On the Skeleton of the Dual Cut Polytope

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ABSTRACT. The cut polytope is the \( \binom{n}{2} \)-dimensional convex polytope generated by all cuts of the complete graph on \( n \) nodes. One of the applications of the cut polytope, the polyhedral approach to the maximum cut problem, leads to the study of its facets which are known only up to \( n = 7 \) where they number 116 764. For \( n \leq 7 \), we describe the skeleton of the dual of the cut polytope, in particular, we give its adjacency relations and diameter. We also give similar results for a relative of the cut polytope, the cut cone, and new results on the size of the facets of the cut polytope.

1. Introduction

We first recall the definitions of the cut polytope \( \text{Cut}P_n \) and its relative the cut cone \( \text{Cut}_n \). Then we present some applications in combinatorial optimization and some geometric and combinatorial properties of the cut polytope.

Given a subset \( S \) of \( V_n = \{1, 2, \ldots, n\} \), the cut determined by \( S \) consists of the pairs \( (i, j) \) of elements of \( V_n \) such that exactly one of \( i, j \) is in \( S \). \( \delta(S) \) denotes both the cut and its incidence vector in \( \mathbb{R}^{\binom{n}{2}} \); that is, \( \delta(S)_{ij} = 1 \) if exactly one of \( i, j \) is in \( S \) and 0 otherwise for \( 1 \leq i < j \leq n \). By abuse of language, we use the term cut for both the cut itself and its incidence vector. The cut polytope \( \text{Cut}P_n \) is the convex hull of all \( 2^{n-1} \) cuts, and the cut cone \( \text{Cut}_n \) is the conic hull of all \( 2^{n-1} - 1 \) nonzero cuts.

Those polyhedra were considered by many authors, see \([2, 5, 8, 10-16]\) and references there. One of the motivations for the study of the cut polytope and cut cone comes from their applications in combinatorial optimization, see for instance \([11]\). Given a graph \( G = (V_n, E) \) and nonnegative weight \( w_e, e \in E \), assigned to its edges, the max-cut problem consists of finding a cut \( \delta(S) \) whose weight \( \sum_{e \in \delta(S)} w_e \) is as large as possible. By setting \( w_e = 0 \) if \( e \) is not an edge of \( G \), we can consider the complete graph on \( V_n \). Then the max-cut problem

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can be stated as a linear programming problem over the cut polytope $\text{Cut}P_n$ as follows:

$$\begin{align*}
\max & \quad w^T \cdot x \\
\text{subject to} & \quad x \in \text{Cut}P_n.
\end{align*}$$

This polyhedral approach to the max-cut problem leads to the study of the facets of the cut polytope $\text{Cut}P_n$.

$\text{Cut}P_n$ is a $\binom{n}{2}$-dimensional 0–1 polytope with $2^{n-1}$ vertices. The polytope $\text{Cut}P_3$ is combinatorially equivalent to the tetrahedron and $\text{Cut}P_4$ is combinatorially equivalent to the 6-dimensional cyclic polytope with 8 vertices. More generally, $\text{Cut}P_n$ is a 3-neighbourly polytope [14]. Another remarkable feature of the cut polytope $\text{Cut}P_n$ is the high number of isometric symmetries it enjoys. The symmetry group of the cut polytope $\text{Is}P_n$ is induced, see [8], by permutations on $V_n = \{1, \ldots, n\}$ and switching reflections which were introduced in [2, 5]. Given a cut $\delta(S)$ and a facet $F$ induced by the inequality $v \cdot x \leq v_0$, the switching reflection of $F$ by the cut $\delta(S)$ is the facet induced by the inequality $v^S \cdot x \leq v_0 - v \cdot \delta(S)$, where $v^S_{ij} = -v_{ij}$ if $(i, j) \in \delta(S)$ and $v^S_{ij} = v_{ij}$ otherwise. The group $\text{Is}(\text{Cut}P_n)$ is isomorphic to $\text{Aut}(\square_n)$, the automorphism group of the folded $n$-cube. We recall that the folded $n$-cube is the graph whose vertices are the partitions of $\{1, \ldots, n\}$ into two subsets, two partitions being adjacent when their common refinement contains a set of size one [3]. The symmetries of $\text{Cut}P_n$ preserve adjacency and linear independence. Throughout this paper we use the fact that the facets of $\text{Cut}P_n$ are partitioned into orbits of its symmetry group, that is, into classes of facets equivalent under permutation or switching.

