ELSEVIER

Discrete Applied Mathematics 115 (2001) 3–14

DISCRETE APPLIED MATHEMATICS

# On the binary solitaire cone

David Avis[a], Antoine Deza[b,c,*]

[a] *McGill University, School of Computer Science, 3480 University, Montréal, Canada H3A 2A7*
[b] *Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo, Japan*
[c] *Ecole des Hautes Etudes en Sciences Sociales, Centre d'Analyse et de Mathématique Sociales, 54 bd Raspail Paris, France*

Received 18 May 1999; received in revised form 4 February 2000; accepted 3 July 2000

**Abstract**

The *solitaire cone* $S_B$ is the cone of all feasible fractional Solitaire Peg games. Valid inequalities over this cone, known as pagoda functions, were used to show the infeasibility of various peg games. The link with the well studied dual metric cone and the similarities between their combinatorial structures (see [3]) leads to the study of a dual cut cone analogue; that is, the cone generated by the $\{0,1\}$-valued facets of the solitaire cone. This cone is called *binary solitaire cone* and denoted as $\mathscr{B}S_B$. We give some results and conjectures on the combinatorial and geometric properties of the binary solitaire cone. In particular we prove that the extreme rays of $S_B$ are extreme rays of $\mathscr{B}S_B$ strengthening the analogy with the dual metric cone whose extreme rays are extreme rays of the dual cut cone. Other related cones are also considered.


## 1. Introduction and basic properties

### *1.1. Introduction*

Peg solitaire is a peg game for one player which is played on a board containing a number of holes. The most common modern version uses a cross shaped board with 33 holes—see Fig. 1—although a 37 hole board is common in France. Computer versions of the game now feature a wide variety of shapes, including rectangles and triangles. Initially the central hole is empty, the others contain pegs. If in some row (column, respectively) two consecutive pegs are adjacent to an empty hole in the same row (column, respectively), we may make a *move* by removing the two pegs and placing

* Corresponding author.
*E-mail addresses:* avis@mutt.cs.mcgill.ca (D. Avis), deza@is.titech.ac.jp, deza@ehess.fr (A. Deza).

0166-218X/01/$ - see front matter 
PII: S0166-218X(01)00211-6

one peg in the empty hole. The objective of the game is to make moves until only one peg remains in the central hole. Variations of the original game, in addition to being played on different boards, also consider various alternate starting and finishing configurations.

The game itself has uncertain origins, and different legends attest to its discovery by various cultures. An authoritative account with a long annotated bibliography can be found in the comprehensive book of Beasley [4]. The book mentions an engraving of Berey, dated 1697, of a lady with a Solitaire board. The modern mathematical study of the game dates to the 1960s at Cambridge University. The group was led by Conway who has written a chapter in [5] on various mathematical aspects of the subject. One of the problems studied by the Cambridge group is the following basic *feasibility* problem of peg solitaire:

> For a given board $B$, starting configuration $c$ and finishing configuration $c'$, determine if there is a legal sequence of moves from $c$ to $c'$.

The complexity of the feasibility problem for the game played on a $n$ by $n$ board was shown by Uehara and Iwata [11] to be NP-complete, so easily checked necessary and sufficient conditions for feasibility are unlikely to exist. One of the tools used to show the infeasibility of certain starting and finishing configurations is a polyhedral cone called the *solitaire cone* $S_B$, corresponding to some given board $B$.

### 1.2. Basic properties

For ease of notation, we will mostly be concerned with rectangular boards which we represent by 0–1 matrices. A zero represents an empty hole and a one represents a peg. For example, let $c=[1\ \ 0\ \ 1\ \ 1]$ and $c'=[0\ \ 0\ \ 1\ \ 0]$ be starting and finishing positions for the 1 by 4 board. This game is *feasible*, involving two moves and the intermediate position $[1\ \ 1\ \ 0\ \ 0]$. For any move on an $m$ by $n$ board $B$ we can define an $m$ by $n$ *move matrix* which has 3 nonzero entries: two entries of $-1$ in the positions from which pegs are removed and one entry of 1 for the hole receiving the new peg. The two moves involved in the previous example are represented by $m_1=[0\ \ 1\ \ -1\ \ -1]$ and $m_2=[-1\ \ -1\ \ 1\ \ 0]$. Clearly $c'=c+m_1+m_2$. By abuse of language, we use the term *move* for both the move itself and the move matrix. In general it is easily seen

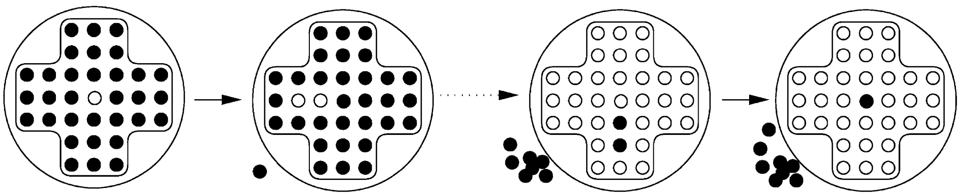


Fig. 1. A feasible English solitaire peg game with possible first and last moves.

