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Title:

The complexity of the envelope
of line and plane arrangements

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Abstract

A facet of an hyperplane arrangement is called external if it belongs to exactly one bounded cell. The set of all external facets forms the envelope of the arrangement. The number of external facets of a simple arrangement defined by n hyperplanes in dimension d is hypothesized to be at least $d\binom{n-2}{d-1}$. In this note we show that, for simple arrangements of 4 lines or more, the minimum number of external facets is equal to $2(n-1)$, and for simple arrangements of 5 planes or more, the minimum number of external facets is between $\frac{n(n-2)+6}{3}$ and $(n-4)(2n-3)+5$.

1 Introduction

Let $\mathcal{A}_{d,n}$ be a simple arrangement formed by n hyperplanes in dimension d. We recall that an arrangement is called simple if $n \geq d+1$ and any d hyperplanes intersect at a distinct point. The closures of connected components of the complement of the hyperplanes forming $\mathcal{A}_{d,n}$ are called the cells, or d-faces, of the arrangement. For $k = 0, \ldots, d-1$, the k-faces of $\mathcal{A}_{d,n}$ are the k-faces of its cells. A facet is a $(d-1)$-face of $\mathcal{A}_{d,n}$, and a facet belonging to exactly one bounded cell is called an external facet. Equivalently, an external facet is a bounded facet which belongs to an unbounded cell. For $k = 0, \ldots, d-2$, an external k-face is a k-face belonging to an external facet. Let $f_k^0(\mathcal{A}_{d,n})$ denote the number of external k-faces of $\mathcal{A}_{d,n}$. The set of all external facets forms the envelope of the arrangement. It was hypothesized in [1] that any simple arrangement $\mathcal{A}_{d,n}$ has at least $d\binom{n-2}{d-1}$ external facets. In Section 2, we show that a simple arrangement of n lines has at least $2(n-1)$ external facets for $n \geq 4$, and that this bound is tight. In section 3, we show that a simple arrangement of n planes has at least $\frac{n(n-2)+6}{3}$ external facets for $n \geq 5$, and exhibit a simple plane arrangement with $(n-4)(2n-3)+5$ external facets. For polytopes and arrangements, we refer to the books of Edelsbrunner [3], Grünbaum [6] and Ziegler [7] and the references therein.

2 The complexity of the envelope of line arrangements

2.1 A lower bound

Proposition 2.1. *For $n \geq 4$, a simple line arrangement has at least $2(n-1)$ external facets.*

Proof. The external vertices of a line arrangement can be divided into three types, namely v_2, v_3 and v_4, corresponding to external vertices respectively incident to 2, 3, and 4 bounded edges. Let us assign to each external vertex v a weight of 1 and redistribute it to the 2 lines intersecting at v the following way: If v is incident to exactly 1 unbounded edge, then give weight 1 to the line containing this edge, and weight 0 to the other line containing v; if v is incident to 2 or 0 unbounded edges, then give weight 0.5 to each of the 2 lines intersecting at v. See Figure 1 for an illustration of the weight distribution. A total of $f_0^0(\mathcal{A}_{2,n})$ weights is distributed and we can also count this quantity line-wise. The end vertices of a line being of type v_2 or v_3, we have three types of lines, $h_{2,2}, h_{2,3}$ and $h_{3,3}$, according to the possible types of their end-vertices. As a line of type $h_{3,3}$ contains 2 vertices of type v_3, its weight is at least 2. Similarly the weight of a line of type $h_{2,2}$ weight is at least 1. Remarking that a line of type $h_{2,3}$ contains at least one vertex of type v_4 yields that the weight of a line of type $h_{2,3}$ is at least $1 + 0.5 + 0.5 = 2$. For $n \geq 4$ the number of lines of type $h_{2,2}$ is at most 2 as otherwise the envelope would be convex which is impossible, see for example [4]. Therefore, counting the total distributed weight line-wise, we have $f_0^0(\mathcal{A}_{2,n}) \geq 2n - 2$. Since for a line arrangement the number of external facets $f_1^0(\mathcal{A}_{2,n})$ is equal to the number of external vertices $f_0^0(\mathcal{A}_{2,n})$, we have $f_1^0(\mathcal{A}_{2,n}) \geq 2(n-1)$. □