The paper is organized as follows. In §2 we present some results on the facets of the cut polytope. Then in §3 and §4, we describe the skeleton of the dual cut polytope $\text{Cut}^*P_n$ for $n \leq 7$, respectively the skeleton of the dual cut cone $\text{Cut}^*_n$ for $n \leq 6$. In §5, we give some results and conjectures on the size and the adjacencies of the facets of the $\text{Cut}P_n$. A general reference for the graph theory used in this paper is [3].

2. Facets of the Cut Polytope

Even if the determination of all the facets of the cut polytope $\text{Cut}P_n$ and the cut cone $\text{Cut}_n$ for any $n$ seems to be hopeless, a wide range of facets has been already found. In particular, they are all known for $n \leq 7$, see [1, 4, 5, 10, 15]. Since it turns out that the facets of the cut polytope $\text{Cut}P_n$ are switchings of the facets of the cut cone $\text{Cut}_n$, see [2], it is enough to determine all the facets of $\text{Cut}_n$ to obtain all the facets of $\text{Cut}P_n$.

To ease the notation we define the following two functions of $x \in R^{n\choose 2}$. For $b = (b_1, \ldots, b_m) \in N^m$, $m \leq n$, and $C$ a cycle with nodeset a subset of $\{1, \ldots, n\}$:
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\[ Q(b) \cdot x = \sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \]

and \[ K(C) \cdot x = \sum_{(ij) \in C} x_{ij} \]

With this notation, the following 11 inequalities define facets of the cut polytope \( \text{Cut}P_n \) for \( n \geq m \):

1. \[ Q(1,1,1) \cdot x \leq 2. \]
2. \[ Q(1,1,1,1) \cdot x \leq 6. \]
3. \[ Q(2,1,1,1,1) \cdot x \leq 12. \]
4. \[ Q(1,1,1,1,1) \cdot x \leq 12. \]
5. \[ Q(2,2,1,1,1,1) \cdot x \leq 20. \]
6. \[ Q(3,2,2,1,1,1) \cdot x - K(1,2,3) \cdot x \leq 28. \]
7. \[ Q(3,1,1,1,1,1) \cdot x \leq 20. \]
8. \[ Q(1,1,1,1,2,1,1) \cdot x - \sum_{i=1,3,5,6} x_{i7} - \sum_{i=2,4,5,7} x_{i6} \leq 12. \]
9. \[ Q(1,1,1,1,1,1) \cdot x - K(1,2,3,4,5) \cdot x \leq 10. \]
10. \[ Q(1,1,1,1,1,1,1) \cdot x - K(1,\ldots,7) \cdot x - 2(x_{2,5} + x_{2,7} + x_{4,7}) \leq 6. \]
11. \[ Q(2,2,1,1,1,1,1) \cdot x - K(1,2,3,4) \cdot x \leq 18. \]

For \( m \leq n \), let \( F^n_i \) denote the facet of \( \text{Cut}P_n \) induced by the inequality \((i)\), and \( O^n_i \) denote the orbit of \( F^n_i \), that is, the class of facets of \( \text{Cut}P_n \) which are equivalent to \( F^n_i \) under permutation and switching. For computational purpose we choose the above 11 representatives \( F^n_i \) such that the right hand side of the inequality \((i)\) is maximal. It turns out that, up to permutation, they are unique such representatives of \( O^n_1 \ldots O^n_{11} \) except for \( O^n_{10} \) which contains two nonzero switchings of \( F^n_{10} \) with right hand side equal to 6. The 16 facets of \( \text{Cut}P_4 \) form the orbit \( O^4_1 \), the 56 facets \( \text{Cut}P_5 \) are partitioned into the 2 orbits \( O^5_1 \) and \( O^5_2 \), the 368 facets of \( \text{Cut}P_6 \) are partitioned into the 3 orbits \( O^6_1, O^6_2 \) and \( O^6_3 \), and the 116 764 facets of \( \text{Cut}P_7 \) are partitioned into the 11 orbits \( O^7_1, O^7_2,\ldots, O^7_{11} \).

