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On the Skeleton of the Dual Cut Polytope

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ABSTRACT. The cut polytope is the $\binom{n}{2}$-dimensional convex polytope generated by all cuts of the complete graph on n nodes. One of the applications of the cut polytope, the polyhedral approach to the maximum cut problem, leads to the study of its facets which are known only up to $n = 7$ where they number 116 764. For $n \leq 7$, we describe the skeleton of the dual of the cut polytope, in particular, we give its adjacency relations and diameter. We also give similar results for a relative of the cut polytope, the cut cone, and new results on the size of the facets of the cut polytope.

1. Introduction

We first recall the definitions of the *cut polytope* $CutP_n$ and its relative the *cut cone* Cut_n. Then we present some applications in combinatorial optimization and some geometric and combinatorial properties of the cut polytope.

Given a subset S of $V_n = \{1, 2, \ldots, n\}$, the *cut* determined by S consists of the pairs (i, j) of elements of V_n such that exactly one of i, j is in S. $\delta(S)$ denotes both the cut and its incidence vector in $\mathbb{R}^{\binom{n}{2}}$; that is, $\delta(S)_{ij} = 1$ if exactly one of i, j is in S and 0 otherwise for $1 \leq i < j \leq n$. By abuse of language, we use the term cut for both the cut itself and its incidence vector. The cut polytope $CutP_n$ is the convex hull of all 2^{n-1} cuts, and the cut cone Cut_n is the conic hull of all $2^{n-1} - 1$ nonzero cuts.

Those polyhedra were considered by many authors, see [**2, 5, 8, 10-16**] and references there. One of the motivations for the study of the cut polytope and cut cone comes from their applications in combinatorial optimization, see for instance [**11**]. Given a graph $G = (V_n, E)$ and nonnegative weight w_e, $e \in E$, assigned to its edges, the *max-cut problem* consists of finding a cut $\delta(S)$ whose weight $\sum_{e\in\delta(S)} w_e$ is as large as possible. By setting $w_e = 0$ if e is not an edge of G, we can consider the complete graph on V_n. Then the max-cut problem

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can be stated as a linear programming problem over the cut polytope $CutP_n$ as follows:

$$\begin{cases} \max \quad w^T \cdot x \\ x \in CutP_n. \end{cases}$$

This polyhedral approach to the max-cut problem leads to the study of the facets of the cut polytope $CutP_n$.

$CutP_n$ is a $\binom{n}{2}$-dimensional $0-1$ polytope with 2^{n-1} vertices. The polytope $CutP_3$ is combinatorially equivalent to the tetrahedron and $CutP_4$ is combinatorially equivalent to the 6-dimensional cyclic polytope with 8 vertices. More generally, $CutP_n$ is a 3-neighbourly polytope [**14**]. Another remarkable feature of the cut polytope $CutP_n$ is the high number of isometric symmetries it enjoys. The symmetry group of the cut polytope $Is(CutP_n)$ is induced, see [**8**], by permutations on $V_n = \{1, \ldots, n\}$ and *switching reflections* which were introduced in [**2, 5**]. Given a cut $\delta(S)$ and a facet F induced by the inequality $v \cdot x \leq v_0$, the switching reflection of F by the cut $\delta(S)$ is the facet induced by the inequality $v^S \cdot x \leq v_0 - v \cdot \delta(S)$, where $v^S_{ij} = -v_{ij}$ if $(i,j) \in \delta(S)$ and $v^S_{ij} = v_{ij}$ otherwise. The group $Is(CutP_n)$ is isomorphic to $Aut(\square_n)$, the automorphism group of the folded n-cube. We recall that the folded n-cube is the graph whose vertices are the partitions of $\{1, \ldots, n\}$ into two subsets, two partitions being adjacent when their common refinement contains a set of size one [**3**]. The symmetries of $CutP_n$ preserve adjacency and linear independence. Throughout this paper we use the fact that the facets of $CutP_n$ are partitioned into orbits of its symmetry group, that is, into classes of facets equivalent under permutation or switching.

