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Solitaire Lattices

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Abstract. One of the classical problems concerning the *peg solitaire* game is the feasibility issue. Tools used to show the infeasibility of various peg games include valid inequalities, known as *pagoda-functions*, and the so-called *rule-of-three*. Here we introduce and study another necessary condition: the *solitaire lattice criterion*. While the lattice criterion is shown to be equivalent to the rule-of-three for the classical English 33-board and French 37-board as well as for any $m \times n$ board, the lattice criterion is stronger than the rule-of-three for games played on more complex boards. In fact, for a wide family of boards presented in this paper, the lattice criterion exponentially outperforms the rule-of-three.

1. Introduction

Peg solitaire is a peg game for one player which is played on a board containing a number of holes. The most common modern version uses a cross shaped board with 33 holes – see Fig. 1 – although a 37 hole board is common in France. Computer versions of the game now feature a wide variety of shapes, including rectangles and triangles. Initially the central hole is empty, the others contain pegs. If in some row (column) two consecutive pegs are adjacent to an empty hole in the same row (column), we may make a *move* by removing the two pegs and placing one peg in the empty hole. The objective of the game is to make moves until only one peg remains in the central hole. Variations of the original game, in addition to being played on different boards, also consider various alternative starting and finishing configurations.

The game itself has uncertain origins, and different legends attest to its discovery by various cultures. An authoritative account with a long annotated bibliography can be found in the comprehensive book of Beasley [4]. The book mentions an engraving of Berey, dated 1697, of a lady with a solitaire board. The book also contains a quotation of Leibniz which was written for the Berlin Academy in 1710. Apparently the first theoretical study of the game that was

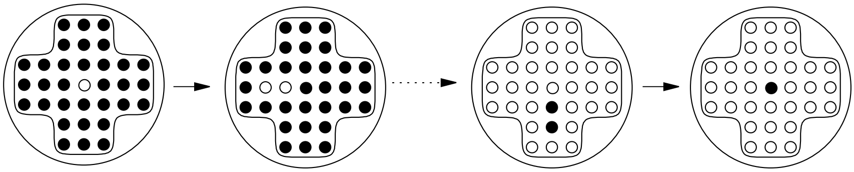


Fig. 1. A feasible English peg solitaire game with possible first and last moves

published was done in 1841 by Suremain de Missery, and was reported in a paper by Vallot [11]. The modern mathematical study of the game dates back to the sixties, see [5, Chapter 23].

The fundamental problem of peg solitaire is the following *feasibility problem* (see Definition 2.1 in the sequel for a formal definition):

Peg Solitaire Feasibility Problem. Given a board B and a pair of configurations (c, c') on B, determine if the pair (c, c') is *feasible*, that is, if there is a legal sequence of moves transforming c into c'.

Well known constructions used to prove that some pair (c, c') is infeasible include the so-called *rule-of-three* (presented in detail in Section 2) and the *pagoda-functions*; see Remark 2.4. In this article we introduce and study another necessary condition for feasibility: the *solitaire lattice criterion*. While the lattice criterion is shown to be equivalent to the rule-of-three for the 33-board and 37-board as well as for any $m \times n$ boards, it is stronger for games played on more complex boards such as the one given in Fig. 4 and 8. The solitaire lattice criterion even exponentially outperforms the rule-of-three for various classes of boards such as the *hook boards* depicted in Fig. 6 and 7. In this article, we determine the solitaire lattice of any rectangular board. It is a challenging problem for further investigation to characterize the *Gröbner bases* of solitaire lattices (see [10] for a state-of-the-art exposition to Gröbner bases theory and [7, 8] for several illustrative Gröbner bases characterizations).

2. The Rule-of-Three

In this section we introduce some terminology used throughout this paper, and recall the so-called rule-of-three (cf. [4, 5]), a classical construction used to test solitaire game feasibility. The rule-of-three was apparently first exposed in 1841 by Suremain de Missery; see Beasley's book [4] for a detailed historical background. The rule-of-three can be used, for example, to show that on the classical cross shaped English 33-board, starting with the initial configuration c_0 of Fig. 1, the only reachable final configurations with *exactly one* peg are c'_0 of Fig. 1 and four others (see discussion following Proposition 2.2).

2.1. Basic Definitions

The *board* of a peg solitaire game is a *finite* subset $B \subset \mathbb{Z}^2$ of the lattice of integer points in the plane. Thus, B stands for the set of locations (i,j) of holes of the board on which the game is played. For example, the classical 33-board is:

$$B = \{(i,j) : -1 \le i \le 1,\ -3 \le j \le 3\} \bigcup \{(i,j) :\ -3 \le i \le 3,\ -1 \le j \le 1\}\ .$$

A *configuration* c on the board is an integer vector $c \in \mathbb{Z}^B \subset \mathbb{R}^B$. When $c \in \{0,1\}^B$, it can be interpreted as a physical configuration of pegs on the board. The *complement* of a $\{0,1\}$-configuration $c \in \{0,1\}^B$ is defined to be the configuration $\bar{c} := \mathbb{1} - c$ where $\mathbb{1} = (1,1,\ldots,1) \in \mathbb{R}^B$ is the all-ones configuration.

For each $(i,j) \in B$ let $e_{i,j}$ be the (i,j)th unit vector in $\mathbb{R}^B$, which is in particular the configuration with a unique peg in the (i,j)th position. A *move* or a *jump* $m_{i,j}$ over (i,j) is a vector in $\mathbb{R}^B$ which has 3 non-zero entries: two entries of -1 in the positions from which pegs are removed and one entry of 1 for the hole receiving the new peg. A move $m_{i,j}$ over (i,j) could be one of the following:

Right move:	$r_{i,j} = e_{i+1,j} - e_{i,j} - e_{i-1,j}$.
Left move:	$l_{i,j} = e_{i-1,j} - e_{i,j} - e_{i+1,j}$.
Down move:	$d_{i,j} = e_{i,j-1} - e_{i,j} - e_{i,j+1}$.
Up move:	$u_{i,j} = e_{i,j+1} - e_{i,j} - e_{i,j-1}$.

We can now make the Peg Solitaire Feasibility Problem precise.