Before describing \( \Omega_n \), the skeleton of the dual cut polytope, for \( n \leq 7 \) in the next section, we present an equality found by Michel Deza, Martin Grötschel and Monique Laurent relating the number of facets of the cut polytope \( \text{Cut}P_n \) and the cut cone \( \text{Cut}_n \):

**Lemma 2.1.** For any facet \( F \) containing the origin, with \( |F| \) denoting the size of \( F \), that is the number of cuts belonging to \( F \), we have:

\[ |O(F)| \cdot |F| = |O(F)_{in \ \text{Cut}_n}| \cdot 2^{n-1} \]
PROOF. Let \( \{F_1, \ldots, F_m\} \) be an ordering of \( O(F) \) and \( b_i \delta(S) = 1 \) if the cut \( \delta(S) \in F_i \) and 0 otherwise, then we have:

\[
\sum_{i,S} b_i \delta(S) = \sum_S (\sum_i b_i \delta(S)) = \sum_S (\sum_i b_i) = \sum_S (|O(F)_{in \text{ Cut}_n}|) = |O(F)_{in \text{ Cut}_n}| \cdot 2^{n-1},
\]

and also,

\[
\sum_{i,S} b_i \delta(S) = \sum_i (\sum_S b_i) = \sum_i (|F_i|) = \sum_i (|F|) = |O(F)| \cdot |F|,
\]

which completes the proof.

3. Skeleton of the Dual Cut Polytope for \( n \leq 7 \)

Any pair of facets of \( \text{OutP}_n \) are adjacent in \( \Omega_n \), the skeleton of the dual cut polytope \( \text{CutP}_n^* \), if and only if their intersection is a face of codimension 2. For \( n \leq 7 \), the size of the orbits of the facets in the cut polytope \( \text{CutP}_n \) was deduced from their corresponding size in the cut cone \( \text{Cut}_n \) found in [12] using Lemma 2.1. Since permutations and switching reflections preserve adjacency and linear independence, we can describe the properties of facets of \( \text{CutP}_n \) by considering a representative facet of each orbit \( O_i^n \), we choose the facets \( F_i^n \) for \( n \leq 7 \).

3.1. Skeleton of \( \text{CutP}_4^* \). The polytope \( \text{CutP}_4 \) is combinatorially equivalent to the 6-dimensional cyclic polytope with 8 vertices. A pair of facets of \( \text{CutP}_4 \) are adjacent in \( \Omega_4 \) if and only if they are non-conflicting. Two facets are called conflicting if there exists a pair \( i,j \) such that the two facets have nonzero coordinates of distinct signs at the position \( i,j \). For example, the facets induced by the inequalities \( Q(1,1,1) \cdot x \leq 2 \) and \( Q(-1,1,1) \cdot x \leq 0 \) are conflicting at pair \( (1,2) \). The notion of conflicting facets was introduced in [16, 17]. The graph \( \Omega_4 \) formed by the 16 facets of \( \text{CutP}_4 \), is the \( (4 \times 4) \)-grid, see [6], that is, the line graph \( L(K_{4,4}) \) which is also \( L(\square_4) \). This graph is a strongly regular graph with parameters \( v = 16, k = 6, \lambda = 2 \) and \( \mu = 2 \), where \( v \) denotes the number of nodes, \( k \) the valency of each node, \( \lambda \) the number of nodes adjacent to two adjacent nodes and \( \mu \) the number of nodes adjacent to two non-adjacent nodes. The diameter \( \delta(\Omega_4) \) of \( \text{CutP}_4^* \), is 2. Since \( \text{CutP}_4 \) is simplicial, all of its faces, including the 16 facets and 48 faces of codimension 2, are simplices.