that if $c, c'$ define a feasible game of $k$ moves there exist move matrices $m_1, \ldots, m_k$ such that

$$c' - c = \sum_{i=1}^{k} m_i. \tag{1.1}$$

Clearly, Eq. (1.1) is necessary but not sufficient for the feasibility of a peg game. For example, take $c = [1\ 1\ 1\ 1]$ and $c' = [0\ 0\ 0\ 1]$. We have $c' - c = [-1\ \ -1\ 1\ 0] + [0\ 1\ \ -1\ \ -1] + [0\ \ -1\ \ -1\ 1]$, but $c, c'$ do not define a feasible game; in fact there are no legal moves! Let us relax the conditions of the original peg game to allow a fractional (positive or negative) number of pegs to occupy any hole. We call this game the *fractional game*, and call the original game the 0–1 *game* (in a 0–1 game we require that in every position of the game a hole is either empty or contains a single peg). A *fractional move matrix* is obtained by multiplying a move matrix by any positive scalar and is defined to correspond to the process of adding a move matrix to a given position. For example, let $c = [1\ 1\ 1]$, $c' = [1\ 0\ 1]$. Then $c' - c = [0\ \ -1\ 0] = \frac{1}{2}[-1\ \ -1\ 1] + \frac{1}{2}[1\ \ -1\ \ -1]$ is a feasible fractional game and can be expressed as the sum of two fractional moves, but is not feasible as a 0 –1 game.

Let $B$ be a board and $n_B$ the total number of possible moves on the board. The *solitaire cone* $S_B$ is the set of all nonnegative combinations of the $n_B$ corresponding move matrices. Thus $c' - c \in S_B$ if:

$$c' - c = \sum_{i=1}^{n_B} y_i m_i, \quad y_i \geqslant 0, \ i = 1, \ldots, n_B. \tag{1.2}$$

In the above definition it is assumed that the $h_B$ holes in the board $B$ are ordered in some way and that $c' - c$ and $m_i$ are $h_B$-vectors. When $B$ is a rectangular $m$ by $n$ board $B_{m,n}$ it is convenient to display $c' - c$ and $m_i$ as $m$ by $n$ matrices, although of course all products should be interpreted as dot products of the corresponding $mn$-vectors. For $n \geqslant 4$ or $m \geqslant 4$, the solitaire cone $S_{m,n}$ associated to the $m$ by $n$ board is a pointed full-dimensional cone and the moves of the solitaire cone are extreme rays; see [3] for a detailed study of the solitaire cone. The following result obtained in 1961 is credited to Boardman (who apparently has not published anything on the subject) by Beasley [4, p. 87]. We identify $c' - c$ with the fractional game defined by $c$ and $c'$.

**Proposition 1.1.** *Eq.* (1.2) ($c' - c \in S_B$) *is necessary and sufficient for the feasibility of the fractional game*; *that is, the solitaire cone* $S_B$ *is the cone of all feasible fractional games.*

The condition $c' - c \in S_B$ is therefore a necessary condition for the feasibility of the original peg game and, more usefully, provides a certificate for the infeasibility of certain games. The certificate of infeasibility is any inequality valid for $S_B$ which is violated by $c' - c$. According to [4, p. 71], these inequalities "were developed by J.H. Conway and J.M. Boardman in 1961, and were called *pagoda functions* by Conway…".

They are also known as *resource counts*, and are discussed in some detail in [5]. The strongest such inequalities are induced by the facets of $S_B$.

Other tools to show the infeasibility of various peg games include the so-called *rule-of-three* which simply amounts to color the board by diagonals of $\alpha$, $\beta$ and $\gamma$ (in either direction). Then, with $\#\alpha$ ($\#\beta, \#\gamma$ resp.) denoting the number of pegs in an $\alpha$-colored ($\beta, \gamma$ resp.) holes, one can check that the parity of $\#\alpha - \#\beta$, $\#\beta - \#\gamma$ and $\#\gamma - \#\alpha$ is an invariant for the moves. The rule-of-three was apparently first published in 1841 by Suremain de Missery; see Beasley's book [4] for a detailed historical background. Another necessary condition generalizing the rule-of-three—the *solitaire lattice criterion*—is to check if $c' - c$ belong to the *solitaire lattice* generated by all integer linear combinations of moves, that is:

$$c' - c = \sum_{i=1}^{n_B} y_i m_i, \quad y_i \in \mathbb{Z}, \ \ i = 1, \dots, n_B.$$

While the lattice criterion is shown to be equivalent to the rule-of-three for the classical English 33-board and French 37-board as well as for any $m \times n$ board, the lattice criterion is stronger than the rule-of-three for games played on more complex boards. In fact, for a wide family of boards the lattice criterion exponentially outperforms the rule-of-three, see [7].

The solitaire cone is generated by a set of extreme rays, each of which is all zero except for three nonzero components which are $1, -1, -1$. In [3], the solitaire cone is related to another cone with the same property, the *flow cone* which is dual to the much studied *metric cone* which arises in the study of multicommodity flows; see, for example, [1,6,8,10].

