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On inventory allocation for periodic review assemble-to-order systems

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ABSTRACT

As shown in Deza et al. (2018), for a periodic review Assemble-To-Order (ATO) system that aims to maximize reward, lowering the degree of component commonality may yield a higher Type-II service level. This is achieved via separating inventories of all the shared components for different products. We further study the optimal bill-of-materials (BOM) structure for two-product ATO systems with arbitrary number of components. The inventory of a common component can be dedicated or shared between different products. We show that an optimal BOM can be found between the following two extremal configurations: either two products share all common components, or they do not share any common component.

1. Introduction

Akçay and Xu [2] studied a periodic review assemble-to-order (ATO) system with an independent base stock policy and a first-come-first-served (FCFS) allocation rule. They formulated a two-stage stochastic integer nonlinear program where the base stock levels and the component allocation are optimized jointly. They showed that the component allocation problem is an NP-hard multidimensional knapsack problem and proposed an order-based component allocation heuristic rule that commits a component to an order only if it leads to the fulfillment of the order within the committed time window. They concluded that their order-based component allocation rule outperforms the component-based allocation rules, such as the fixed-priority and fair-shared rules, see [1,11]. Huang and de Kok [7] studied periodic-review ATO systems with linear holding and backlogging costs, installation stock policy, and a FCFS allocation rule. They introduced the concept of multimatching which refers to the coupling of multiple component units and product units. They showed that the FCFS allocation rule decouples the problem of optimal component allocation over time into deterministic period-by-period component allocation optimization problems. Huang [6] evaluated the impact of two non-FCFS allocation rules in a periodic review ATO system with component base stock policy; i.e., the last-come-first-served-within-one-period rule and the product-based-priority-within-time-windows rule. He proposed three benchmark mathematical programming models to test the non-FCFS allocation rules and concluded that both rules can not only outperform FCFS allocation rule in certain areas, but also better address the differences in customer service requirements. Doğru et al. [5] investigated a continuous review W system and concluded that the FCFS base stock policy is typically suboptimal. They also provided a lower bound for the optimal objective value and developed a policy attaining the lower bound under some symmetry condition for the

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cost parameters and a so-called *balanced capacity* condition for the solution. Jaarsveld and Scheller–Wolf [8] developed a heuristic algorithm for large scale continuous review ATO systems which improves as the average newsvendor fractiles increase. They showed that, for large scale ATO systems, the best FCFS rule is nearly optimal, and proposed a no-holdback allocation rule which can outperform the best FCFS rule. Deza et al. [4] studied the impact of component commonality on periodic review ATO systems. They showed that lowering component commonality may yield a higher type-II service level. The lower degree of component commonality is achieved via separating inventories of the same component for different products. They substantiated this property via computational and theoretical approaches. They showed that for low service levels the use of separate inventories of the same component for different products could achieve a higher reward than with shared inventory. Finally considering a simple ATO system consisting of one component shared by two products, they characterized the budget ranges such that the use of separate inventories is beneficial, as well as the budget ranges such that component commonality is beneficial. For more details and literature review, please refer to Deza et al. [4] and Liang [10].

A natural research question arising from [4] is how to allocate inventories in ATO systems optimally to achieve higher reward. In this paper, we study this problem for a periodic review ATO system with an independent base policy and a FCFS allocation rule. We analyze the formulation of Akçay and Xu [2] which jointly optimizes the base stock levels and the component allocation. In particular, we consider two-product stochastic models with arbitrary number of common components and show that either full component commonality or non-component commonality does not work worse than partial component commonality. Components with common function can be replaced by a single one; such universal component is called *common*. A common component is called *dedicated* if it is used to assemble only one product, and *shared* if it is shared by more than one product. A product-specific component that is irreplaceable is called *non-common*. In Section 2, we detail the formulations. The main results are presented in Section 3, the proofs are given in Section 4, and a few future directions are presented in Section 5.

2. The stochastic programming model

2.1. Akçay and Xu formulation

Following the model proposed by Akçay and Xu [2], we assume:

(1) a periodic review system,
(2) an independent base stock policy is used for each component,
(3) the product demands are satisfied by a FCFS rule,
(4) the product demands are correlated within each period, while the demands over different periods are independent,
(5) the replenishment lead time for each component is constant,
(6) a product reward is collected if the assembly is completed within the given time window.