3.2. Skeleton of \( \text{CutP}_5^* \). The 10-dimensional polytope \( \text{CutP}_5 \) has 16 vertices and 56 facets which are partitioned into the 2 orbits \( O_1^5 \) and \( O_2^5 \). As for \( \text{CutP}_4 \), a pair of facets of \( \text{CutP}_5 \) are adjacent in \( \Omega_5 \) if and only if they are non-conflicting. In Figure 1 we give the adjacency table of \( \Omega_5 \), \( V_F \) the valency and \( |F| \) the size
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<table>
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<tr>
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<td>$</td>
<td>F^5_i</td>
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</table>

**Figure 1.** Adjacencies in the skeleton of $CutP^*_5$

of the facets, and $|O^5_i|$ the size of the orbits of $CutP_5$, see [6]. For example, in the left column of the upper table in Figure 1, 10 means that any facet $F$ of $O^5_2$ is adjacent to 10 facets of $O^5_1$. The neighbours of $F^5_2$ are the 10 permutations of $F^5_1$. The diameter $\delta(\Omega_5)$ of $CutP^*_5$ is 2. The simplex facets of $CutP_5$ are all the 16 facets of the orbit $O^5_2$ and, among its 640 faces of codimension 2, exactly 400 are simplices.

### 3.3. Skeleton of $CutP^*_6$

The 15-dimensional polytope $CutP_6$ contains 32 vertices and 368 facets which are partitioned into the 3 orbits $O^6_1$, $O^6_2$ and $O^6_3$. Using switching reflections and permutations it is only tedious but easy to obtain, as for $CutP_5$, the tables given in Figure 2. The simplex facets of $CutP_6$ are all the 192 facets of the orbit $O^6_3$.

<table>
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<tr>
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<tr>
<td>$</td>
<td>F^6_i</td>
<td>$</td>
<td>24</td>
</tr>
</tbody>
</table>

**Figure 2.** Adjacencies in the skeleton of $CutP^*_6$

**Corollary 3.1.** $CutP_6$ has exactly 10 480 faces of codimension 2.

**Proof.** The number of faces of codimension 2 of a polytope is half of the total valency of the skeleton of its dual. Since we know the valency of all the 368 nodes of $\Omega_6$, the result is a straightforward calculation. Moreover, one can check that exactly 4 800 of these faces of codimension 2 are simplices.

**Corollary 3.2.** The diameter $\delta(\Omega_6)$ of $CutP^*_6$ is 3.
PROOF. For any pair of facets $F$ and $F'$ of $CutP_6$ we have to find a path in $\Omega_6$ of length shorter than 3. Since the diameter of the restriction of $\Omega_6$ to $O_1^6$ is 2, see [6], we can assume that $F$ and $F'$ do not belong to $O_1^6$. If they both belong to $O_2^6$ then, since they both have more than half the size of $O_1^6$ neighbors in $O_2^6$, we can find a facet in $O_1^6$ adjacent to $F$ and $F'$. If $F \in O_2^6$ and $F' \in O_1^6$, then one can easily check that the facet induced by either the inequality $Q(1,0,0,0,1,1) \cdot x \leq 2$ or the inequality $Q(0,0,1,1,1) \cdot x \leq 2$ is adjacent to $F$ and $F'$. If they both belong to $O_2^6$, we can suppose without loss of generality that $F$, respectively $F'$, is the facet induced by the inequality $Q(2,1,1,1,1,1) \cdot x \leq 2$, respectively the inequality $Q(-1,2,-1,1,1,1) \cdot x \leq 2$. Then one can check that $F$ and $F'$ have no common neighbor and that the facets $G$ and $G'$, respectively induced by the inequality $Q(1,0,0,1,1) \cdot x \leq 2$ and $Q(-1,1,1) \cdot x \leq 0$, are adjacent and respectively adjacent to $F$ and $F'$.