## 2. Facets of the solitaire cone

For simplicity we consider rectangular boards and, to avoid the special effects created by the boundary, we study their toric closures which are simply called *toric boards*. In other words, the toric $m$ by $n$ board is an $m$ by $n$ rectangular board with additional jumps which traverse the boundary. Note that the associated toric solitaire cone $S_{m\times n}$ is pointed and full-dimensional for $m \geqslant 3$ or $n \geqslant 3$. Let $B$ be a rectangular $m$ by $n$ board, with $m \geqslant 3$ or $n \geqslant 3$. Using the notation described following Eq. (1.2), we will represent the coefficients of the facet inducing inequality

$$az \leqslant 0 \tag{2.3}$$

by the $m$ by $n$ array $a = [a_{i,j}]$. Inequality (2.3) holds for every $z \in S_{m\times n}$. It is a convenient abuse of terminology to refer to $a$ as a *facet* of $S_{m\times n}$. Three consecutive row or column elements of an $m$ by $n$ array are denoted by $(t_1, t_2, t_3)$ and called a *consecutive triple* of row or column indices. For example both $t_1 = i, j$, $t_2 = i, j+1$, $t_3 = i, j+2$ and $t_1 = i+2, j$, $t_2 = i+1, j$, $t_3 = i, j$ are consecutive triples. Using this notation we see that a move matrix for $B$ is an $m$ by $n$ matrix that is all zero except for elements of some consecutive triple which take the values $1, -1, -1$. Each consecutive triple defines a *triangle inequality*

$$a_{t_1} \leqslant a_{t_2} + a_{t_3}. \tag{2.4}$$

The definition of *consecutive triple* is extended by allowing row indices to be taken modulo $m$ and column indices to be taken modulo $n$. For example, for a 4 by 4 toric board both $t_1 = 3,1,\ t_2 = 3,2,\ t_3 = 3,4$ and $t_1 = 3,4,\ t_2 = 4,4,\ t_3 = 1,4$ are consecutive triples (see Fig. 2). Similarly we extend the definition of a consecutive string of entries to include strings that traverse the boundary.

The $\{0,1\}$-valued facets the solitaire cone are considerably more complex than the 0–1 facets of the dual metric cone, which are generated by cuts in the complete graph. For a toric board $B$, a complete characterization of 0–1 facets of $S_{m\times n}$ was given in [3]. Let $a$ be an $m$ by $n$ 0–1 matrix. We define the 1-*graph* $G_a$ on $a$ as follows: vertices of $G_a$ correspond to the ones, and two ones are adjacent if the corresponding coefficients are in some consecutive triple where the remaining coefficient is zero. Note that in fact there must be at least two such triples since if $(t_1, t_2, t_3)$ is such a triple then so is $(t_3, t_2, t_1)$. Theorem 2.1 characterizes $\{0,1\}$-valued facets of $S_{m\times n}$, see Fig. 3 for an illustration.

**Theorem 2.1.** *Let $B$ be the $m$ by $n$ toric board. A $m$ by $n$ 0–1 matrix $a$ is a facet of $S_{m\times n}$ if and only if* (i) *no nonzero row or column contains two consecutive zeroes, and* (ii) *the* 1-*graph $G_a$ is connected.*

Theorem 2.1 is proved in [3], and we give only a brief outline here. For the sufficiency, a facet generating procedure is used that makes use of the fact that any zero in the matrix $a$ causes the two elements on either side to be equal, by the triangle inequalities, implying an edge in $G_a$. It is shown that conditions (i) and (ii) ensure the procedure terminates with a facet. For the necessity, if condition (i) fails $a$ violates a triangle inequality. If condition (ii) fails, another matrix is generated that is not a scalar multiple of $a$, yet satisfies the same tight triangle inequalities, contradicting the fact that $a$ is a facet. Theorem 2.1 is useful for proving large classes of 0–1 matrices

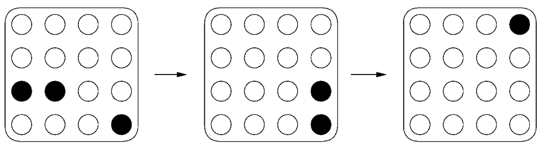

Fig. 2. Moves on the 4 by 4 toric board.

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Fig. 3. Two pagoda functions of $S_{4\times 4}$, only the first one being a facet.

are facets. Let $x=(x_1,\ldots,x_m)$ and $y=(y_1,\ldots,y_n)$ be two vectors. We say the $m$ by $n$ matrix $a$ is the *product* of $x$ and $y$ if for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ $a_{i,j}=x_i y_j$. A simple application of Theorem 2.1 gives:

**Corollary 2.2**

1. A 0–1 *n-vector is a facet of the* 1 *by n toric board if and only if it has no pair of consecutive zeroes, no string of five or more ones, and at most one string of four ones.*
2. *The product of two* 0–1 *facets of the* 1 *by m and* 1 *by n toric boards gives a* 0–1 *facet of the m by n toric board.*

**Proposition 2.2.** *Let $h(n)$ be the number of $\{0,1\}$-valued facets of $S_{1\times n}$. For $n \geqslant 7$, $(n+18)1.46^{n-8} \leqslant h(n) \leqslant (n+19)1.47^{n-6}$.*