The paper is organized as follows. In §2 we present some results on the facets of the cut polytope. Then in §3 and §4, we describe the skeleton of the dual cut polytope $CutP_n^*$ for $n \leq 7$, respectively the skeleton of the dual cut cone Cut_n^* for $n \leq 6$. In §5, we give some results and conjectures on the size and the adjacencies of the facets of the $CutP_n$. A general reference for the graph theory used in this paper is [**3**].

2. Facets of the Cut Polytope

Even if the determination of all the facets of the cut polytope $CutP_n$ and the cut cone Cut_n for any n seems to be hopeless, a wide range of facets has been already found. In particular, they are all known for $n \leq 7$, see [**1, 4, 5, 10, 15**]. Since it turns out that the facets of the cut polytope $CutP_n$ are switchings of the facets of the cut cone Cut_n, see [**2**], it is enough to determine all the facets of Cut_n to obtain all the facets of $CutP_n$.

To ease the notation we define the following two functions of $x \in I\!R^{\binom{n}{2}}$. For $b = (b_1, \ldots, b_m) \in I\!N^m$, $m \leq n$, and C a cycle with nodeset a subset of $\{1, \ldots, n\}$:

$$Q(b) \cdot x = \sum_{1 \le i < j \le n} b_i b_j x_{ij}$$

and
$$K(C) \cdot x = \sum_{(ij) \in C} x_{ij}$$

With this notation, the following 11 inequalities define facets of the cut polytope $CutP_n$ for $n \ge m$:

(1) $Q(1,1,1) \cdot x \le 2.$

(2) $Q(1,1,1,1,1) \cdot x \le 6.$

(3) $Q(2,1,1,1,1,1) \cdot x \le 12.$

(4) $Q(1,1,1,1,1,1,1) \cdot x \le 12.$

(5) $Q(2,2,1,1,1,1,1) \cdot x \le 20.$

(6) $Q(3,2,2,1,1,1,1) \cdot x - K(1,2,3) \cdot x \le 28.$

(7) $Q(3,1,1,1,1,1,1) \cdot x \le 20.$

(8) $Q(1,1,1,1,2,1,1) \cdot x - \sum_{i=1,3,5,6} x_{i7} - \sum_{i=2,4,5,7} x_{i6} \le 12.$

(9) $Q(1,1,1,1,1,1,1) \cdot x - K(1,2,3,4,5) \cdot x \le 10.$

(10) $Q(1,1,1,1,1,1,1) \cdot x - K(1,\ldots,7) \cdot x - 2(x_{2,5} + x_{2,7} + x_{4,7}) \le 6.$

(11) $Q(2,2,1,1,1,1,1) \cdot x - K(1,2,3,4) \cdot x \le 18.$

For $m \le n$, let F_i^n denote the facet of $CutP_n$ induced by the inequality (i), and O_i^n denote the orbit of F_i^n, that is, the class of facets of $CutP_n$ which are equivalent to F_i^n under permutation and switching. For computational purpose we choose the above 11 representatives F_i^n such that the right hand side of the inequality (i) is maximal. It turns out that, up to permutation, they are unique such representatives of $O_1^n \ldots O_{11}^n$ except for O_{10}^7 which contains two nonzero switchings of F_{10}^7 with right hand side equal to 6. The 16 facets of $CutP_4$ form the orbit O_1^4, the 56 facets $CutP_5$ are partitioned into the 2 orbits O_1^5 and O_2^5, the 368 facets of $CutP_6$ are partitioned into the 3 orbits O_1^6, O_2^6 and O_3^6, and the 116 764 facets of $CutP_7$ are partitioned into the 11 orbits $O_1^7, O_2^7, \ldots, O_{11}^7$.