Definition 2.1. *A pair (c,c') of configurations on a board B is feasible if there is a sequence $m^1,\ldots,m^k \in \mathbb{Z}^B$ of moves on B such that $c' = c + \sum_{i=1}^k m^i$ and such that $c + \sum_{j=1}^i m^j \in \{0,1\}^B$ for $i = 1,\ldots,k$.*

For instance, the English 33-board admits 76 moves. The moves $r_{-1,0}$ and $u_{0,-1}$ are the first and last moves – see Fig. 1 – in some sequence of moves transforming the initial configuration $c_0 = \mathbb{1} - e_{0,0}$ to its complementary final configuration $c'_0 = e_{0,0}$ in that classical game.

2.2. The Rule-of-Three

Let $\mathbb{Z}_2 := \{a,b,c,e\}$ be the Abelian group with identity e and addition table

$$a+a = b+b = c+c = e, \quad a+b = c, \quad a+c = b, \quad b+c = a\ .$$

Define the following two maps $g_1, g_2 : \mathbb{Z}^2 \longrightarrow \mathbb{Z}_2$, which simply color the integer lattice $\mathbb{Z}^2$ by diagonals of a, b and c in either direction; see Fig. 2:

$$g_1(i,j) := \begin{cases} a & \text{if } (i+j) \equiv 0 \pmod 3 \\ b & \text{if } (i+j) \equiv 1 \pmod 3, \\ c & \text{if } (i+j) \equiv 2 \pmod 3 \end{cases} \quad g_2(i,j) := \begin{cases} a & \text{if } (i-j) \equiv 0 \pmod 3 \\ b & \text{if } (i-j) \equiv 1 \pmod 3 \\ c & \text{if } (i-j) \equiv 2 \pmod 3 \end{cases}\ .$$

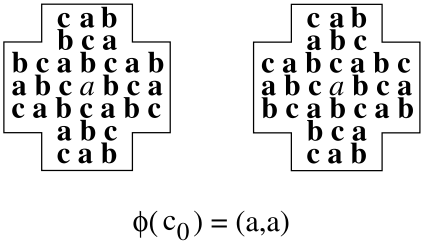


Fig. 2. The score of the initial configuration of the English board

For any board $B \subset \mathbb{Z}^2$, define the *score map* to be the $\mathbb{Z}$-module homomorphism

$$\begin{aligned} \phi : \mathbb{Z}^B &\longrightarrow \mathbb{Z}_2^2 \\ e_{i,j} &\mapsto \big(g_1(i,j)\ ,\ g_2(i,j)\big)\ . \end{aligned}$$

Thus, the score of a configuration $c \in \mathbb{Z}^B$ is given by

$$\phi(c) = \sum_{(i,j)\in B} c_{i,j} \cdot \big(g_1(i,j)\ , g_2(i,j)\big)\ .$$

Since the board B under discussion will always be clear from the context, we use the notation ϕ for any board. For instance, the score of the configuration $e_{0,0}$ of one peg in the center of the English 33-board has the score $\phi(e_{0,0}) = (a,a)$, as is also the score of its complement $\mathbb{1} - e_{0,0}$; see Fig. 2. The score of the board B is defined to be

$$\phi(B) := \phi(\mathbb{1}) = \sum_{(i,j)\in B} \phi(e_{i,j}) = \sum_{(i,j)\in B} \big(g_1(i,j)\ ,\ g_2(i,j)\big)\ .$$

Note that $\phi(B) = \phi(c) + \phi(\bar{c})$ for any configuration c and its complement $\bar{c}$. Thus, the score of the English 33-board is $\phi(B) = \phi(e_{0,0}) + \phi(\mathbb{1} - e_{0,0}) = (a,a) + (a,a) = (e,e)$.

It is easy to verify that any feasible move $m_{i,j}$ on any board B has the identity score $\phi(m_{i,j}) = (e,e)$. This gives the following proposition.

Proposition 2.2. [The rule-of-three]. *A necessary condition for a pair of configuration (c,c') to be feasible is that $\phi(c'-c) = (e,e)$, namely, $c'-c \in Ker(\phi)$.*

Using Proposition 2.2, it can be shown that on the cross shaped English 33-board, starting with the initial configuration c_0 of Fig. 1, the only reachable final configurations with *exactly one* peg are c'_0 (given in Fig. 1), c'_1, c'_2, c'_3 and c'_4 with, respectively, a final peg in position $(0,0), (-3,0), (0,3), (3,0)$ and $(0,-3)$.

Proposition 2.3. *Let B be any board. A necessary condition for the configurations pair $(c,\bar{c})$ to be feasible, with $\bar{c} = \mathbb{1} - c$ the complement of c, is that the board score is $\phi(B) = (e,e)$.*

Proof. It follows from Proposition 2.2 that $(c, \bar{c})$ is feasible only if $\phi(\bar{c}) = \phi(c)$, which is equivalent to $\phi(B) = \phi(c) + \phi(\bar{c}) = \phi(c) + \phi(c) = (e, e)$. □

An application of Proposition 2.3 is that complementary games are not feasible on the following *French* board given in Fig. 3 since $\phi(B) = (a, a)$. A Board B satisfying $\phi(B) = (e, e)$ is called a *null-class* board in [4]. Clearly, a board is null if and only if, under each of the labellings g_1, g_2, the numbers of occurrences of each of a, b, c are all of the same parity; this can be used in characterizing certain classes of shapes of null boards.

Remark 2.4. Among other constructions used to prove that a pair (c, c') is not feasible are the *pagoda-functions*; see [4, 5]. The strongest such linear inequalities are the facets of the solitaire cone $Cone(B)$ defined to be the conic hull of all moves $m_{i,j}$. Note that all moves are extreme rays as they belong to the intersection of the hyperplane $x_1 + x_2 + \cdots + x_{|B|} = -1$ and the sphere $x_1^2 + x_2^2 + \cdots + x_{|B|}^2 = 3$. See [1] for some combinatorial properties of this cone which is simply the cone of all fractional feasible games.

