	14
(3,6)	11	22
(3,7)	2,628	32
(3,8)	9,276,595	44
(4,7)	135	47
(4,8)	9,276,595	84

Table 3. Entries for $\Phi_{\mathcal{A}}(d,n)$ for $d \leq 4$ and $n \leq 8$

Proposition 7. *The entry for for* $(d,n) = (3,8)$ *disproves the hypothesized inequality* $\Phi_{\mathcal{A}}(d,n) \geq d\binom{n-2}{d-1}$ *as we have* $\Phi_{\mathcal{A}}(3,8) = 44 < 45 = 3\binom{6}{2}$.

In addition of providing a promising lower bound for $\Delta_{\mathcal{A}}(d,n)$, the covering extensions of $\mathcal{A}^*_{d,n-1}$ might provide a good upper bound for $\Phi_{\mathcal{A}}(d,n)$. See Table 4 where the value of $\Phi_{\mathcal{A}}(d,n)$ is compared with the minimum number of external facets $\Phi^c_{\mathcal{A}}(d,n)$ over all covering extensions of $\mathcal{A}^*_{d,n-1}$ as well as with $\phi(\mathcal{A}^*_{d,n})$, the number of external facets of the cyclic arrangement $\mathcal{A}^*_{d,n}$, and with the hypothesized lower bound $d\binom{n-2}{d-1}$. In particular, we obtain an upper bound for $\Phi_{\mathcal{A}}(3,9)$. We have $\Phi_{\mathcal{A}}(3,6) = \Phi^c_{\mathcal{A}}(3,6)$ and $\Phi_{\mathcal{A}}(4,7) = \Phi^c_{\mathcal{A}}(4,7)$, and all arrangements achieving $\Phi_{\mathcal{A}}(3,6)$ and $\Phi_{\mathcal{A}}(4,7)$ are covering arrangements.

(d,n)	$\Phi_{\mathcal{A}}(d,n)$	$\Phi^c_{\mathcal{A}}(d,n)$	$\phi(\mathcal{A}^*_{d,n})$	$d\binom{n-2}{d-1}$
(3,6)	22	22	24	18
(3,7)	32	34	40	30
(3,8)	44	48	60	45
(3,9)	?	64	84	63
(4,7)	47	47	50	40
(4,8)	84	88	100	80

Table 4. Covering extensions as an upper bound for $\Phi_{\mathcal{A}}(d,n)$

Remark 8. *While the covering extensions turn out to provide only lower, respectively upper, bound for $\Delta_{\mathcal{A}}(d,n)$, respectively $\Phi_{\mathcal{A}}(d,n)$, the hypothesized relation between maximizing $\Delta_{\mathcal{A}}(d,n)$ and minimizing $\Phi_{\mathcal{A}}(d,n)$ is computationally substantiated by the existence for $d \leq 4$ and $n \leq 8$ of at least one simple arrangement simultaneously maximizing $\Delta_{\mathcal{A}}(d,n)$ and minimizing $\Phi_{\mathcal{A}}(d,n)$.*

The known entries for $\Delta_{\mathcal{A}}(d,n)$ are summarized in Table 5.

	2	3	4	5	6	$\dots$	$n-d$	$\dots$
2	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{17}{10}$	$\frac{26}{15}$	$\frac{38}{21}$	$\dots$	$2-\frac{2\lceil\frac{n}{2}\rceil}{(n-1)(n-2)}$	
3	$\frac{3}{2}$	2	$\frac{9}{4}$	$\frac{17}{7}$	$\geq\frac{67}{28}$			
4	$\frac{8}{5}$	$\frac{11}{5}$	$\frac{19}{7}$					
$\vdots$	$\vdots$							
d	$\frac{2d}{d+1}$							
$\vdots$	$\vdots$							

Table 5. Known entries for $\Delta_{\mathcal{A}}(d,n)$

Acknowledgments The authors are grateful to the anonymous referees for helpful suggestions. Research supported for the first and last authors by an NSERC Discovery grant, a MITACS grant and the Canada Research Chairs program. The second and third authors were partially supported by KAKENHI.

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