Before describing Ω_n, the skeleton of the dual cut polytope, for $n \le 7$ in the next section, we present an equality found by Michel Deza, Martin Grötschel and Monique Laurent relating the number of facets of the cut polytope $CutP_n$ and the cut cone Cut_n:

LEMMA 2.1. *For any facet F containing the origin, with $|F|$ denoting the size of F, that is the number of cuts belonging to F, we have:*

$$|O(F)| \cdot |F| = |O(F)_{in\ Cut_n}| \cdot 2^{n-1}$$

PROOF. Let $\{F_1, \ldots, F_m\}$ be an ordering of $O(F)$ and $b_{iS} = 1$ if the cut $\delta(S) \in F_i$ and 0 otherwise, then we have:

$$\begin{aligned}\sum_{i,S} b_{iS} &= \sum_S (\sum_i b_{iS}) = \sum_S (\sum_i b_{i\emptyset}) \\ &= \sum_S (|O(F)_{in\ Cut_n}|) = |O(F)_{in\ Cut_n}| \cdot 2^{n-1},\end{aligned}$$

and also,

$$\begin{aligned}\sum_{i,S} b_{iS} &= \sum_i (\sum_S b_{iS}) = \sum_i (|F_i|) \\ &= \sum_i (|F|) = |O(F)| \cdot |F|,\end{aligned}$$

which completes the proof.

3. Skeleton of the Dual Cut Polytope for $n \leq 7$

Any pair of facets of $CutP_n$ are adjacent in Ω_n, the skeleton of the dual cut polytope $CutP_n^*$, if and only if their intersection is a face of codimension 2. For $n \leq 7$, the size of the orbits of the facets in the cut polytope $CutP_n$ was deduced from their corresponding size in the cut cone Cut_n found in [**12**] using Lemma 2.1. Since permutations and switching reflections preserve adjacency and linear independence, we can describe the properties of facets of $CutP_n$ by considering a representative facet of each orbit O_i^n, we choose the facets F_i^n for $n \leq 7$.

3.1. Skeleton of $CutP_4^*$. The polytope $CutP_4$ is combinatorially equivalent to the 6-dimensional cyclic polytope with 8 vertices. A pair of facets of $CutP_4$ are adjacent in Ω_4 if and only if they are non-conflicting. Two facets are called conflicting if there exists a pair i, j such that the two facets have nonzero coordinates of distinct signs at the position i, j. For example, the facets induced by the inequalities $Q(1,1,1) \cdot x \leq 2$ and $Q(-1,1,1) \cdot x \leq 0$ are conflicting at pair $(1,2)$. The notion of conflicting facets was introduced in [**16, 17**]. The graph Ω_4 formed by the 16 facets of $CutP_4$, is the (4×4)-grid, see [**6**], that is, the line graph $L(K_{4,4})$ which is also $L(\square_4)$. This graph is a strongly regular graph with parameters $v = 16$, $k = 6$, $\lambda = 2$ and $\mu = 2$, where v denotes the number of nodes, k the valency of each node, λ the number of nodes adjacent to two adjacent nodes and μ the number of nodes adjacent to two non-adjacent nodes. The diameter $\delta(\Omega_4)$ of $CutP_4^*$, is 2. Since $CutP_4$ is simplicial, all of its faces, including the 16 facets and 48 faces of codimension 2, are simplices.

3.2. Skeleton of $CutP_5^*$. The 10-dimensional polytope $CutP_5$ has 16 vertices and 56 facets which are partitioned into the 2 orbits O_1^5 and O_2^5. As for $CutP_4$, a pair of facets of $CutP_5$ are adjacent in Ω_5 if and only if they are non-conflicting. In Figure 1 we give the adjacency table of Ω_5, V_F the valency and $|F|$ the size

	O_1^5	O_2^5	V_F
$F \in O_1^5$	24	4	28
$F \in O_2^5$	10	0	10

i	1	2
$\|O_i^5\|$	40	16
$\|F_i^5\|$	12	10

FIGURE 1. Adjacencies in the skeleton of $CutP_5^*$

of the facets, and $|O_i^5|$ the size of the orbits of $CutP_5$, see [**6**]. For example, in the left column of the upper table in Figure 1, 10 means that any facet F of O_2^5 is adjacent to 10 facets of O_1^5. The neighbours of F_2^5 are the 10 permutations of F_1^5. The diameter $\delta(\Omega_5)$ of $CutP_5^*$ is 2. The simplex facets of $CutP_5$ are all the 16 facets of the orbit O_2^5 and, among its 640 faces of codimension 2, exactly 400 are simplices.