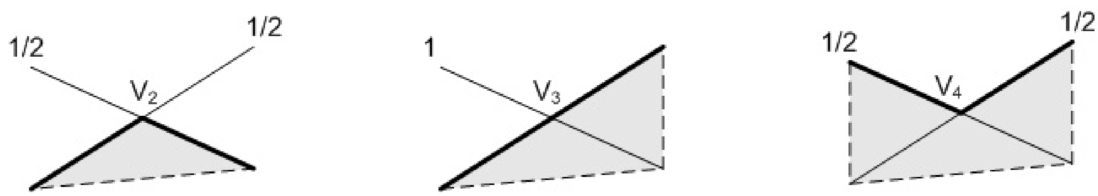

Figure 1: The weight distribution for the lines of an arrangement (the shaded area corresponds to the bounded cells).

2.2 A line arrangement attaining the lower bound

For $n \geq 4$, consider the following simple line arrangement: $\mathcal{A}^o_{2,n}$ is made of the 2 lines h_1 and h_2 forming, respectively, the x_1 and x_2 axis, and $(n-2)$ lines defined by their intersections with h_1 and h_2. We have $h_k \cap h_1 = \{1 + (k-3)\varepsilon, 0\}$ and $h_k \cap h_2 = \{0, 1 - (k-3)\varepsilon\}$ for $k = 3, 4, \ldots, n-1$, and $h_n \cap h_1 = \{2, 0\}$ and $h_n \cap h_1 = \{0, 2 + \varepsilon\}$ where ε is a constant satisfying $0 < \varepsilon < 1/(n-3)$. See Figure 2 for an arrangement combinatorially equivalent to $\mathcal{A}^o_{2,7}$. One can easily check that $\mathcal{A}^o_{2,7}$ has $2(n-1)$ external facets and therefore the lower bound given in Proposition 2.1 is tight.

Proposition 2.2. *For $n \geq 4$, the minimum possible number of external facets of a simple line arrangement is $2(n-1)$.*

3 The complexity of the envelope of plane arrangements

3.1 A lower bound

Proposition 3.1. *For $n \geq 5$, a simple plane arrangement has at least $\frac{n(n-2)+6}{3}$ external facets.*

Proof. Let h_i for $i = 1, 2, \ldots, n$ be the planes forming the arrangement $\mathcal{A}_{3,n}$. For $i = 1, 2, \ldots, n$, the external vertices of the line arrangement $\mathcal{A}_{3,n} \cap h_i$ are external vertices of the plane arrangement $\mathcal{A}_{3,n}$. For $n \geq 5$, the line arrangement $\mathcal{A}_{3,n} \cap h_i$ has at least $2(n-2)$ external facets by Proposition 2.1, i.e., at least $2(n-2)$ external vertices. Since an external vertex of $\mathcal{A}_{3,n}$ belongs to 3 planes, it is counted three times. In other words, the number of external vertices of $\mathcal{A}_{3,n}$ satisfies