**Proof.** The formula can be verified directly for $n \leqslant 11$ by referring to the 5th column of Table 1. We define an $f$-vector to be a $\{0,1\}$-valued vector of length $n$ with no 2 consecutive zeroes, no string of 4 or more ones and starting and ending with a one. We first count $f(n)$, the number of $f$-vectors. Direct calculation shows that: $f(2)=1$, $f(3)=2$, $f(4)=2$, $f(5)=4$. For $n \geqslant 6$, a $f$-vector has the form [1 0 1 ... 1], [ 1 1 0 1 ... 1 ] or [ 1 1 1 0 1 ... 1], where the string 1 ... 1 is an $f$-vector. In other words, we have $f(n)=f(n-2)+f(n-3)+f(n-4)$. It is easy to show by induction that for $n \geqslant 6$,

$$1.46^{n-2} \leqslant f(n) \leqslant 1.47^{n}. \tag{5}$$

Now, for $n \geqslant 12$, by Item (1) of Corollary 3, the number $h(n)$ of $\{0,1\}$-valued facets of $S_{1\times n}$ is the number of toric $\{0,1\}$-valued vectors of length $n$ with no 2 consecutive zeroes, no string of 5 or more ones, and at most one string of 4 ones. Call such vectors *h-vectors*. If an $h$-vector has no string of 4 ones, then it either starts with [ 0 1 ... 1], [ 1 0 1 ... ], [ 1 1 0 1 ... 1 0 ], [ 1 1 0 1 ... 1 0 1 ] or [ 1 1 1 0 1 ... 1 0 ] where the string 1 ... 1 is an $f$-vector. In other words, we have $2f(n-1)+f(n-4)+2f(n-5)=2f(n-3)+3f(n-4)+4f(n-5)$ $h$-vectors without a string of 4 ones. We have $nf(n-6)$ $h$-vectors with one string of 4 ones as each is of the form [ ... 1 0 1 1 1 1 0 1 ...]. Therefore, the total number of $h$-vectors for $n \geqslant 8$ is given by

$$h(n)=2f(n-3)+3f(n-4)+4f(n-5)+nf(n-6). \tag{6}$$

The proposition follows by substituting the asymptotic bounds for $f$ obtained above in this equation.

## 3. The binary solitaire cone and other relatives

The link with the dual metric cone and the similarities between their combinatorial structures—see [3]—leads to the study of a dual cut cone analogue; that is, the *binary* *solitaire cone* $\mathscr{B}S_B$ generated by the $\{0,1\}$-valued facets of the solitaire cone. We give some results and conjectures on the combinatorial and geometric properties of the binary solitaire cone. In particular we prove that the extreme rays of $S_B$ are extreme rays of $\mathscr{B}S_B$ strengthening the analogy with the dual metric cone, for which the extreme rays are also extreme rays of the dual cut cone. Other related cones are also considered.

### 3.1. The binary solitaire cone

The dual cut cone is generated by the $\{0,1\}$-valued facets of the dual metric cone. Similarly, we consider the cone generated by the $\{0,1\}$-valued facets of the solitaire cone. This cone is called *binary solitaire cone* and denoted $\mathscr{B}S_B$.

We present in details some small dimensional cases and give some results and conjectures on the combinatorial and geometric properties of the binary solitaire cone. In particular, we investigate the diameter, adjacency and incidence relationships of the binary solitaire cone $\mathscr{B}S_{m\times n}$ and its dual $\mathscr{B}S^*_{m\times n}$. Two extreme rays (resp. facets) of a polyhedral cone are *adjacent* if they belong to a face of dimension (resp. codimension) two. The number of rays (resp. facets) adjacent to the ray $r$ (resp. facet $F$) is denoted $A_r$ (resp. $A_F$). A ray and a facet are *incident* if the ray belongs to the facet. We denote by $I_r$ (resp. $I_F$) the number of facets (resp. rays) incident to the ray $r$ (resp. facet $F$). The diameter of $\mathscr{B}S_B$ (its dual $\mathscr{B}S^*_B$ resp.), that is, the smallest number $\delta$ such that any two vertices can be connected by a path with at most $\delta$ edges, is $\delta(\mathscr{B}S_B)$ ($\delta(\mathscr{B}S^*_B)$ resp.); see Fig. 4.

Finding all extreme rays of the cone $\mathscr{B}S_B$ (such as the 930 048 rays of $\mathscr{B}S_{4\times 4}$) is an example of a *convex hull* or *vertex enumeration* problem, for which various computer programs are available. The computational results in this paper were obtained using the double description method *cdd* implemented by Fukuda [9], and the reverse search method *lrs* implemented by Avis [2]. The diameters of cones were computed using *graphy* implemented by Fukuda [9].

| Board | $\#rays$ | $I_r$ | $A_r$ | $\#facets$ | $I_F$ | $A_F$ | $\delta(\mathcal{B}S_B)$ | $\delta(\mathcal{B}S^*_B)$ |
|---|---|---|---|---|---|---|---|---|
| $1\times 3$ | 3 | 2 | 2 | 3 | 2 | 2 | 1 | 1 |
| $1\times 4$ | 8 | 3 | 3 | 6 | 4 | 4 | 3 | 2 |
| $1\times 5$ | 15 | 4∼5 | 4∼5 | 10 | 6∼8 | 5∼7 | 3 | 2 |
| $1\times 6$ | 30 | 5∼6 | 5∼6 | 11 | 12∼16 | 8∼9 | 4 | 2 |
| $1\times 7$ | 42 | 8∼11 | 7∼10 | 21 | 18∼20 | 11∼13 | 3 | 2 |
| $1\times 8$ | 72 | 10∼16 | 9∼14 | 30 | 24∼36 | 14∼18 | 3 | 2 |
| $1\times 9$ | 126 | 9∼25 | 9∼19 | 48 | 32∼54 | 18∼26 | 3 | 2 |
| $1\times 10$ | 200 | 14∼36 | 12∼24 | 67 | 40∼80 | 16∼33 | 4 | 2 |
| $1\times 11$ | 231 | 11∼58 | 11∼26 | 110 | 50∼86 | 24∼45 | 4 | 3 |
| $1\times 12$ | 516 | 12∼85 | 12∼51 | 159 | 60∼172 | 21∼62 | 4 | 3 |
| $3\times 3$ | 18 | 11 | 15 | 15 | 12∼14 | 12∼14 | 2 | 2 |
| $4\times 4$ | 930 048 | 15∼168 | 15∼? | 340 | ? | ? | ? | ? |
| $3\times 4$ | 1 284 | 11∼(30 34) | 11∼369 | 54 | 194∼506 | 35∼52 | 3 | 2 |
| $3\times 5$ | 101 444 | 14∼(118 129) | 14∼14 607 | 240 | ? | ? | ? | ? |