3. The Lattice Criterion

3.1. The Solitaire Lattice

Next, we introduce the main object of study in this article. Let $B \subset \mathbb{Z}^2$ be any board and let M be the set of all possible moves $m_{i,j}$ on B. The *solitaire lattice* of B is the lattice of all integer linear combinations of moves,

$$\mathrm{Lat}(B) := \mathrm{Lat}(M) = \mathbb{Z} \cdot M = \left\{ \sum_{m \in M} z_m \cdot m : z_m \in \mathbb{Z} \right\} \subseteq \mathbb{Z}^B \ .$$

The following proposition is immediate from the definition.

Proposition 3.1. *A necessary condition for a pair of configurations (c, c') to be feasible on a board B is that $c' - c \in Lat(B)$.*

The following proposition states that, for any board B and any two configurations c, c' on B, if $c' - c \in \mathrm{Lat}(B)$ then $c' - c \in \mathrm{Ker}(\phi)$. In other words, it shows that the necessary condition for feasibility provided by Proposition 3.1 is gener-

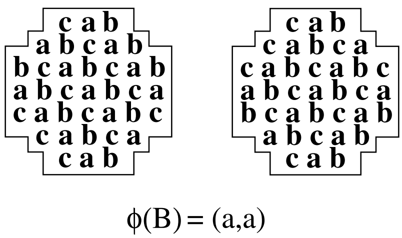


Fig. 3. A non null-class board

ally stronger than the one provided by Proposition 2.2. Therefore it could be more useful in proving non-feasibility, and motivates the close study of the lattice $\mathrm{Lat}(B)$ taken in this paper.

Proposition 3.2. *For any board B, we have $Lat(B) \subseteq Ker(\phi)$.*

Proof. Since ϕ is a homomorphism of $\mathbb{Z}$-modules which maps each lattice generator $m_{i,j} \in M$ to (e,e), it follows that $\phi(v) = (e,e)$ for any $v \in \mathrm{Lat}(B)$. □

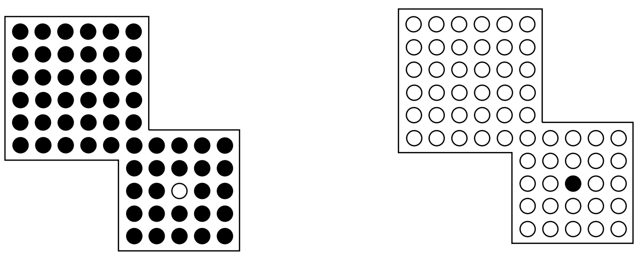
Fig. 4 illustrates that there are games whose infeasibility can be detected by the lattice criterion but not by the rule-of-three. Specifically, it provides an example of a null-class board and a game on it whose associated configuration pair $(c, \bar{c})$ satisfies $\bar{c} - c \in \mathrm{Ker}(\phi)$ but $\bar{c} - c \notin \mathrm{Lat}(B)$; see Section 5.1. This shows that the lattice criterion may be strictly stronger than the rule-of-three.

3.2. The Lattice Criterion Versus the Rule-of-three

We proceed to discuss the relations between the lattice criterion and the rule-of-three, namely, between $\mathrm{Lat}(B)$ and $\mathrm{Ker}(\phi)$, in more detail. Any lattice $L \subseteq Z^d \subset \mathbb{R}^d$ has a *basis*, that is, a free generating set. Any basis is also $\mathbb{R}$-linearly independent. The *rank* of the lattice is the cardinality of any basis, and the lattice is *full rank* if its rank is d, namely, the $\mathbb{R}$-span of the lattice is the entire space $\mathbb{R}^d$. The solitaire lattice of a board B is typically of full rank $|B|$. The *determinant* of a full rank lattice $L \subseteq \mathbb{R}^d$ is defined to be the absolute value $\det(L) := |\det(V)|$ of the determinant of any (ordered) basis $V = [v_1, \ldots, v_d]$ of L. Since $\mathrm{Lat}(B) \subseteq \mathrm{Ker}(\phi)$ for any board B, we always have $\mathbb{Z}^B/\mathrm{Ker}(\phi) \subseteq \mathbb{Z}^B/\mathrm{Lat}(B)$. Therefore, our main task is to compute and compare the indices $[\mathbb{Z}^B : \mathrm{Ker}(\phi)] = |\mathbb{Z}^B/\mathrm{Ker}(\phi)|$ and $[\mathbb{Z}^B : \mathrm{Lat}(B)] = |\mathbb{Z}^B/\mathrm{Lat}(B)|$. For a typical board B, the map ϕ is onto and the lattice $\mathrm{Lat}(B)$ is full rank, giving

$$[\mathbb{Z}^B : \mathrm{Ker}(\phi)] = |\mathrm{Im}(\phi)| = |\mathbb{Z}_2^2| = 16$$

and



Configuration c Configuration $\bar{c}$

Fig. 4. An infeasible game satisfying the rule-of-three but not the solitaire lattice criterion

$$[\mathbb{Z}^B : \mathrm{Lat}(B)] = \det(\mathrm{Lat}(B)) \ .$$

This gives the following useful lemma.

Lemma 3.3. *For a board B such that ϕ is an onto map and $Lat(B)$ is full rank, $Lat(B) = Ker(\phi)$ if and only if $det(Lat(B)) = 16$.*

For example, the board given in Fig. 4, produces a full rank lattice $\mathrm{Lat}(B)$ and an onto map ϕ but $\det(\mathrm{Lat}(B)) = 32$ – see Section 5.1 – and therefore $\mathrm{Lat}(B)$ is strictly contained in $\mathrm{Ker}(\phi)$. The (complementary) game of Fig. 4 is infeasible while satisfying the rule-of-three.

The computation of $[\mathbb{Z}^B : \mathrm{Ker}(\phi)]$ is easy. As the following proposition shows, for any usual board this quantity equals 16.