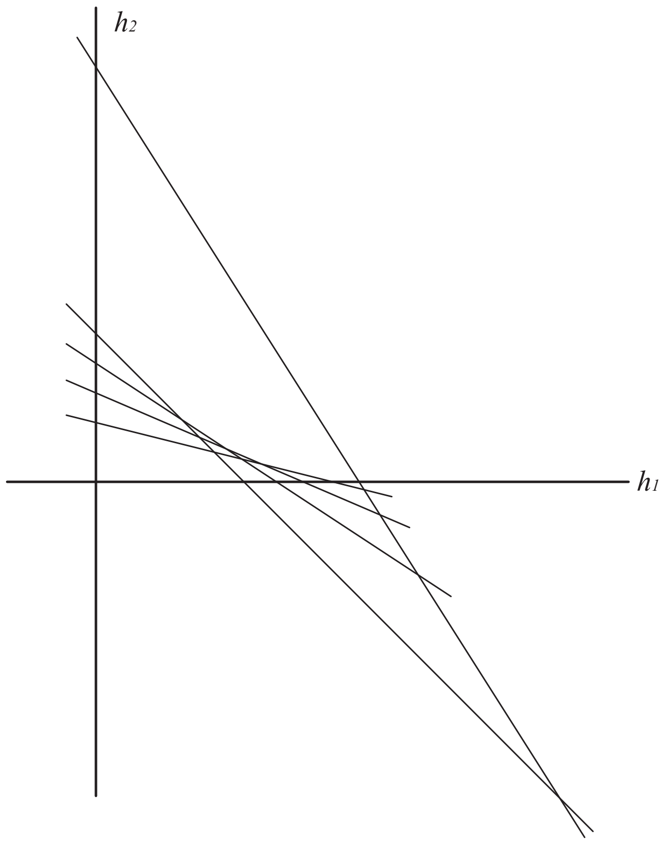

Figure 2: An arrangement combinatorially equivalent to $\mathcal{A}^o_{2,7}$

$f^0_0(\mathcal{A}_{3,n}) \geq \frac{2n(n-2)}{3}$ for $n \geq 5$. As the union of all of the bounded cells is a piecewise linear ball, see [2], the Euler characteristic of the boundary gives $f^0_0(\mathcal{A}_{3,n}) - f^0_1(\mathcal{A}_{3,n}) + f^0_2(\mathcal{A}_{3,n}) = 2$. Since an external vertex belong to at least 3 external edges, we have $2f^0_1(\mathcal{A}_{3,n}) \geq 3f^0_0(\mathcal{A}_{3,n})$. Thus, we have $2f^0_2(\mathcal{A}_{3,n}) \geq f^0_0(\mathcal{A}_{3,n}) + 4$. As $f^0_0(\mathcal{A}_{3,n}) \geq \frac{2n(n-2)}{3}$, it gives $f^0_2(\mathcal{A}_{3,n}) \geq \frac{n(n-2)+6}{3}$ □

3.2 A plane arrangement with few external facets

For $n \geq 5$, we consider following simple plane arrangement: $\mathcal{A}^o_{3,n}$ is made of the 3 planes h_1, h_2 and h_3 corresponding, respectively, to $x_3 = 0$, $x_2 = 0$ and $x_1 = 0$, and $(n-3)$ planes defined by their intersections with the x_1, x_2 and x_3 axis. We have $h_k \cap h_1 \cap h_2 = \{1 + 2(k-4)\varepsilon, 0, 0\}$, $h_k \cap h_1 \cap h_3 = \{0, 1 + (k-4)\varepsilon, 0\}$ and $h_k \cap h_2 \cap h_3 = \{0, 0, 1 - (k-4)\varepsilon\}$ for $k = 4, 5, \ldots, n-1$, and $h_n \cap h_1 \cap h_2 = \{3, 0, 0\}$, $h_n \cap h_1 \cap h_3 = \{0, 2, 0\}$ and $h_n \cap h_2 \cap h_3 = \{0, 0, 3 + \varepsilon\}$ where ε is a constant satisfying $0 < \varepsilon < 1/(n-4)$. See Figure 3 for an illustration of an arrangement combinatorially equivalent to $\mathcal{A}^o_{3,7}$ where, for clarity, only the bounded cells belonging to the positive orthant are drawn.