Fig. 4. Small binary toric boards.

**Theorem 3.1.** *The extreme rays of the solitaire cone, that is, the moves, are extreme rays of the binary solitaire cone.*

**Proof.** Given any extreme ray (move) $c$ of $S_{m\times n}$, let $\mathscr{F}$ be the intersection of all the facets of $\mathscr{B}S_{m\times n}$ containing $c$. We want to prove that any vector $r\in\mathscr{F}$ is a scalar multiple of $c$.

*Case* $m\leqslant 2$: First, take $m=1$. For $n=3,\dots,12$ Theorem 3.1 was checked by computer so we can assume that $n\geqslant 13$. All extreme rays of $S_{1\times n}$ being equivalent up to scrolling and reversing, we can assume that $c=[-1\ \ -1\ \ 1\ \ 0\ \ \dots\ \ 0]$. For $j=4,\dots,n$, consider the two inequalities defined by $f^1_{1j}r\leqslant 0$ and $f^0_{1j}r\leqslant 0$ as given below where the boxed value is the $j$th coordinate

$$f^1_{1j}=1,0,1,0,1,0,\dots,0,1,0,1,\boxed{1},1,0,1,0,\dots$$
$$f^0_{1j}=1,0,1,0,1,0,\dots,0,1,0,1,\boxed{0},1,0,1,0,\dots$$

Since the associated 1-graphs $G_{f^1_{1j}}$ and $G_{f^0_{1j}}$ are connected, Theorem 2.1 gives that those inequalities induce 2 facets $F^1_{1j}$ and $F^0_{1j}$ of $\mathscr{B}S_{1\times n}$. As clearly $f^1_{1j}c=f^0_{1j}c=0$, we have $F^1_{1j}\cap F^0_{1j}\subset\mathscr{F}$. Therefore, any vector $r\in\mathscr{F}$ satisfies $f^1_{1j}r=f^0_{1j}r=0$ for $j=4,\dots,n$. This implies $r_j=0$ for $j=4,\dots,n$. Moreover, the two inequalities defined by $f_{1,1}r\leqslant 0$ and $f_{1,2}r\leqslant 0$ as given below

$$f_{1,1}=0,1,1,0,1,0,1,\dots\qquad f_{1,2}=1,0,1,0,1,0,1,\dots$$

clearly induce 2 facets also belonging to $\mathscr{F}$. It implies $f_{1,1}r=f_{1,2}r=0$, that is, $r_2+r_3=r_1+r_3=0$. In other words $r=r_3\times c$, which completes the proof. Since the case $m=2$ is almost equivalent to the case $m=1$, we next consider case $m=3$.

*Case* $m=3$: For $n=3,4,5$ Theorem 3.1 was checked by computer so we can assume that $n\geqslant 6$. The two cases $c'_{1,1}=c'_{1,2}=-c'_{1,3}=-1$ and $c''_{1,1}=c''_{2,1}=-c''_{3,1}=-1$ being essentially the same, we can assume that $c=c'$. For $i=1,2,3$ and $j=4,\dots,n$, consider the inequalities defined by $f^1_{ij}r\leqslant 0$ ($f^0_{ij}r\leqslant 0$ resp.) as given below where the boxed value is the $ij$th coordinate. The coordinates of $f^0_{ij}$ differ from $f^1_{ij}$ only for the $ij$th coordinate which is set to 0.

$$f^1_{ij}=\begin{bmatrix}1&0&1&1&\dots&1&0&1&1&0&1&1&1&0&1&1&0&\dots\\0&1&1&0&\dots&0&1&1&0&1&\boxed{1}&1&0&1&1&0&1&\dots\\1&1&0&1&\dots&1&1&0&1&1&1&0&1&1&0&1&1&\dots\end{bmatrix}.$$

Similarly to the case $m\leqslant 2$, the associated 1-graphs $G_{f^1_{ij}}$ and $G_{f^0_{ij}}$ are connected and, up to a rotation along the axis $i=2$, we have $f^1_{ij}c'=f^0_{ij}c'=0$; that is, the induced facets satisfy $F^1_{ij}\cap F^0_{ij}\subset\mathscr{F}$. Therefore, any vector $r\in\mathscr{F}$ satisfies $f^1_{ij}r=f^0_{ij}r=0$ for $i=1,2,3$ and $j=4,\dots,n$. This implies $r_{ij}=0$ for $i=1,2,3$ and $j=4,\dots,n$. Slightly modified $f^1_{ij}$ and $f^0_{ij}$ for $i=2,3$ and $j=1,2,3$ give $r_{ij}=0$ for $i=2,3$ and $j=1,2,3$.