3.3. Skeleton of $CutP_6^*$. The 15-dimensional polytope $CutP_6$ contains 32 vertices and 368 facets which are partitioned into the 3 orbits O_1^6, O_2^6 and O_3^6. Using switching reflections and permutations it is only tedious but easy to obtain, as for $CutP_5$, the tables given in Figure 2. The simplex facets of $CutP_6$ are all the 192 facets of the orbit O_3^6.

	O_1^6	O_2^6	O_3^6	V_F
$F \in O_1^6$	58	60	24	142
$F \in O_2^6$	50	10	10	70
$F \in O_3^6$	10	5	0	15

i	1	2	3
$\|O_i^6\|$	80	96	192
$\|F_i^6\|$	24	20	15

FIGURE 2. Adjacencies in the skeleton of $CutP_6^*$

COROLLARY 3.1. *$CutP_6$ has exactly* 10 480 *faces of codimension* 2.

PROOF. The number of faces of codimension 2 of a polytope is half of the total valency of the skeleton of its dual. Since we know the valency of all the 368 nodes of Ω_6, the result is a straightforward calculation. Moreover, one can check that exactly 4 800 of these faces of codimension 2 are simplices.

COROLLARY 3.2. *The diameter $\delta(\Omega_6)$ of $CutP_6^*$ is* 3.

PROOF. For any pair of facets F and F' of $CutP_6$ we have to find a path in Ω_6 of length shorter than 3. Since the diameter of the restriction of Ω_6 to O_1^6 is 2, see [**6**], we can assume that F and F' do not belong to O_1^6. If they both belong to O_2^6 then, since they both have more than half the size of O_1^6 neighbours in O_1^6, we can find a facet in O_1^6 adjacent to F and F'. If $F \in O_3^6$, for example $F = F_3^6$, and $F' \in O_2^6$, then one can easily check that the facet induced by either the inequality $Q(1,0,0,1,1) \cdot x \leq 2$ or the inequality $Q(0,0,1,1,1) \cdot x \leq 2$ is adjacent to F and F'. If they both belong to O_3^6, we can suppose without loss of generality that F, respectively F', is the facet induced by the inequality $Q(2,1,1,1,1,1) \cdot x \leq 12$, respectively the inequality $Q(-1,2,-1,1,1,1) \cdot x \leq 2$. Then one can check that F and F' have no common neighbour and that the facets G and G', respectively induced by the inequality $Q(1,0,0,1,1) \cdot x \leq 2$ and $Q(-1,1,1) \cdot x \leq 0$, are adjacent and respectively adjacent to F and F'.

REMARK 3.3. *While the restriction of Ω_6 to the facets of O_1^6 forms a graph of diameter 2, see [**6**], its restriction to the facets of O_2^6 forms a graph of diameter 3 with parameters $v = 96, k = 10, \lambda = 4$ and $\mu \in \{0,1,2\}$, which is locally $\bar{K}_{5,5}$. More generally, the inequality $\sum_{1 \leq i < j \leq 2t+1} x_{ij} \leq t(t+1)$ induces a facet F_t of $CutP_{2t+2}$ and the facets belonging to the orbit O_t of F_t form a graph which is localy the complement of $K_{2t+1,2t+1}$; two facets of O_t being adjacent if and only if they are non-conflicting.*