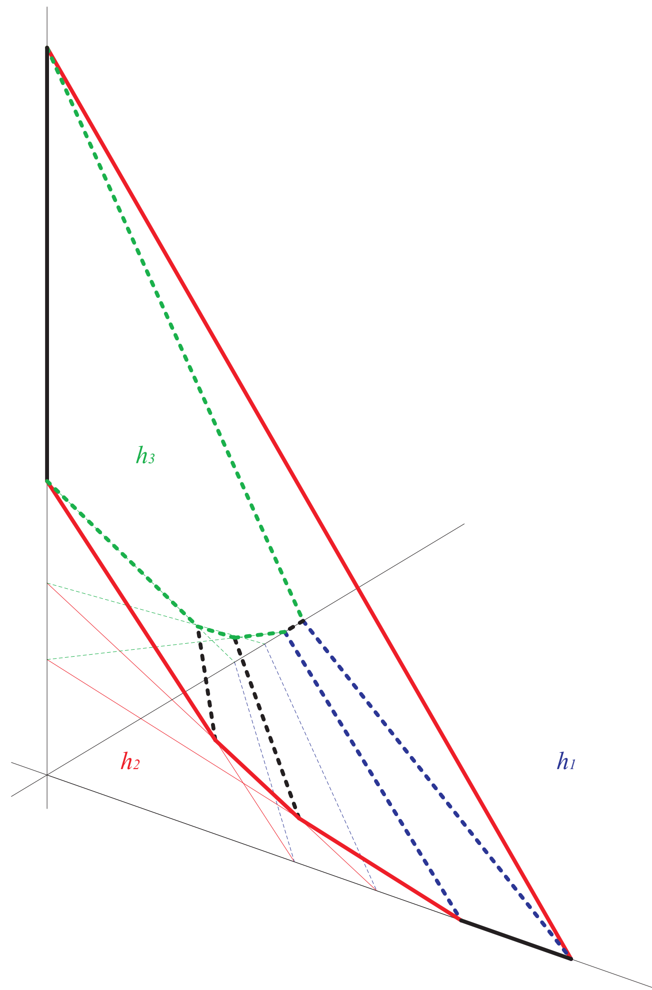

Figure 3: An arrangement combinatorially equivalent to $\mathcal{A}^o_{3,7}$

We first check by induction that the arrangement $\mathcal{A}^*_{3,n}$ formed by the first n planes of $\mathcal{A}^o_{n+1,3}$ has $2(n-2)(n-3)$ external facets. The arrangement $\mathcal{A}^*_{3,n}$ is combinatorially equivalent to the plane cyclic arrangement which is dual to the cyclic polytope, see [5] for combinatorial properties of the (projective) cyclic arrangement in general dimension. See Figure 4 for an illustration of $\mathcal{A}^*_{3,6}$. Let H_3^+ denote the half-space defined by h_3 and containing the positive orthant, and H_3^- the other half-space defined by h_3. The union of the bounded cells of $\mathcal{A}^*_{3,n}$ in H_3^- is combinatorially equivalent to the bounded cells of $\mathcal{A}^*_{3,n-1}$ and therefore has $2(n-3)(n-4)$ facets on its boundary by induction hypothesis, including $\binom{n-3}{2}$ bounded facets contained in h_3. These $\binom{n-3}{2}$ bounded facets also belong to a bounded cell of $\mathcal{A}^*_{3,n}$ in H_3^- and therefore are not external facets of $\mathcal{A}^*_{3,n}$. Thus, the number of external facets of $\mathcal{A}^*_{3,n}$ belonging to a bounded cell in H_3^- is $2(n-3)(n-4) - \binom{n-3}{2}$. The union of the bounded cells of $\mathcal{A}^*_{3,n}$ in H_3^+ can be viewed as a simplex cut by $n-4$ sliding down planes. It has $2\binom{n-2}{2} + 2(n-3) = n(n-3)$ facets on its boundary, including the $\binom{n-3}{2}$

bounded facets contained in h_3 belonging to a bounded cell of $\mathcal{A}^*_{3,n}$ in H_3^-. Thus, the number of external facets of $\mathcal{A}^*_{3,n}$ belonging to a bounded cell in H_3^+ is $n(n-3) - \binom{n-3}{2}$. Therefore, $\mathcal{A}^*_{3,n}$ has $n(n-3) + 2(n-3)(n-4) - 2\binom{n-3}{2} = 2(n-2)(n-3)$ external facets. We now consider how the addition of h_n to $\mathcal{A}^*_{3,n-1}$ impacts the number of external facets. This impact is similar in nature to the addition of h_n to the first $n-1$ lines of $\mathcal{A}^o_{2,n}$. The addition of h_n creates $\binom{n}{2}$ new bounded cells: one above h_1 that we call the n-*shell*, and the other ones being below h_1. The n-shell turns $n-4$ external facets of $\mathcal{A}^*_{3,n-1}$ above h_1 into internal facets of $\mathcal{A}^o_{3,n}$, and adds 3 external facets. For each external facet of $\mathcal{A}^*_{3,n-1}$ belonging to h_1 which is turned into an internal facet of $\mathcal{A}^o_{3,n}$, one external facet of $\mathcal{A}^o_{3,n}$ on h_n and not incident to h_1 is added. Below h_1, the addition of h_n creates $3(n-4)+2$ new external facets of $\mathcal{A}^o_{3,n}$ with an edge on h_1. Finally, $n-4$ new external facets belonging to h_1 and bounded by h_n are created from unbounded facets of $\mathcal{A}^*_{3,n-1}$. Thus, the total number of external facets of $\mathcal{A}^o_{3,n}$ is $2(n-3)(n-4) - (n-4) + 3 + (3(n-4)+2) + (n-4) = (n-4)(2n-3) + 5$.