Moreover, the two inequalities defined by $f_{1,1}r \leqslant 0$ and $f_{1,2}r \leqslant 0$ as given below

$$f_{1,1}=\begin{bmatrix} 1\ 0\ 1\ 0\ 1 \dots 0\ 1\ 0 \dots \\ 0\ 0\ 0\ 0\ 0 \dots 0\ 0\ 0 \dots \\ 1\ 0\ 1\ 0\ 1 \dots 0\ 1\ 0 \dots \end{bmatrix} \quad f_{1,2}=\begin{bmatrix} 0\ 1\ 1\ 0\ 1 \dots 1\ 0\ 1 \dots \\ 0\ 0\ 0\ 0\ 0 \dots 0\ 0\ 0 \dots \\ 0\ 1\ 1\ 0\ 1 \dots 1\ 0\ 1 \dots \end{bmatrix}$$

implies $r_{1,1}=r_{1,2}=-r_{1,3}$, that is, $r=r_3\times c'$, which completes the proof.

*Case* $m\geqslant 4$: Theorem 3.1 was checked by computer for $n=4$ so we can assume that $n\geqslant 5$. We can take $c_{ij}=0$ except $c_{1,1}=c_{1,2}=-c_{1,3}=-1$. For $i=3,\dots,m-1$ or $j=4,\dots,n$, consider the inequalities defined by $f_{ij}^1 r\leqslant 0$ ($f_{ij}^0 r\leqslant 0$ resp.) as given below where the boxed value is the $ij$th coordinate. The coordinates of $f_{ij}^0$ differ from $f_{ij}^1$ only for the $ij$th coordinate which is set to 0.

$$f_{ij}^1=\begin{bmatrix} 1\ 0\ 1\ 0 \dots 0\ 1\ 0\ 1\ 0\ 1 & 1 & 1\ 0\ 1\ 0\ 0 \dots \\ 0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0\ 0\ 0 & 0 & 0\ 0\ 0\ 0\ 0 \dots \\ \vdots\ \vdots\ \vdots\ \vdots \dots \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots \dots \\ 1\ 0\ 1\ 0 \dots 0\ 1\ 0\ 1\ 0\ 1 & 1 & 1\ 0\ 1\ 0\ 0 \dots \\ 0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0\ 0\ 0 & 0 & 0\ 0\ 0\ 0\ 0 \dots \\ 1\ 0\ 1\ 0 \dots 0\ 1\ 0\ 1\ 0\ 1 & 1 & 1\ 0\ 1\ 0\ 0 \dots \\ 1\ 0\ 1\ 0 \dots 0\ 1\ 0\ 1\ 0\ 1 & \boxed{1} & 1\ 0\ 1\ 0\ 0 \dots \\ 1\ 0\ 1\ 0 \dots 0\ 1\ 0\ 1\ 0\ 1 & 1 & 1\ 0\ 1\ 0\ 0 \dots \\ 0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0\ 0\ 0 & 0 & 0\ 0\ 0\ 0\ 0 \dots \\ 1\ 0\ 1\ 0 \dots 0\ 1\ 0\ 1\ 0\ 1 & 1 & 1\ 0\ 1\ 0\ 0 \dots \\ \vdots\ \vdots\ \vdots\ \vdots \dots \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots & \vdots & \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots \dots \end{bmatrix}.$$

Clearly, we have $f_{ij}^1 c=f_{ij}^0 c=0$. By filling the gaps with the following $\{0,1\}$-valued matrices (or their transposes), the associated 1-graphs $G_{f_{ij}^1}$ and $G_{f_{ij}^0}$ are both connected. Therefore, any vector $r\in\mathscr{F}$ satisfies $r_{ij}=0$ for $i=3,\dots,m-1$ or $j=4,\dots,n$.

$$\begin{bmatrix} 1\ 0\ 1\ 0 \dots 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0 \dots \\ 0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \dots \\ 1\ 0\ 1\ 0 \dots 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0 \dots \\ 0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \dots \\ \vdots\ \vdots\ \vdots\ \vdots \dots \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots \dots \end{bmatrix} \begin{bmatrix} 1\ 0\ 1\ 0 \dots 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0 \dots \\ 1\ 0\ 1\ 0 \dots 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0 \dots \\ 0\ 0\ 0\ 0 \dots 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \dots \\ 1\ 0\ 1\ 0 \dots 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0 \dots \\ \vdots\ \vdots\ \vdots\ \vdots \dots \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots \dots \end{bmatrix}.$$