Proposition 3.4. *If a board B contains a 2×2 sub-board then $[\mathbb{Z}^B : Ker(\phi)] = |Im(\phi)| = 16$.*

Proof. One can easily check that the 16 possible $\{0, 1\}$-configurations on a 2×2 board are mapped by ϕ precisely onto the 16 elements of $\mathbb{Z}_2^2$. $\square$

3.3. Testing the Lattice Criterion Efficiently

Let B be a board and let $V = \{v_1, \ldots, v_d\}$ be a basis of the solitaire lattice $\mathrm{Lat}(B)$. Then for any two given configurations c, c' on B, the necessary condition $c' - c \in \mathrm{Lat}(B)$ for feasibility provided by Proposition 3.1 holds if and only if the linear system $\sum_{i=1}^{d} \lambda_i v_i = c' - c$ is solvable (over $\mathbb{R}$) and its unique solution is an integer vector $\lambda \in \mathbb{Z}^d$. This can be efficiently tested, say by Gaussian elimination. Thus, in order to efficiently facilitate the lattice criterion for game feasibility, we need to determine a basis of $\mathrm{Lat}(B)$. One way to find a basis is by applying the *Hermite normal form algorithm* (cf. Schrijver [9]) to any generating set of the lattice. However, it is more efficient and illuminating to *characterize* a canonical basis for the lattice whenever possible. This approach is taken, for instance, in Lovasz' outstanding work [6] on the *matching lattice*. Our goal in the next section is to provide such a canonical basis for the solitaire lattice of any rectangular board. In fact, the basis we provide is precisely the *Hermite basis* of the lattice which would have been produced by the Hermite normal form algorithm. However, our characterization will allow to write it down at once, without applying the time consuming Hermite normal form algorithm.

4. The Hermite Basis of the Solitaire Lattice

We start by defining the Hermite basis of an arbitrary full rank lattice. Any integer matrix A of full row rank can be transformed by a sequence of unimodular column operations (multiplication of a column by -1, exchange of two columns, or addition of an integer multiple of one column to another), to a *unique* matrix of the form $[V\ 0]$ where V is square, lower triangular, non-negative, and each row of V has a single maximum which is on the main diagonal. The matrix $[V\ 0]$ is called the *Hermite normal form* (or HNF for short) of the matrix A. Moreover, the square matrix V appearing in the HNF depends only on the lattice $\mathrm{Lat}(A)$ generated by the columns of A. We define the *Hermite basis* $\mathrm{Hermite}(L)$ of a full rank lattice $L \subseteq \mathbb{Z}^d$ to be the canonical ordered basis consisting of the columns of V where $[V\ 0]$ is the HNF of any matrix A whose columns generate L. To ease the notation, in the following we simply identify the Hermite basis and the square matrix V. For a thorough discussion of the Hermite normal form and its applications see the book [9]. We proceed to characterize the canonical Hermite basis of the solitaire lattice of any rectangular board. In what follows we are going to represent elements of $\mathbb{Z}^B \cong \mathbb{Z}^{|B|}$ by column vectors, where the coordinates are ordered by the lexicographic top-to-bottom left-to-right order on $B \subset \mathbb{Z}^2$, that is, $(i,j) \in B$ precedes $(r,s) \in B$ if either $j > s$ or $j = s$ and $i < r$.

4.1. Playing on a $1 \times n$ Board

We consider first the $1 \times n$ board $B_{1,n} := ((0,0),(1,0),\ldots,(n-1,0))$ shown in Fig. 5 where, without loss of generality, we take the leftmost hole to be the origin $(0,0)$. It has $2(n-2)$ moves which are precisely the $n-2$ right moves $r_{i,0}$ and $n-2$ left moves $l_{i,0}$ for $1 \le i \le n-2$.

Let $F(n)$ denote the Fibonacci sequence $(\ldots,-3,2,-1,1,0,1,1,2,3,\ldots)$ defined for all integers by $F(n) = F(n-1) + F(n-2)$ with $F(0) = 0$ and $F(-1) = F(1) = 1$. Let $f(n)$ denote the symmetric sequence $(\ldots,1,0,1,1,0,1,1,0,1,\ldots)$ obtained by reducing the Fibonacci sequence modulo 2, so $f(n) = 0$ if $n \equiv 0 \pmod 3$ and $f(n) = 1$ otherwise. The following theorem provides a characterization of the Hermite basis of any $1 \times n$ board $B_{1,n}$ and shows that $\mathrm{Lat}(B_{1,n}) = \mathrm{Ker}(\phi)$.

Theorem 4.1. *Let $B_{1,n}$ be any $1 \times n$ board with $n \ge 4$. Then the following hold for the solitaire lattice $Lat(B_{1,n})$:*

1. *The Hermite basis of $Lat(B_{1,n})$ is*

$$V_{1,n} = \left[\begin{array}{ccc|cc} & I_{n-2} & & & 0 \\ \hline f(n-1) & \ldots & f(2) & 2 & 0 \\ f(n-2) & \ldots & f(1) & 0 & 2 \end{array}\right] = \left[\begin{array}{ccccccc|cc} & & & I_{n-2} & & & & & 0 \\ \hline \ldots 1 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 0 \\ \ldots 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 2 \end{array}\right].$$

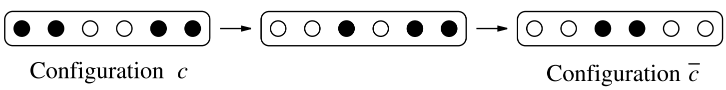


Fig. 5. A feasible game on the 1×6 board

2. *The solitaire lattice $Lat(B_{1,n})$ is characterized by:*

$$c \in Lat(B_{1,n}) \iff \begin{cases} \sum\{c_{i,0} : 0 \le i \le n-1,\ i \not\equiv 0 \pmod 3\} \equiv 0 \pmod 2 \\ \sum\{c_{i,0} : 0 \le i \le n-1,\ i \not\equiv 1 \pmod 3\} \equiv 0 \pmod 2 \end{cases}$$

3. *$Lat(B_{1,n})$ has full rank with determinant $det(Lat(B_{1,n})) = 4$.*
4. *$[\mathbb{Z}^{B_{1,n}} : Ker(\phi)] = 4 = [\mathbb{Z}^{B_{1,n}} : Lat(B_{1,n})]$ hence $Lat(B_{1,n}) = Ker(\phi)$.*