REMARK 3.3. While the restriction of $\Omega_6$ to the facets of $O_1^6$ forms a graph of diameter 2, see [6], its restriction to the facets of $O_2^6$ forms a graph of diameter 3 with parameters $v = 96$, $k = 10$, $\lambda = 4$, and $\mu \in \{0,1,2\}$, which is locally $K_{5,5}$. More generally, the inequality $\sum_{1 \leq i < j \leq 2t+1} x_{ij} \leq t(t + 1)$ induces a facet $F_t$ of $CutP_{2t+2}$ and the facets belonging to the orbit $O_t$ of $F_t$ form a graph which is locally the complement of $K_{2t+1,2t+1}$; two facets of $O_t$ being adjacent if and only if they are non-conflicting.

### 3.4 Skeleton of $CutP_7^*$

$$
\begin{array}{cccccccccccc}
\text{Facets} & O_1^7 & O_2^7 & O_3^7 & O_4^7 & O_5^7 & O_6^7 & O_7^7 & O_8^7 & O_9^7 & O_{10}^7 & O_{11}^7 & V_F \\
F \in O_1^7 & 112 & 264 & 672 & 64 & 240 & 432 & 48 & 3456 & 1728 & 2304 & 2112 & 11432 \\
F \in O_2^7 & 110 & 180 & 100 & 4 & 40 & 120 & 0 & 840 & 240 & 480 & 480 & 2594 \\
F \in O_3^7 & 70 & 25 & 10 & 2 & 10 & 20 & 2 & 60 & 0 & 0 & 40 & 239 \\
F \in O_4^7 & 140 & 21 & 42 & 0 & 21 & 0 & 0 & 252 & 0 & 420 & 896 \\
F \in O_5^7 & 25 & 10 & 10 & 1 & 10 & 0 & 0 & 0 & 0 & 20 & 76 \\
F \in O_6^7 & 9 & 6 & 4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 21 \\
F \in O_7^7 & 15 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 \\
F \in O_8^7 & 12 & 7 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 21 \\
F \in O_9^7 & 15 & 5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 21 \\
F \in O_{10}^7 & 14 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 \\
F \in O_{11}^7 & 11 & 6 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 21 \\
\end{array}
$$

$$
\begin{array}{cccccccccccc}
\text{Facets} & O_1^7 & O_2^7 & O_3^7 & O_4^7 & O_5^7 & O_6^7 & O_7^7 & O_8^7 & O_9^7 & O_{10}^7 & O_{11}^7 & V_F \\
F \in O_1^7 & 112 & 264 & 672 & 64 & 240 & 432 & 48 & 3456 & 1728 & 2304 & 2112 & 11432 \\
F \in O_2^7 & 110 & 180 & 100 & 4 & 40 & 120 & 0 & 840 & 240 & 480 & 480 & 2594 \\
F \in O_3^7 & 70 & 25 & 10 & 2 & 10 & 20 & 2 & 60 & 0 & 0 & 40 & 239 \\
F \in O_4^7 & 140 & 21 & 42 & 0 & 21 & 0 & 0 & 252 & 0 & 420 & 896 \\
F \in O_5^7 & 25 & 10 & 10 & 1 & 10 & 0 & 0 & 0 & 0 & 20 & 76 \\
F \in O_6^7 & 9 & 6 & 4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 21 \\
F \in O_7^7 & 15 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 \\
F \in O_8^7 & 12 & 7 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 21 \\
F \in O_9^7 & 15 & 5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 21 \\
F \in O_{10}^7 & 14 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 \\
F \in O_{11}^7 & 11 & 6 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 21 \\
\end{array}
$$