3.4 Skeleton of $CutP_7^*$.

	O_1^7	O_2^7	O_3^7	O_4^7	O_5^7	O_6^7	O_7^7	O_8^7	O_9^7	O_{10}^7	O_{11}^7	V_F
$F \in O_1^7$	112	264	672	64	240	432	48	3 456	1 728	2 304	2 112	11 432
$F \in O_2^7$	110	180	100	4	40	120	0	840	240	480	480	2 594
$F \in O_3^7$	70	25	10	2	10	20	2	60	0	0	40	239
$F \in O_4^7$	140	21	42	0	21	0	0	0	252	0	420	896
$F \in O_5^7$	25	10	10	1	0	10	0	0	0	0	20	76
$F \in O_6^7$	9	6	4	0	2	0	0	0	0	0	0	21
$F \in O_7^7$	15	0	6	0	0	0	0	0	0	0	0	21
$F \in O_8^7$	12	7	2	0	0	0	0	0	0	0	0	21
$F \in O_9^7$	15	5	0	1	0	0	0	0	0	0	0	21
$F \in O_{10}^7$	14	7	0	0	0	0	0	0	0	0	0	21
$F \in O_{11}^7$	11	6	2	1	1	0	0	0	0	0	0	21

i	1	2	3	4	5	6	7	8	9	10	11
$\lvert O_i^7 \rvert$	140	336	1 344	64	1 344	6 720	448	40 320	16 128	32 040	26 880
$\lvert F_i^7 \rvert$	48	40	30	35	26	21	21	21	21	21	21

FIGURE 3. Adjacencies in the skeleton of $CutP_7^*$

The 21-dimensional polytope $CutP_7$ contains 64 vertices and 116 764 facets which are partitioned into the 11 orbits $O_1^7, O_2^7 \ldots O_{11}^7$. The adjacencies for the 6 orbits of simplex facets $O_6^7, O_7^7 \ldots O_{11}^7$ could be derived from the list of facets of the simplex facets of Cut_7 given in [**13**]. The adjacencies for the orbit O_1^7 and O_4^7 can be deduced from the results given respectively in [**6**] and [**7**], the remaining adjacencies were checked by computer. The complete adjacency table is given in Figure 3. The simplex facets of $CutP_7$ are all the 113 536 facets of the orbits $O_6^7, O_7^7 \ldots O_{11}^7$. The local graph induced by a simplex facet F of $CutP_7$ is the clique K_{21} for $F \in O_7^7, O_9^7$ or O_{10}^7, $K_{21} - K_2$ for $F \in O_6^7$ or O_{11}^7, and $K_{21} - K_3$ for $F \in O_8^7$.

COROLLARY 3.4. *$CutP_7$ has exactly* 2 668 512 *faces of codimension* 2.

PROOF. As for $CutP_6$, since we know the valencies of Ω_7, the result is a straightforward calculation. Moreover, using a computer, we found that exactly 2 438 016 of these faces of codimension 2 are simplices.

COROLLARY 3.5. *The diameter $\delta(\Omega_7)$ of $CutP_7^*$ satisfies:* $3 \leq \delta(\Omega_7) \leq 4$.

PROOF. The diameter of the restriction of Ω_7 to O_1^7 is 2, see [**6**]. Then, since every facet of $CutP_7$ is adjacent to a facet of O_1^7, we have $\delta(\Omega_7) \leq 4$. The two facets, respectively induced by the inequality $Q(3,1,1,1,1,1,1) \cdot x \leq 20$ and $Q(-1,3,-1,1,1,1,1) \cdot x \leq 6$, having no common neighbour, we have $\delta(\Omega_7) \geq 3$.

REMARK 3.6. *It was conjectured in* [**6**] *that the facets of O_1^n form a dominating set in Ω_n, that is, every facet of the cut polytope is adjacent to a facet belonging to the orbit O_1^n. Since the diameter of the restriction of Ω_n to O_1^n is* 2 [**6**], *it would imply that the diameter of the dual cut polytope satisfies $\delta(\Omega_n) \leq 4$.*

4. Skeleton of the Dual Cut Cone for $n \leq 6$

The results about Ω'_n, the skeleton of the dual cut cone Cut_n^*, are similar to the ones concerning the cut polytope. The differences result from the fact that the symmetry group of the cut cone Cut_n is not fully determined. Clearly all permutations of $\{1, \ldots, n\}$ are isometric symmetries of Cut_n, that is, we have $Sym(n) \subset Is(Cut_n)$, and the equality probably holds.