Proof. To prove (1), let $A_{1,n}$ be the integer $n \times 2(n-2)$ integer matrix whose columns are the generators $r_{1,0}, \dots, r_{n-2,0}, l_{1,0}, \dots, l_{n-2,0}$ of the lattice $\mathrm{Lat}(B_{1,n})$. Applying to $A_{1,n}$ the following unimodular column operations,

$$r^1_{i,0} := -\sum_{k=i}^{n-2} F(i-k-1) r_{k,0}, \quad l^1_{i,0} := -\sum_{k=i}^{n-2} F(k-i+1) l_{k,0}, \qquad 1 \le i \le n-2 \ ,$$

we obtain the matrix:

$$A'_{1,n} = \left[\begin{array}{cccc|cccc} & I_{n-2} & & & & -I_{n-2} & & \\ \hline -F(1-n) & -F(2-n) & \dots & -F(-2) & F(n-1) & F(n-2) & \dots & F(2) \\ -F(2-n) & -F(3-n) & \dots & -F(-1) & F(n-2) & F(n-3) & \dots & F(1) \end{array}\right].$$

Applying now

$$r^2_{i,0} := r^1_{i,0}, \quad l^2_{i,0} := l^1_{i,0} + r^1_{i,0}, \qquad 1 \le i \le n-2 \ ,$$

we obtain

$$A''_{1,n} = \left[\begin{array}{cccc|ccccc} & I_{n-2} & & & & & 0_{n-2} & & \\ \hline -F(1-n) & -F(2-n) & \dots & -F(-2) & \dots\ 2F(6) & 0 & 2F(4) & 0 & 2 \\ -F(2-n) & -F(3-n) & \dots & -F(-1) & \dots\ \ 0 & 2F(4) & 0 & 2 & 0 \end{array}\right].$$

Using the last two columns $l^2_{n-2,0}$ and $l^2_{n-3,0}$ to round modulo 2 the other columns, we obtain the Hermite normal form of $A_{1,n}$. Then, dropping all zero columns and interchanging the last two columns, we get the desired Hermite basis $V_{1,n} := \mathrm{Hermite}(B_{1,n})$ of $\mathrm{Lat}(B_{1,n})$ as claimed in (1).

Next we prove (2) from (1) (note that (1) also follows from (4) below). We have that $c \in \mathrm{Lat}(B_{1,n})$ if and only if $c = V \cdot \lambda$ for some $\lambda = (\lambda_0, \dots, \lambda_{n-1}) \in \mathbb{Z}^n$. This holds if and only if $\lambda_i = c_{i,0}$ for $0 \le i \le n-3$ and

$$
\begin{aligned}
c_{n-2,0} &= 2\lambda_{n-2} + \sum_{i=0}^{n-3} f(n-1-i) \cdot c_{i,0} \\
&= 2\lambda_{n-2} + c_{n-3,0} + c_{n-5,0} + \cdots + f(n-1) \cdot c_{0,0} \ ,
\end{aligned}
$$

$$
\begin{aligned}
c_{n-1,0} &= 2\lambda_{n-1} + \sum_{i=0}^{n-3} f(n-2-i) \cdot c_{i,0} \\
&= 2\lambda_{n-1} + c_{n-3,0} + c_{n-4,0} + \cdots + f(n-2) \cdot c_{0,0} \ .
\end{aligned}
$$

It follows that $c \in \mathrm{Lat}(B_{1,n})$ if and only if

$$
\sum_{i=0}^{n-1} f(n-1-i) \cdot c_{i,0} \ \equiv \ 0 \ (\mathrm{mod}\ 2) \quad \text{and} \quad \sum_{i=0}^{n-1} f(n-2-i) \cdot c_{i,0} \ \equiv \ 0 \ (\mathrm{mod}\ 2)
$$

which translates to (2). For (3) it is clear that $\mathrm{Lat}(B_{1,n})$ has full rank $n = |B|$, and its determinant is the product of the entries on the main diagonal of the Hermite basis hence equals 4. Finally, it is easy to see that $|\mathrm{Im}(\phi)| = 4$ which, together with $\det(\mathrm{Lat}(B_{1,n})) = 4$ implies (4). □

4.2. Playing on an $m \times n$ Board

Since the $2 \times n$ board amounts to two independent $1 \times n$ boards, we have, taking the lower leftmost hole to be $(0, 0)$:

Theorem 4.2. *Let $B_{2,n}$ be any $2 \times n$ board with $n \geq 4$. Then the following hold for the solitaire lattice $Lat(B_{2,n})$:*

1. *The Hermite basis of $Lat(B_{2,n})$ is*

$$
V_{2,n} = \left[\begin{array}{c|c} V_{1,n} & 0_n \\ \hline 0_n & V_{1,n} \end{array} \right].
$$

2. *The solitaire lattice $Lat(B_{2,n})$ is characterized by:*

$$
c \in Lat(B_{2,n}) \iff \begin{cases}
\sum \{ c_{i,0} : 0 \leq i \leq n-1, i \not\equiv 0 \ (\mathrm{mod}\ 3) \} \ \equiv \ 0 \ (\mathrm{mod}\ 2) \\
\sum \{ c_{i,0} : 0 \leq i \leq n-1, i \not\equiv 1 \ (\mathrm{mod}\ 3) \} \ \equiv \ 0 \ (\mathrm{mod}\ 2) \\
\sum \{ c_{i,1} : 0 \leq i \leq n-1, i \not\equiv 0 \ (\mathrm{mod}\ 3) \} \ \equiv \ 0 \ (\mathrm{mod}\ 2) \\
\sum \{ c_{i,1} : 0 \leq i \leq n-1, i \not\equiv 1 \ (\mathrm{mod}\ 3) \} \ \equiv \ 0 \ (\mathrm{mod}\ 2)
\end{cases}
$$

3. *$Lat(B_{2,n})$ has full rank with determinant $det(Lat(B_{2,n})) = 16$.*
4. *$[\mathbb{Z}^{B_{2,n}} : Ker(\phi)] = 16 = [\mathbb{Z}^{B_{2,n}} : Lat(B_{2,n})]$ hence $Lat(B_{2,n}) = Ker(\phi)$.*

We now proceed to derive analogs of Theorems 4.1 and 4.2 for any rectangular $m \times n$ board $B_{m,n}$, with $n \geq 4$ or $m \geq 4$. We take the lower leftmost hole to be $(0, 0)$. The set of moves is the $2m(n-2)$ row moves $r_{i,j}, l_{i,j}$ and the $2n(m-2)$ column moves $u_{i,j}, d_{i,j}$ for $1 \leq j \leq n-2$ and $1 \leq i \leq m-2$.