**Figure 3.** Adjacencies in the skeleton of $CutP_7^*$
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The 21-dimensional polytope Cut$_7$ contains 64 vertices and 116 764 facets which are partitioned into the 11 orbits $O_1^7$, $O_2^7$ \ldots $O_{11}^7$. The adjacencies for the 6 orbits of simplex facets $O_6^7$, $O_7^7$ \ldots $O_{11}^7$ could be derived from the list of facets of the simplex facets of Cut$_7$ given in [13]. The adjacencies for the orbit $O_1^7$ and $O_4^7$ can be deduced from the results given respectively in [6] and [7], the remaining adjacencies were checked by computer. The complete adjacency table is given in Figure 3. The simplex facets of Cut$_7$ are all the 113 536 facets of the orbits $O_6^7$, $O_7^7$ \ldots $O_{11}^7$. The local graph induced by a simplex facet $F$ of Cut$_7$ is the clique $K_{21}$ for $F \in O_7^7$, $O_9^7$ or $O_{10}^7$, $K_{21} - K_2$ for $F \in O_6^7$ or $O_{11}^7$, and $K_{21} - K_3$ for $F \in O_8^7$.

**Corollary 3.4.** Cut$_7$ has exactly 2 668 512 faces of codimension 2.

**Proof.** As for Cut$_6$, since we know the valencies of $\Omega_7$, the result is a straightforward calculation. Moreover, using a computer, we found that exactly 2 438 016 of these faces of codimension 2 are simplices.

**Corollary 3.5.** The diameter $\delta(\Omega_7)$ of Cut$_7^n$ satisfies: $3 \leq \delta(\Omega_7) \leq 4$.

**Proof.** The diameter of the restriction of $\Omega_7$ to $O_1^7$ is 2, see [6]. Then, since every facet of Cut$_7$ is adjacent to a facet of $O_1^7$, we have $\delta(\Omega_7) \leq 4$. The two facets, respectively induced by the inequality $Q(3, 1, 1, 1, 1, 1, 1) \cdot x \leq 20$ and $Q(-1, 3, -1, 1, 1, 1) \cdot x \leq 6$, having no common neighbour, we have $\delta(\Omega_7) \geq 3$.

**Remark 3.6.** It was conjectured in [6] that the facets of $O_1^n$ form a dominating set in $\Omega_n$, that is, every facet of the cut polytope is adjacent to a facet belonging to the orbit $O_1^n$. Since the diameter of the restriction of $\Omega_n$ to $O_1^n$ is 2 [6], it would imply that the diameter of the dual cut polytope satisfies $\delta(\Omega_n) \leq 4$.

4. **Skeleton of the Dual Cut Cone for $n \leq 6$**

The results about $\Omega'_n$, the skeleton of the dual cut cone Cut$_n^*$, are similar to the ones concerning the cut polytope. The differences result from the fact that the symmetry group of the cut cone Cut$_n$ is not fully determined. Clearly all permutations of $\{1, \ldots, n\}$ are isometric symmetries of Cut$_n$, that is, we have $Sym(n) \subset Is(Cut_n)$, and the equality probably holds.

Then, as for the cut polytope, we can use the fact that the facets of the cut cone are partitioned into the orbits of $Sym(n)$. We choose the 4 representatives $G_1^n$, $G_2^n$, $G_3^n$ and $G_3'n$, defined by the following inequalities, and denote by $U_i^n$ the orbit of the facet $G_i^n$,

\begin{align*}
(1) & \quad Q(-1, 1, 1) \cdot x \leq 0. \\
(2) & \quad Q(-1, -1, 1, 1, 1) \cdot x \leq 0. \\
(3) & \quad Q(-2, -1, 1, 1, 1) \cdot x \leq 0. \\
(3') & \quad Q(2, 1, 1, -1, -1) \cdot x \leq 0.
\end{align*}
Skeleton of $\text{Cut}_4^*$ and $\text{Cut}_5^*$. The graph $\Omega_4$ formed by the 12 facets of $\text{Cut}_4$ is the $(4 \times 3)$-grid, see [6], that is, the line graph $L(K_{4,3})$. This graph is a regular graph with parameters $v = 12$, $k = 5$, $\lambda = 2$ or 1 and $\mu = 2$. The diameter of $\text{Cut}_4^*$ is 2. The 40 facets of $\text{Cut}_5$ are partitioned into the 2 orbits $U_1^5$ and $U_2^5$. The adjacency table of $\Omega_5$, the valency and size of the facets, and the size of the orbits $\text{Cut}_5$ are given in Figure 4. $\delta(\Omega_5) = 2$ and $\text{Cut}_5$ has 375 faces of codimension 2. The simplex facets of $\text{Cut}_5$ are all the 10 facets of the orbit $U_2^5$.