Slightly modified $f_{ij}^1$ and $f_{ij}^0$ for $i=2,m$ and $j=1,2,3$ give $r_{ij}=0$ for $i=2,m$ and $j=1,2,3$. For example, the following inequalities set $r_{2,2}$ to 0

$$f_{2,2}^1=\begin{bmatrix} 0 & 1 & 1\ 1\ 0\ 1\ 0\ 1 \dots \\ 1 & \boxed{1} & 1\ 0\ 1\ 0\ 1\ 0 \dots \\ 0 & 1 & 0\ 1\ 0\ 1\ 0\ 1 \dots \\ 1 & 0 & 1\ 0\ 1\ 0\ 1\ 0 \dots \\ 0 & 1 & 0\ 1\ 0\ 1\ 0\ 1 \dots \\ \vdots & \vdots & \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots \dots \end{bmatrix} \quad f_{2,2}^0=\begin{bmatrix} 0 & 1 & 1\ 1\ 0\ 1\ 0\ 1 \dots \\ 1 & \boxed{0} & 1\ 0\ 1\ 0\ 1\ 0 \dots \\ 0 & 1 & 0\ 1\ 0\ 1\ 0\ 1 \dots \\ 1 & 0 & 1\ 0\ 1\ 0\ 1\ 0 \dots \\ 0 & 1 & 0\ 1\ 0\ 1\ 0\ 1 \dots \\ \vdots & \vdots & \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots \dots \end{bmatrix}.$$

Finally, the following two inequalities set $r_{1,1} = r_{1,2} = -r_{1,3}$,

$$f_{1,1} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix} \qquad f_{1,2} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

that is, $r = r_3 \times c$, which completes the proof. □

**Corollary 3.2.** *The binary solitaire cone is full-dimensional.*

Out of the 930 048 extreme rays of $\mathscr{B}S_{4\times4}$, the 64 extreme rays of $S_{4\times4}$, that is, the moves, reached the highest incidence $I_r^{\max} = 168$ which is almost three times larger than the second highest incidence $I_r^{\text{submax}} = 57$. Similarly, out of the 101 444 extreme rays of $\mathscr{B}S_{3\times5}$, the 15 vertical moves of $S_{3\times5}$, reached the highest adjacency $A_r^{\max} = 14$ 607 while the average adjacency is $A_r^{\text{ave}} \simeq 33.16$. These computational results and other similarities with the metric cone—see [3]—lead us to the following conjectures:

**Conjecture 3.3**

1. *For $n \geqslant 3$ and $m \geqslant 3$, the moves form a dominating set in the skeleton of $\mathscr{B}S_{m\times n}$.*
2. *The incidence of the moves is maximal in the skeleton of $\mathscr{B}S_{m\times n}$.*
3. *For $m, n$ large enough, at least one ray $r$ of $\mathscr{B}S_{m\times n}$ is simple, (that is, $I_r = mn - 1$).*

Item (1) of Conjecture 3.3 holds for $\mathscr{B}S_{3\times4}$ and is false for $m \leqslant 2$. The smallest 1 by $n$ board for which the conjecture fails is the 1 by 10 board. Item (2) holds for all cones presented in Fig. 4 and is false if we replace the incidence by the adjacency as, for example, for $\mathscr{B}S_{3\times5}$. If true, item (3) would imply that the edge connectivity, the minimal incidence and the minimal adjacency of the skeleton of $\mathscr{B}S_{m\times n}$ are equal to $nm - 1$. This holds for $\mathscr{B}S_{3\times4}$, $\mathscr{B}S_{3\times5}$ and $\mathscr{B}S_{4\times4}$.

### *3.2. The trellis solitaire cone*

The $\{0,1\}$-valued facets of the solitaire cone have much less structure than the set of cut metrics. In fact, the cut metrics are related to products of vectors of length $n$. This motivates the next definition. Let $f$ and $g$ be $\{0,1\}$-valued vectors of length $m$ and $n$ respectively, and let $c_{ij} = f_i \cdot g_j$ for $i = 1, \dots, m$, $j = 1, \dots, n$. If $c \cdot x \leqslant 0$ defines a facet of $\mathscr{B}S_{m\times n}$, we call it a *trellis facet*. The *trellis solitaire cone* $\mathscr{T}S_B$ is generated by all of the *trellis facets* of the binary solitaire cone $\mathscr{B}S_B$. See Item (2) of Corollary 2.2 for an easy construction of trellis facets. For example, among the two facets of $\mathscr{B}S_{3\times5}$ given in Fig. 5, only the right one is a trellis facet.

**Conjecture 3.4.** *The binary trellis solitaire cone is full-dimensional.*

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 | 1 |

| | | | | |
|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 |

Fig. 5. A facet and a trellis facet of $\mathscr{B}S_{3\times 5}$.

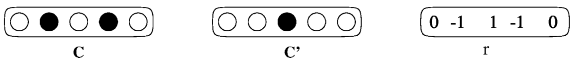


Fig. 6. The extreme ray $r$ of $\mathscr{C}S_{1\times 5}$ corresponding to the move from $c$ to $c'$.

| Board | $\#rays$ | $I_r$ | $A_r$ | $\#facets$ | $I_F$ | $A_F$ | $\delta(\mathcal{C}S_B)$ | $\delta(\mathcal{C}S_B^*)$ |
|---|---|---|---|---|---|---|---|---|
| 1 × 7 | 21 | 26∼33 | 16∼17 | 91 | 6∼12 | 6∼32 | 2 | 4 |
| 1 × 8 | 24 | 119∼130 | 19∼20 | 404 | 7∼12 | 7∼44 | 2 | 5 |
| 3 × 3 | 18 | 11 | 15 | 15 | 12∼14 | 12∼14 | 2 | 2 |
| 3 × 4 | 48 | 904∼1 192 | 39∼45 | 3 576 | 11∼33 | 11∼738 | 2 | ? |

Fig. 7. Small complete toric boards.