The Hermite normal form of $\mathrm{Lat}(B_{m,n})$ can be deduced from the one of $\mathrm{Lat}(B_{1,n})$ by the following construction. For $n \geq 3$, let define the $n \times n$ $\{0,1\}$-matrix $F_{1,n}$ by:

$$\begin{cases} F_{1,3} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ F_{1,n} = V_{1,n} - I_n \qquad \text{for} \quad n \geq 4. \end{cases}$$

We have:

Lemma 4.3. *For $n \geq 4$ or $m \geq 4$, the Hermite basis of $Lat(B_{m,n})$ is $V_{m,n} = F_{m,n} + I_{mn}$ where $F_{m,n}$ is an $mn \times mn$ $\{0,1\}$-matrix derived from $F_{1,m}$ by replacing each 0 by 0_n and each 1 by $F_{1,n}$.*

Proof. Let first consider the $2m(n-2)$ row moves. Since each row is a $1 \times n$ board, Theorem 4.1 gives that each set of $2(n-2)$ row moves can be replaced (after dropping all 0 columns) by the $n \times n$ matrix $V_{1,n} = F_{1,n} + I_n$. In other words, the $2m(n-2)$ row moves can be replaced by the following $mn \times mn$ matrix:

$$R_{m,n} = \left[\begin{array}{c|c|c|c|c|c|c} F_{1,n}+I_n & & & & & & \\ \hline & F_{1,n}+I_n & & & & & \\ \hline & & F_{1,n}+I_n & & & & \\ \hline & & & \ddots & & & \\ \hline & & & & F_{1,n}+I_n & & \\ \hline & & & & & F_{1,n}+I_n & \\ \hline & & & & & & F_{1,n}+I_n \end{array}\right].$$

On the other hand, let us write the $2n(m-2)$ column moves the following way. First the down moves in the following order: $d_{0,m-2}, d_{1,m-2}, \ldots, d_{n-1,m-2}, d_{0,m-3}, d_{1,m-3}, \ldots, d_{n-1,m-3}, \ldots, d_{0,1}, d_{1,1}, \ldots, d_{n-1,1}$ and then the corresponding up moves in the same order. Multiplying all the column moves by -1, we get:

$$L_{m,n} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} I_n & & & & & -I_n & & & & \\ \hline I_n & I_n & & & & I_n & -I_n & & & \\ \hline -I_n & I_n & I_n & & & I_n & I_n & -I_n & & \\ \hline & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots & \\ \hline & & -I_n & I_n & I_n & & & I_n & I_n & -I_n \\ \hline & & & -I_n & I_n & & & & I_n & I_n \\ \hline & & & & -I_n & & & & & I_n \end{array}\right].$$

Identifying each I_n by 1, we recognize the row moves of a 1 my m board which can be replaced (after dropping all 0 columns) by $V_{1,m} = F_{1,m} + I_m$. Then easy combinations with $R_{m,n}$ give:

$$V_{m,n} = \left[\begin{array}{c|c|c|c|c|c|c} I_n & & & & & & \\ \hline & I_n & & & & & \\ \hline & & I_n & & & & \\ \hline & & & \ddots & & & \\ \hline & & & & I_n & & \\ \hline \dots & \dots & \dots & 0_n & F_{1,n} & F_{1,n}+I_n & \\ \hline \dots & \dots & \dots & F_{1,n} & F_{1,n} & 0_n & F_{1,n}+I_n \end{array}\right],$$

that is, $V_{m,n} = F_{m,n} + I_{mn}$ and completes the proof. □

For example, for the 4×5 board:

$$F_{1,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \text{ leads to } F_{4,5} = \begin{bmatrix} 0_5 & 0_5 & 0_5 & 0_5 \\ 0_5 & 0_5 & 0_5 & 0_5 \\ 0_5 & F_{1,5} & F_{1,5} & 0_5 \\ F_{1,5} & F_{1,5} & 0_5 & F_{1,5} \end{bmatrix}$$

where

$$F_{1,5} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Directly from Lemma 4.3, we get:

Theorem 4.4. *Let $B_{m,n}$ be any $m \times n$ board with $n \geq 4$ or $m \geq 4$. Then the following hold for the solitaire lattice $Lat(B_{m,n})$:*

1. *The Hermite basis of $Lat(B_{m,n})$ is*

$$V_{m,n} = F_{m,n} + I_{mn} \quad .$$

2. *The solitaire lattice $Lat(B_{m,n})$ is characterized by:*

$$c \in \mathrm{Lat}(B_{m,n}) \Longleftrightarrow \begin{cases} \sum\{c_{i,0} : 0 \leq i \leq n-1,\ i \not\equiv 0 \pmod 3\} \equiv 0 \pmod 2 \\ \sum\{c_{i,0} : 0 \leq i \leq n-1,\ i \not\equiv 1 \pmod 3\} \equiv 0 \pmod 2 \\ \sum\{c_{i,1} : 0 \leq i \leq n-1,\ i \not\equiv 0 \pmod 3\} \equiv 0 \pmod 2 \\ \sum\{c_{i,1} : 0 \leq i \leq n-1,\ i \not\equiv 1 \pmod 3\} \equiv 0 \pmod 2 \end{cases}$$

3. *$Lat(B_{m,n})$ has full rank with determinant $det(Lat(B_{m,n})) = 16$.*

4. $[\mathbb{Z}^{B_{m,n}} : Ker(\phi)] = 16 = [\mathbb{Z}^{B_{m,n}} : Lat(B_{m,n})]$ *hence* $Lat(B_{m,n}) = Ker(\phi)$.

Note that Theorem 4.4 can be extended to the case $m = 1, 2$ simply by setting $F_{1,1} = I_1$ and $F_{1,2} = I_2$.

5. The Lattice Criterion Beats the Rule-of-Three

Building on the results just established on rectangular boards, we can now proceed to exhibit a large class of boards for which $\mathrm{Lat}(B) \neq \mathrm{Ker}(\phi)$.