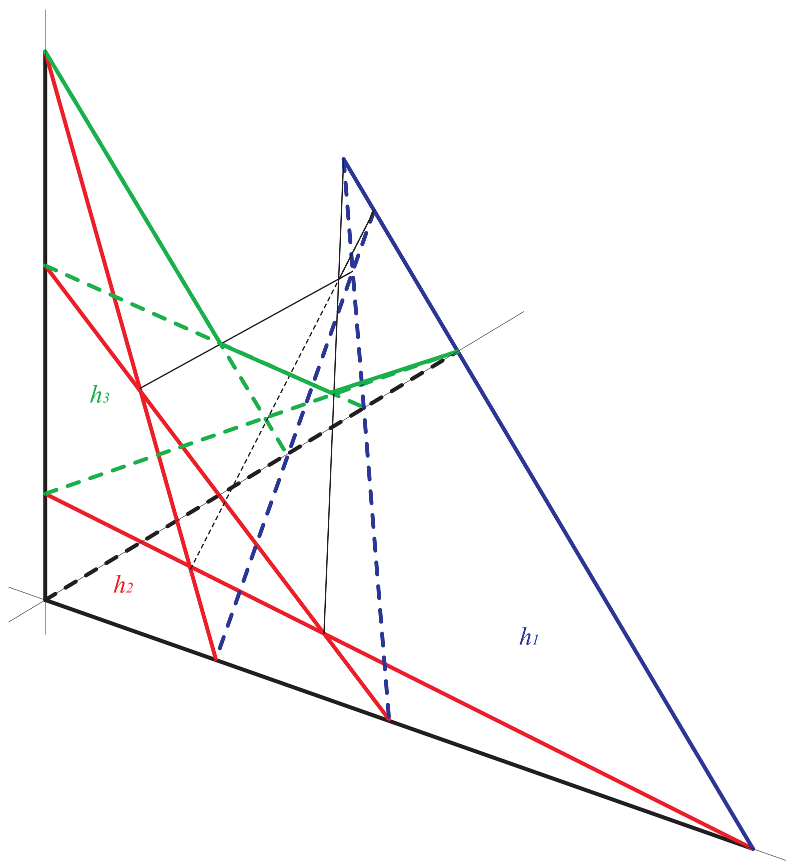

Figure 4: An arrangement combinatorially equivalent to $\mathcal{A}^*_{3,6}$

Remark 3.1. *We do not believe that $\mathcal{A}^o_{3,n}$ minimizes the number of external facets. Among the 43 simple combinatorial types of arrangements formed by 6 planes, the minimum number of external facets is* 22 *while $\mathcal{A}^o_{3,6}$ has 23 external facets. See Figure 5 for an illustration of the combinatorial type of one of the two simple arrangements with* 6 *planes having 22 external facets. The far away* *vertex on the right and 3 bounded edges incident to it are cut off (same for the far away vertex on the left) so the 10 bounded cells of the arrangement appear not too small.*

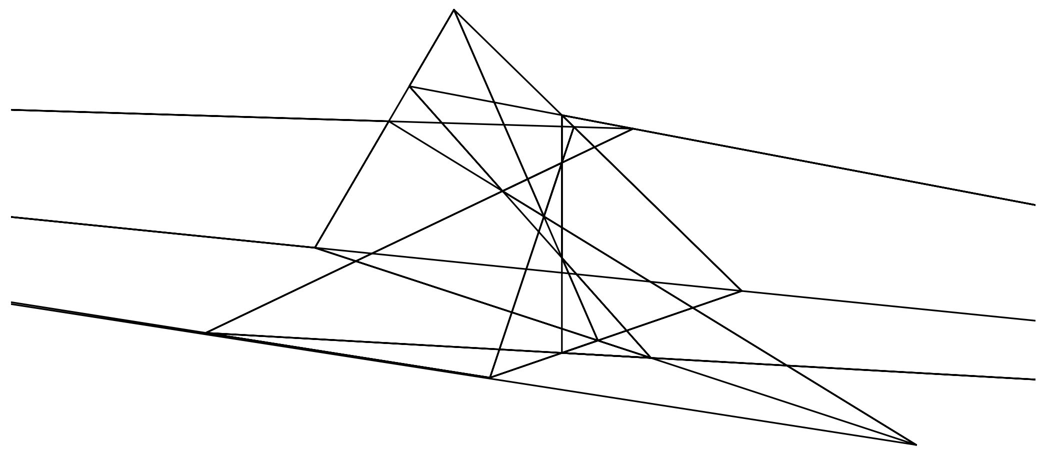

Figure 5: An arrangement formed by 6 planes and having 22 external facets

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