Then, as for the cut polytope, we can use the fact that the facets of the cut cone are partitioned into the orbits of $Sym(n)$. We choose the 4 representatives G_1^n, G_2^n, G_3^n and $G_{3'}^n$ defined by the following inequalities, and denote by U_i^n the orbit of the facet G_i^n,

(1) $$Q(-1,1,1) \cdot x \leq 0.$$

(2) $$Q(-1,-1,1,1,1) \cdot x \leq 0.$$

(3) $$Q(-2,-1,1,1,1,1) \cdot x \leq 0.$$

(3') $$Q(2,1,1,-1,-1,-1) \cdot x \leq 0.$$

	U_1^5	U_2^5	V_F
$G \in U_1^5$	19	3	22
$G \in U_2^5$	9	0	9

i	1	2
$\|U_i^5\|$	30	10
$\|G_i^5\|$	11	9

FIGURE 4. Adjacencies in the skeleton of Cut_5^*

Skeleton of Cut_4^* and Cut_5^*. The graph Ω'_4 formed by the 12 facets of Cut_4 is the (4×3)-grid, see [**6**], that is, the line graph $L(K_{4,3})$. This graph is a regular graph with parameters $v = 12$, $k = 5$, $\lambda = 2$ or 1 and $\mu = 2$. The diameter of Cut_4^* is 2. The 40 facets of Cut_5 are partitioned into the 2 orbits U_1^5 and U_2^5. The adjacency table of Ω'_5, the valency and size of the facets, and the size of the orbits Cut_5 are given in Figure 4. $\delta(\Omega'_5) = 2$ and Cut_5 has 375 faces of codimension 2. The simplex facets of Cut_5 are all the 10 facets of the orbit U_2^5.

Skeleton of Cut_6^*. The 210 facets of Cut_6 are partitioned into the 4 orbits U_1^6, U_2^6, U_3^6 and $U_{3'}^6$. The adjacencies of Ω'_6 are given in Figure 5, $\delta(\Omega'_6) = 3$ and Cut_6 has 5 190 faces of codimension 2. The simplex facets of Cut_6 are all the 90 facets of the orbits U_3^6 and $U_{3'}^6$.

	U_1^6	U_2^6	U_3^6	$U_{3'}^6$	V_F
$G \in U_1^6$	45	39	5	9	98
$G \in U_2^6$	39	8	2	5	54
$G \in U_3^6$	10	4	0	0	14
$G \in U_{3'}^6$	9	5	0	0	14

i	1	2	3	3′
$\|U_i^6\|$	60	60	30	60
$\|G_i^6\|$	23	19	14	14

FIGURE 5. Adjacencies in the skeleton of Cut_6^*

REMARK 4.1. *The* 38 780 *facets of Cut_7 are partitioned into* 36 *orbits of* $\mathit{Sym}(7)$.

5. On the Shape of the Cut Polytope

In this section, we give a tight upper bound and some conjectures on the size and the adjacency of the facets of the cut polytope.

LEMMA 5.1. *Any facet F of the cut polytope satisfies $|F| \leq 3.2^{n-3}$ with equality if and only if $F \in O_1^n$.*

PROOF. Let F be a facet of $CutP_n$ induced by the inequality:

$$\sum_{1 \leq i < j \leq n} v_{ij} x_{ij} \leq a, \tag{1}$$

v_{kl} a nonzero coordinate of F, and $F(S)$ the value of the left hand side of (1) on the cut $\delta(S)$. With $S \cap \{i,j\} = \emptyset$, we have:

$$F(S \cup \{i\}) + F(S \cup \{j\}) - F(S) - F(S \cup \{i,j\}) = 2\ v_{ij} \neq 0.$$