Skeleton of $\text{Cut}_6^*$. The 210 facets of $\text{Cut}_6$ are partitioned into the 4 orbits $U_1^6$, $U_2^6$, $U_3^6$ and $U_{3'}^6$. The adjacencies of $\Omega_6$ are given in Figure 5, $\delta(\Omega_6) = 3$ and $\text{Cut}_6$ has 5 190 faces of codimension 2. The simplex facets of $\text{Cut}_6$ are all the 90 facets of the orbits $U_3^6$ and $U_{3'}^6$.

<table>
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<tr>
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</tr>
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<td>$</td>
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<td>$</td>
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<td>19</td>
</tr>
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</table>

Figure 5. Adjacencies in the skeleton of $\text{Cut}_6^*$

Remark 4.1. The 38 780 facets of $\text{Cut}_7$ are partitioned into 36 orbits of $\text{Sym}(7)$.

5. On the Shape of the Cut Polytope

In this section, we give a tight upper bound and some conjectures on the size and the adjacency of the facets of the cut polytope.
Lemma 5.1. Any facet $F$ of the cut polytope satisfies $|F| \leq 3.2^{n-3}$ with equality if and only if $F \in O_n$.

Proof. Let $F$ be a facet of $CutP_n$ induced by the inequality:

$$\sum_{1 \leq i < j \leq n} v_{ij} x_{ij} \leq a,$$

$v_{kl}$ a nonzero coordinate of $F$, and $F(S)$ the value of the left hand side of (1) on the cut $\delta(S)$. With $S \cap \{i, j\} = \emptyset$, we have:

$$F(S \cup \{i\}) + F(S \cup \{j\}) - F(S) - F(S \cup \{i, j\}) = 2v_{ij} \neq 0.$$

This implies that no more than 3 of any such 4 cuts belong to $F$, and therefore that no facet contains more than $\frac{3}{4} 2^{n-1} = 3.2^{n-3}$ vertices. Reversely, let $F$ be a facet of $CutP_n$ containing $3.2^{n-3}$ vertices. Without loss of generality, we can assume that $a$, the right hand side of (1) is 0. We first show that any facet $F$ of $CutP_n$ has at least 3 nonzero coordinates. If $F$ has only 2 nonzero coordinates, since $CutP_n$ lies in the positive orthant of $IR_+^{n}$, we can assume that $F$ is induced by the inequality $v_{ij} - \alpha v_{kl} \leq 0$ with $\alpha \geq 1$. The point $p$ with all coordinates equal to $\frac{1}{3}$, except $p_{kl} = \frac{1}{3\alpha}$, belongs to both the facet $F$ and the relative interior of $P$, which is impossible. Then we prove that a facet $F$ containing $3.2^{n-3}$ vertices has at most 3 nonzero coordinates. Let consider any 3 nonzero coordinates of $F$ $v_{ij}, v_{kl}$ and $v_{st}$. Suppose that the six indices $i, j, k, l, s$ and $t$ are distinct, that is, $|\{i, j, k, l, s, t\}| = 6$. In that case, we have:

(2) for $S \cap \{i, j\} = \emptyset$, $F(S \cup \{i\}) + F(S \cup \{j\}) - F(S) - F(S \cup \{i, j\}) = 2v_{ij} \neq 0$,

(3) for $S \cap \{k, l\} = \emptyset$, $F(S \cup \{k\}) + F(S \cup \{l\}) - F(S) - F(S \cup \{k, l\}) = 2v_{kl} \neq 0$,

(4) for $S \cap \{s, t\} = \emptyset$, $F(S \cup \{s\}) + F(S \cup \{t\}) - F(S) - F(S \cup \{s, t\}) = 2v_{st} \neq 0$.