Using Lemma 4.3, we can efficiently compute the Hermite normal basis of more complex boards. The algorithm consists in covering a board B by overlapping rectangular sub-boards B_i, $i = 1, \ldots, I$. Then we append each matrix $A^i_{m_i,n_i}$ $i = 1, \ldots, I$ consisting of the (row and column) moves within each sub-board B_i and add the *cross moves*, that is, the moves from one sub-board to another. Since the number of cross moves is, in general, quite small, the computation of the resulting Hermite normal form of the initial board B is, in general, a bit tedious but easy. This algorithm also provides a rough upper bound for the determinant, that is, $\det(\mathrm{Lat}(B)) \leq \prod_{i=1}^{I} \det(\mathrm{Lat}(B_i)) \leq 2^{4I}$.

Starting with the 60 holes board given in Fig. 4, we illustrate this method by some examples of solitaire lattices with determinant 2^k for arbitrary $k \geq 2$.

5.1. *A Board with* $Lat(B) \neq Ker\ (\phi)$

The board B_{60} given in Fig. 4, is made of two overlapping square boards $B_{6,6}$ and $B_{5,5}$ with 4 cross-moves centered on the common hole. Using Theorem 4.4, the 60×160 integer matrix whose columns are the generators (moves) of $\mathrm{Lat}(B_{60})$ can be brought by unimodular column operation to:

$$\left[\begin{array}{cccc|cccc|cccc}
1 & & & & & & & & & & & \\
 & \ddots & & & & & & & & & & \\
 & & V_{6,6} & & & & & & & & -1 & 1 \\
 & & & \ddots & & & & & -1 & 1 & & \\
\hline
\cdots & 1 & 0 & 2 & 1 & & & & -1 & -1 & -1 & -1 \\
\hline
 & & & & & \ddots & & & 1 & -1 & & \\
 & & & & & & V_{5,5} & & & & 1 & -1 \\
 & & & & & & & \ddots & & & & \\
 & & & & & & & 2 & & & &
\end{array}\right],$$

which leads by Gaussian elimination to the Hermite basis $V_{60} = F_{60} + I_{60}$ where F_{60} is:

$$\left[\begin{array}{l|l|l|l|l|l}
0_{28} & & & & & \\ \hline
\ldots\ 1\ \ 1\ \ 0\ 1 & 1 & & & & \\
\ldots\ 0\ \ 0\ \ 0\ 0 & 0\ 0 & & & & \\ \hline
 & & 0_{23} & & & \\ \hline
\ldots\ 1\ \ 0\ \ 1\ 1 & 0\ 1 & 0\ \star\ 0\ 0\ 0\ 0\ 0\ \ 1\ \ 0\ 1\ 1\ 0\ 1\ 0\ 1 & 1 & & \\
\ldots\ 0\ \ 0\ \ 0\ 0 & 0\ 0 & 1\ \star\ 0\ 0\ 0\ 0\ 0\ \ 0\ \ 1\ 1\ 0\ 1\ 0\ 1\ 1 & 0\ 1 & & \\ \hline
 & & & & 0_3 & \\ \hline
\ldots\ 1\ \ 0\ \ 1\ 1 & 0\ 1 & 0\ \ldots\ 1\ 0\ 1\ 1\ 0\ \ 1\ \ 0\ 1\ 1\ 0\ 0\ 0\ 0 & 0\ 0 & 1\ 0\ 1 & 1 \\
\ldots\ 0\ \ 0\ \ 0\ 0 & 0\ 0 & 0\ \ldots\ 0\ 1\ 1\ 0\ 1\ \ 0\ \ 1\ 1\ 0\ 1\ 0\ 0\ 0 & 0\ 0 & 0\ 1\ 1 & 0\ 1
\end{array}\right].$$

We get the following statement which, in particular, excludes any complementary game $\{c, \bar{c}\}$ while, B_{60} being a null-class board, complementary games satisfy the rule-of-three; see Fig. 4 for such an example.

Corollary 5.1.

1. *$Lat(B_{60})$ has full rank with determinant $det(Lat(B_{60})) = 32$.*
2. *$[\mathbb{Z}^{B_{60}} : Ker(\phi)] = 16 \neq 32 = [\mathbb{Z}^{B_{60}} : Lat(B_{60})]$ hence $Lat(B_{60}) \neq Ker(\phi)$.*
3. *$\mathbb{1} \notin Lat(B_{60})$, that is, any complementary game $\{c, \bar{c}\}$ is infeasible on B_{60}.*

Proof. (1) and (2) are a direct consequence of the computation of the Hermite basis V_{60} of $\mathrm{Lat}(B_{60})$. Since, for example, the last row of the Hermite basis V_{60} has an even number of entry of 1, $\mathbb{1} = c + \bar{c} \notin \mathrm{Lat}(B_{60})$, that is, $c - \bar{c} \notin \mathrm{Lat}(B_{60})$ and gives (3). On the other hand, B_{60} is a null-class board, that is, $\phi(\mathbb{1}) = \phi(c - \bar{c}) = (e, e)$. □

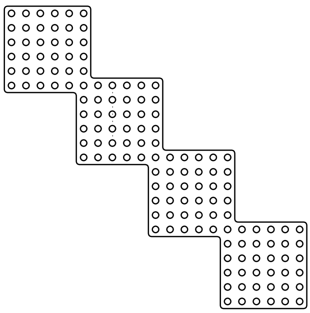


Fig. 6. A hook board B with associated lattice satisfying $\det(\mathrm{Lat}(B)) = 128$

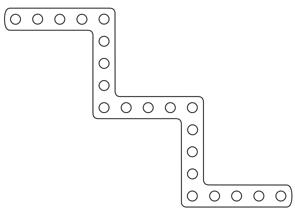


Fig. 7. A 5-hook board B with associated lattice satisfying $\det(\mathrm{Lat}(B)) = 64$

5.2. Hook Boards with Exponential Determinant

The computation of the Hermite normal form of B_{60} done in Section 5.1 can be easily extended to any set of k rectangular boards pairwise overlapping on a common corner. The resulting board – see for example Fig. 6 – will give a lattice with determinant 2^{k+3}. Similarly, any *k-hook board* – one which consists of a set of k linear boards pairwise overlapping on one common hole (see Fig. 7) – creates a lattice with determinant 2^{k+1}.