This implies that no more than 3 of any such 4 cuts belong to F, and therefore that no facet contains more than $\frac{3}{4}\ 2^{n-1} = 3.2^{n-3}$ vertices. Reversely, let F be a facet of $CutP_n$ containing 3.2^{n-3} vertices. Without loss of generality, we can assume that a, the right hand side of (1) is 0. We first show that any facet F of $CutP_n$ has at least 3 nonzero coordinates. If F has only 2 nonzero coordinates, since $CutP_n$ lies in the positive orthant of $I\!R^{\binom{n}{2}}$, we can assume that F is induced by the inequality $v_{ij} - \alpha v_{kl} \leq 0$ with $\alpha \geq 1$. The point p with all coordinates equal to $\frac{1}{3}$, except $p_{kl} = \frac{1}{3\alpha}$, belongs to both the facet F and the relative interior of P, which is impossible. Then we prove that a facet F containing 3.2^{n-3} vertices has at most 3 nonzero coordinates. Let consider any 3 nonzero coordinates of F v_{ij}, v_{kl} and v_{st}. Suppose that the six indices i, j, k, l, s and t are distinct, that is, $|\{i,j,k,l,s,t\}| = 6$. In that case, we have:

(2) for $S \cap \{i,j\} = \emptyset$, $F(S \cup \{i\}) + F(S \cup \{j\}) - F(S) - F(S \cup \{i,j\}) = 2v_{ij} \neq 0$,

(3) for $S \cap \{k,l\} = \emptyset$, $F(S \cup \{k\}) + F(S \cup \{l\}) - F(S) - F(S \cup \{k,l\}) = 2v_{kl} \neq 0$,

(4) for $S \cap \{s,t\} = \emptyset$, $F(S \cup \{s\}) + F(S \cup \{t\}) - F(S) - F(S \cup \{s,t\}) = 2v_{st} \neq 0$.

Since F contains 3/4 of the total number of vertices of $CutP_n$, exactly 3 terms of the left hand side of the equations (2), (3) or (4) are null. Suppose that v_{ij} and v_{kl} have the same sign (otherwise we consider v_{ij} and v_{st} or v_{kl} and v_{st}). Then the equations (2) with $S = \emptyset$ and (3) with $S = \{i\}$ imply that $F(\{i\})$ cannot be the only nonzero term of both the equations (2) and (3), that is $F(\{i\}) = 0$. In the same way, the equations (2) with $S = \{s\}$, respectively $S = \{t\}$ and $\{k,l\}$, and (3) with $S = \{i,s\}$, respectively $S = \{i,t\}$ and $\{j\}$, imply that $F(\{i,s\})$, respectively $F(\{i,t\})$ and $F(\{i,s,t\})$, is null, which contradicts the equation (4) with $S = \{i\}$. Similarly $|\{i,j,k,l,s,t\}| = 5$ or 4 also leads to a contradiction. Therefore F has exactly 3 nonzero coordinates v_{ij}, v_{ik} and v_{jk}. We have:

(5) for $S \cap \{i,j\} = \emptyset$, $F(S \cup \{i\}) + F(S \cup \{j\}) - F(S) - F(S \cup \{i,j\}) = 2v_{ij} \neq 0$,

(6) for $S \cap \{i,k\} = \emptyset$, $F(S \cup \{i\}) + F(S \cup \{k\}) - F(S) - F(S \cup \{i,k\}) = 2v_{ik} \neq 0$,

(7) for $S \cap \{j,k\} = \emptyset$, $F(S \cup \{j\}) + F(S \cup \{k\}) - F(S) - F(S \cup \{j,k\}) = 2v_{jk} \neq 0$.

Then, the equations (5) with $S = \{k\}$ and (6) with $S = \{j\}$ imply that $v_{ij} = v_{ik}$; (6) with $S = \emptyset$ and (7) with $S = \{i\}$ imply that $v_{ik} = -v_{jk}$. This means that F is the facet of O_1^n induced by the inequality $x_{jk} - x_{ij} - x_{ik} \leq 0$.