Since $F$ contains $3/4$ of the total number of vertices of $CutP_n$, exactly 3 terms of the left hand side of the equations (2), (3) or (4) are null. Suppose that $v_{ij}$ and $v_{kl}$ have the same sign (otherwise we consider $v_{ij}$ and $v_{st}$ or $v_{kl}$ and $v_{st}$). Then the equations (2) with $S = \emptyset$ and (3) with $S = \{i\}$ imply that $F(\{i\})$ cannot be the only nonzero term of both the equations (2) and (3), that is $F(\{i\}) = 0$. In the same way, the equations (2) with $S = \{s\}$, respectively $S = \{t\}$ and $\{k, l\}$, and (3) with $S = \{i, s\}$, respectively $S = \{i, t\}$ and $\{j\}$, imply that $F(\{i, s\})$, respectively $F(\{i, t\})$ and $F(\{i, s, t\})$, is null, which contradicts the equation (4) with $S = \{i\}$. Similarly $|\{i, j, k, l, s, t\}| = 5$ or 4 also leads to a contradiction. Therefore $F$ has exactly 3 nonzero coordinates $v_{ij}, v_{ik}$ and $v_{jk}$. We have:

(5) for $S \cap \{i, j\} = \emptyset$, $F(S \cup \{i\}) + F(S \cup \{j\}) - F(S) - F(S \cup \{i, j\}) = 2v_{ij} \neq 0$,

(6) for $S \cap \{i, k\} = \emptyset$, $F(S \cup \{i\}) + F(S \cup \{k\}) - F(S) - F(S \cup \{i, k\}) = 2v_{ik} \neq 0$,

(7) for $S \cap \{j, k\} = \emptyset$, $F(S \cup \{j\}) + F(S \cup \{k\}) - F(S) - F(S \cup \{j, k\}) = 2v_{jk} \neq 0$. 


Then, the equations (5) with $S = \{k\}$ and (6) with $S = \{j\}$ imply that $v_{ij} = v_{ik}$; (6) with $S = \emptyset$ and (7) with $S = \{i\}$ imply that $v_{ik} = -v_{jk}$. This means that $F$ is the facet of $O^n_1$ induced by the inequality $x_{jk} - x_{ij} - x_{ik} \leq 0$.

**Remark 5.2.** The $4(n\choose3)$ facets of $\text{Cut}P_n$ belonging to $O^n_1$ contain $3.2^{n-3}$ cuts, that is $3/4$ of the total number of vertices of the cut polytope. Those facets are the extreme opposite of being simplices. We think that the shape of the cut polytope is essentially given by its non-simplex facets, in particular by the facets of $O^n_1$ (see Remark 3.6), and that the huge majority of the facets of $\text{Cut}P_n$ are simplices which only "polish" it. This belief is shared by designers of the cutting plane methods who hope that the "few nice" classes of facets they use will be sufficient to prove the optimality or provide excellent bounds, and that the facets they have no access to contribute very little to the computational behavior of such methods.

**Conjecture 5.3.** For $n \geq 5$, any two simplex facets of the cut polytope are not adjacent in $\Omega_n$. It holds for $n \leq 7$.

**Conjecture 5.4.** The cut polytope $\text{Cut}P_n$ is asymptotically simplicial. In fact, more than 97% of the facets of $\text{Cut}P_7$ and 91% of its faces of codimension 2 are simplices.

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**References**

15. V. P. Grishukhin, All facets of the cut cone $C_n$ for $n = 7$ are known, European Journal of Combinatorics 11 (1990), 115–117.
17. M. Laurent, Graphic vertices of the metric polytope, Research report No. 91737-OR, Institut für Diskrete Mathematik, Universität Bonn.

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