### 3.3. The complete solitaire cone

The *complete solitaire cone* $\mathscr{C}S_B$ is induced by a variation of the Solitaire game. To the classical moves we add the moves which consist of removing two pegs surrounding an empty hole and placing one peg in this empty hole as shown in Fig. 6. The incidence and adjacency relationships and diameters of small dimensional complete solitaire cones are presented in Fig. 7.

Two rays are called *strongly conflicting* in there exist two pairs $i,j$ and $k,l$ such that the two rays have nonzero coordinates of distinct signs at positions $i,j$ and $k,l$ (respectively $i,j$). We have $\mathscr{C}S_{3\times 3} = S_{3\times 3}$ and, by analogy with the classical solitaire cone case, we conjecture:

**Conjecture 3.5.** *For $n \geqslant 7$ and $m \geqslant 7$ a pair of extreme rays of $\mathscr{C}S_{m\times n}$ are adjacent if and only if they are not strongly conflicting.*

If true, Conjecture 3.5 would imply that $\delta(\mathscr{C}S_{m\times n}) = 2$.

### 3.4. The binary complete solitaire cone

In the same way as we did for the solitaire cone, we consider the cone generated by the facets of the complete solitaire cone $\mathscr{C}S_B$ whose coordinates are, up to a constant multiplier, $\{0,1\}$-valued. This cone is called *complete binary solitaire cone* and denoted as $\mathscr{B}\mathscr{C}S_B$. The incidence and adjacency relationships of small dimensional complete binary solitaire cones $\mathscr{B}\mathscr{C}S_B$ are presented in Fig. 8.

| Board | $\#rays$ | $I_r$ | $A_r$ | $\#facets$ | $I_F$ | $A_F$ | $\delta(\mathcal{BCS}_B)$ | $\delta(\mathcal{BCS}_B^*)$ |
|---|---|---|---|---|---|---|---|---|
| $1\times 7$ | 7 | 6 | 6 | 7 | 6 | 6 | 1 | 1 |
| $1\times 8$ | 16 | 9 | 10 | 12 | 12 | 9∼10 | 2 | 2 |
| $3\times 3$ | 18 | 11 | 15 | 15 | 12∼14 | 12∼14 | 2 | 2 |
| $3\times 4$ | 72 | 14∼28 | 15∼58 | 36 | 30∼45 | 30∼35 | 2 | 2 |

Fig. 8. Small binary complete toric boards.

## 4. Conclusion

Theorem 3.1 strengthens the analogy of the solitaire cone with the dual metric cone, for which the extreme rays are also extreme rays of the dual cut cone. On the other hand, so far we have not yet found an analogue of the *hypermetric* facets of the cut cone, that generalize the triangle inequalities. Another open question is the determination of a tighter relaxation of the solitaire cone $S_B$ by some *cuts analogue*. The trellis solitaire cone $\mathscr{T}S_B$ is a candidate as well as the cone generated by the $\{0,1\}$-valued facets with the minimal number of ones. For $S_{4\times 4}$ and $S_{3\times i}$: $i=3,4,5$, these facets have maximal incidence and adjacency in the skeleton of $S^*_{m\times n}$.

## Acknowledgements

Research supported by NSERC Canada and FCAR Québec for the first author and by an Inoue Foundation for Science Fellowship for the second one.

## References

[1] D. Avis, On the extreme rays of the metric cone, Can. J. Math. XXXII 1 (1980) 126–144.
[2] D. Avis, lrs, A revised implementation of the reverse search vertex enumeration algorithm, in: G. Kalai, G.M. Ziegler (eds.), Polytopes-Combinatorics and Computation, DMV-Seminars, 29 Birkhäuser, Berlin, 2000 pp. 177–198.
[3] D. Avis, A. Deza, On the solitaire cone and its relationship to multi-commodity flows, Mathematical Programming 90-1 (2001) 27–57.
[4] J.D. Beasley, The Ins and Outs of Peg Solitaire, Oxford University Press, Oxford, 1992.
[5] J.H. Conway, E.R. Berlekamp, R.K. Guy, Purging Pegs Properly, Winning Ways for your mathematical plays, Vol. 2, Academic Press, New York, 1982, pp. 697–734.
[6] A. Deza, M. Deza, K. Fukuda, On skeletons, diameters and volumes of metric polyhedra, Combinatorics and Computer Science, Lecture Notes in Computer Science, Vol. 1120, Springer, Berlin, 1996, pp. 112–128.
[7] A. Deza, S. Onn, Solitaire lattices, Graphs and Combinatorics, to appear.
[8] M. Deza, M. Laurent, in: Geometry of cuts and metrics, Algorithms and Combinatorics, Vol. 15, Springer, Berlin, 1997.
[9] K. Fukuda, cdd/cdd+ reference manual 0.75, ETHZ, Zürich, Switzerland, 1997.
[10] M. Lomonosov, Combinatorial Approaches to Multiflow Problems, Discrete Appl. Math. 11 (1985) 1–93.
[11] R. Uehara, S. Iwata, Generalized Hi-Q is NP-complete, Trans. IEICE E 73 (1990) 270–273.