REMARK 5.2. *The* $4\binom{n}{3}$ *facets of* $CutP_n$ *belonging to* O_1^n *contain* 3.2^{n-3} *cuts, that is* $3/4$ *of the total number of vertices of the cut polytope. Those facets are the extreme opposite of being simplices. We think that the shape of the cut polytope is essentially given by its non-simplex facets, in particular by the facets of* O_1^n *(see Remark* 3.6*), and that the huge majority of the facets of* $CutP_n$ *are simplices which only "polish" it. This belief is shared by designers of the cutting plane methods who hope that the "few nice" classes of facets they use will be sufficient to prove the optimality or provide excellent bounds, and that the facets they have no access to contribute very little to the computational behavior of such methods.*

CONJECTURE 5.3. *For* $n \geq 5$*, any two simplex facets of the cut polytope are not adjacent in* Ω_n*. It holds for* $n \leq 7$.

CONJECTURE 5.4. *The cut polytope* $CutP_n$ *is asymptotically simplicial. In fact, more than* 97% *of the facets of* $CutP_7$ *and* 91% *of its faces of codimension* 2 *are simplices.*

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REFERENCES

1. D. Avis and Mutt, *All facets of the six point Hamming cone*, European Journal of Combinatorics **10** (1989), 309–312.
2. F. Barahona and A.R. Mahjoub, *On the cut polytope*, Mathematical Programming **36** (1986), 157–173.
3. A.E. Brouwer , A.M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin and New York, 1989.
4. M. Deza, *On the Hamming geometry of unitary cubes*, Doklady Akademii Nauk SSR **134** (1960), 1037–1040 (Russian); English transl. in Soviet Physics **5** (1961), 940–943.
5. ———, *Matrices de formes quadratiques non négatives pour des arguments binaires*, Comptes rendus de l'Académie des Sciences de Paris **277** (1973), 873–875.
6. A. Deza and M. Deza, *The ridge graph of the metric polytope and some relatives*, Polytopes: Abstract, Convex and Computational (T. Bisztriczky, P. McMullen, R. Schneider and A. Ivic Weiss, eds.) (to appear).
7. M. Deza, K. Fukuda and M. Laurent, *The inequicut cone*, Discrete Mathematics **119** (1993), 21–48.
8. M. Deza, V.P. Grishukhin and M. Laurent, *The symmetries of the cut polytope and of some relatives*, Applied Geometry and Discrete Mathematics, the "Victor Klee Festschrift" (P. Gritzmann and B. Sturmfels, eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 4, 1991, pp. 205–220.
9. M. Deza, M. Grötschel and M. Laurent, *Cuts and Metrics*, book in preparation.
10. M. Deza and M. Laurent, *Facets for the cut cone I*, Mathematical Programming **56 (2)** (1992), 121–160.
11. ———, *Applications of cut polyhedra*, Journal of Computational and Applied Mathematics (to appear).

12. ———, *New results on facets of the cut cone*, Journal of Combinatorics, Information and System Sciences (R.C. Bose memorial issue) **17 (1-2)** (1992), 13–38.
13. ———, *The cut cone: simplicial faces and linear dependencies*, Bulletin of the Institute of Mathematics, Academia Sinica **21 (2)** (1993), 143–182.
14. M. Deza, M. Laurent and S. Poljak, *The cut cone III: on the role of triangle facets*, Graphs and Combinatorics **8** (1992), 125–142.
15. V. P. Grishukhin, *All facets of the cut cone C_n for $n = 7$ are known*, European Journal of Combinatorics **11** (1990), 115–117.
16. ———, *Computing extreme rays of the metric cone for seven points*, European Journal of Combinatorics **13** (1992), 153–165.
17. M. Laurent, *Graphic vertices of the metric polytope*, Research report No. 91737-OR, Institut für Diskrete Mathematik, Universität Bonn.
18. M. Padberg, *The boolean quadratic polytope: some characteristics, facets and relatives*, Mathematical Programming **45** (1989), 139–172.

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