Hook boards demonstrate that the solitaire lattice criterion can be *exponentially finer* than the rule-of-three in the sense of the following theorem.

Theorem 5.2. *For hook boards, the solitaire lattice condition exponentially outperforms the rule-of-three, that is, for every k and every k-hook board B, the ratio of the number of congruence classes of $\mathbb{Z}^B$ modulo Lat(B) to the number of congruence classes of $\mathbb{Z}^B$ modulo Ker(ϕ) satisfies*

$$\frac{[\mathbb{Z}^B : Lat(B)]}{[\mathbb{Z}^B : Ker(\phi)]} = 2^{k-1} \quad .$$

5.3. Some Other Boards

Similar computations give that both the classical English 33-board B_{33} and the French 37-board B_{37} satisfy:

1. $\det(\mathrm{Lat}(B_{33})) = 16$, hence $\mathrm{Lat}(B_{33}) = \mathrm{Ker}(\phi)$,
2. $\det(\mathrm{Lat}(B_{37})) = 16$, hence $\mathrm{Lat}(B_{37}) = \mathrm{Ker}(\phi)$.

On the other hand, the following null-class 165-board B_{165} yields to a solitaire lattice satisfying $\det(\mathrm{Lat}(B_{165})) = 256$.

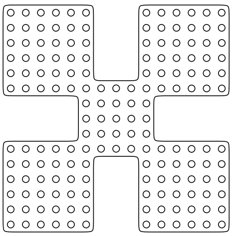


Fig. 8. A null-class board B satisfying $\det(\mathrm{Lat}(B)) = 256$

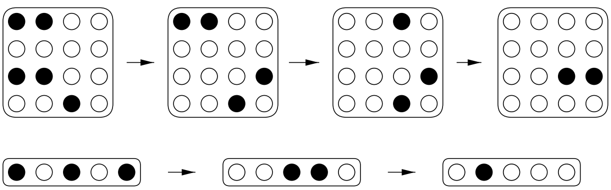


Fig. 9. Moves on the 4×4 and 1×5 toric boards

Fig. 10. An inside move

6. Variations

In order to avoid many special situations caused by the boundary, one can consider the toric closure $\bar{B}$ of a board B. In other words, the toric $m \times n$ board for $m \geq 1$ and $n \geq 3$ is an $m \times n$ rectangular board with additional jumps which traverse the boundary of the board as illustrated for $\bar{B}_{1,5}$ and $\bar{B}_{4,4}$ by Fig. 9.

Clearly the toric closure of $B_{1,n}$ generates the 4 additional moves: $r_{0,0}, l_{0,0}, r_{n-1,0}$ and $l_{n-1,0}$. Appending those 4 moves to the Hermite normal form of $\mathrm{Lat}(B_{1,n})$ given in Section 4.1, one can easily check that:

Proposition 6.1. *Let $\bar{B}_{1,n}$ be any $1 \times n$ toric board with $n \geq 4$. Then the following hold for the solitaire lattice $Lat(\bar{B}_{1,n})$:*

1. *$Lat(\bar{B}_{1,n}) = Lat(B_{1,n})$ for $n \equiv 0 \pmod 3$,*
2. *$Lat(\bar{B}_{1,n}) = \mathbb{Z}^n$ otherwise.*

Similarly, the toric closure of $B_{m,n}$, generates $4mn$ additional moves and Proposition 6.1 immediately implies:

Proposition 6.2. *Let $\bar{B}_{m,n}$ be any $m \times n$ toric board with $n \geq 4$ or $m \geq 4$. Then the following hold for the solitaire lattice $Lat(\bar{B}_{m,n})$:*

1. *$Lat(\bar{B}_{m,n}) = Lat(B_{m,n})$ for $m \equiv n \equiv 0$ $(mod\ 3)$*
2. *$Lat(\bar{B}_{m,n}) = \mathbb{Z}^{mn}$ otherwise.*

Note that $\mathrm{Lat}(\bar{B}_{1,3})$ and $\mathrm{Lat}(\bar{B}_{3,3})$ are full rank lattice while their non-toric counterparts are not.

The rule-of-three can be defined on a toric board $\bar{B}_{m,n}$ if and only if $m \equiv n \equiv 0 \pmod 3$. In other words, for $m \not\equiv 0 \pmod 3$ or $n \not\equiv 0 \pmod 3$, the rule-of-three excludes no configuration, that is, $\mathrm{Ker}(\phi)$ can be set to $\mathbb{Z}^{mn}$. In other words, Propositions 6.1 and 6.2 imply:

Proposition 6.3. *Let $\bar{B}_{m,n}$ be any $m \times n$ toric board with $n \geq 4$ or $m \geq 4$. Then we have: $Lat(\bar{B}_{m,n}) = Ker(\phi)$.*

In another variation of the solitaire game, to the classical moves we add the moves which consist of removing two pegs surrounding an empty hole and placing one peg in this empty hole as showed in Fig. 10.

Let denote those *inside row moves* $i^r_{i,j}$ (respectively, column moves $i^c_{i,j}$) and by $\mathrm{Lat}(\mathscr{C}B)$ the *complete solitaire lattice* generated by all classical and inside moves. The conic hull of all those moves is called the *complete solitaire cone*; see [1] for an interesting link between the complete solitaire game and the multicommodity flow problem. Since $\phi(i^r_{i,j}) = \phi(i^c_{i,j}) = (e, e)$, we clearly have $\mathrm{Lat}(\mathscr{C}B) \subseteq \mathrm{Ker}(\phi)$.

Proposition 6.4. *Let $B_{m,n}$ and $\mathscr{C}B_{m,n}$ be any $m \times n$ board and its complete version with $n \geq 4$ or $m \geq 4$. Then we have: $Lat(\mathscr{C}B_{m,n}) = Lat(B_{m,n})$.*

Proof. We have $i^r_{i,j} = 3l_{i,j} + 2r_{i,j} - 2l_{i,j-1} - 2r_{i,j-1}$, that is, insides moves are integer linear combinations of classical moves. $\square$

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