In addition, the following sequence of events is assumed for each period: inventory position reviewed → new replenishment order of components placed → earlier component replenishment order arrive → demand realized → component allocated and product assembled → associated reward accounted for.

In this model, assembly takes zero time while component lead times are greater than zero. The model is based on a multi-matching approach proposed by Huang [6] and Huang and de Kok [7] where multiple components are matched with multiple products to satisfy demands. In each period within the time window, reward are collected by satisfying product demands. We recall that the time window is the number of periods between the order receiving period and the order fulfillment period. In particular, a time window equal to 0 means that the demand must be fulfilled within the period the order is received; that is, we must have enough components to satisfy the demand within that period in order to collect reward. The base stocks of the ATO system are constrained by a pre-set overall budget. The approach is based on a two-stage decision model. The first stage consists of determining a base stock level for each component, and the second stage consists of determining products that need to be assembled in each period with respect to some constraints reflecting the inventory availability. The first stage decisions are made before the second stage decisions following a two-stage stochastic programming framework, see Birge and Louveaux [3]. The objective of the approach is to maximize the expected total reward collected from the products assembled within given time windows. Note that while all products are eventually assembled within $L+1$ periods, the reward are collected only within the pre-set time windows. The notations are summarized in Table 1.

The second stage corresponds to the allocation problem $\big(Alloc(S,\xi)\big)$, where $S=(S_i)$ is the vector representing base stock levels, $\xi = \{P_{j,k} | j = 1,\ldots,m; k = 0,-1,\ldots,-L\}$ is the vector representing random demands, and $O_{i,k}$ is the number of component i available at period k. Note that $O_{i,k} = (S_i - D_i^{L_i-k})^+$ for $0 \le k \le L_i$ where $D_i^{L_i-k} = \sum_{s=0}^{L_i-k} D_{i,-s}$, and $O_{i,k} = D_{i,0}$ for $L_i+1 \le k \le L+1$ are inferred from the base stock policy and a FCFS rule, see Huang [6] and Huang and de Kok [7].

Table 1
Notations.

n	Number of components
m	Number of products
i, i'	Index of component
j	Index of product
S_i	Base stock level of component i
c_i	Unit base stock level cost of component i
L_i	Lead time of component i
L	Maximum lead time among all components; that is, $L = \max_i L_i$
w_j	Time window of product j
k	Index of period k corresponding to the duration $[k, k+1)$; $k = 0$ implies the current period; negative values of k imply previous periods
$x_{j,k}$	Number of product j assembled in period k
$r_{j,k}$	Reward for satisfying the demand for product j in period k
$a_{i,j}$	Number of component i used to assemble one unit of product j; that is, the bill-of-materials (BOM)
B	The budget, i.e., $\sum_i (c_i S_i) \leq B$
$P_{j,k}$	Demand of product j at period k
P_j	Demand of product j at the current period; that is, $P_{j,0}$
$D_{i,k}$	Demand of component i at period k; that is, $D_{i,k} = \sum_j \left(a_{i,j} P_{j,k}\right)$
M	Number of independent samples
N	Number of realizations in one sample
l	Index of sample $l = 1, \ldots, M$
h	Index of realization $h = 1, \ldots, N$
d	Number of dedicated components; $d = 0$, respectively $d = n$ implies a full commonality, respectively non-commonality, configuration
x^+	The positive part of x; that is, $x^+ = (\|x\|+x)/2$

$$\begin{aligned} \max \quad & \sum_{j=1}^{m} \sum_{k=0}^{w_j} (r_{j,k}\, x_{j,k}) & & \big(Alloc(S,\xi)\big) \\ & \sum_{k=0}^{w_j} x_{j,k} \leq P_j & & j = 1, \ldots, m \\ \sum_{\mu=0}^{k} \sum_{j=1}^{m} & (a_{i,j}\, x_{j,\mu}) \leq O_{i,k} & & i = 1, \ldots, n, \quad k = 0, \ldots, L+1 \\ & x_{j,k} \in \mathbb{Z}_+ & & j = 1, \ldots, m, \quad k = 0, \ldots, L+1 \end{aligned}$$

The first set of constraints guarantees that assembly will satisfy customer demand. Please note that $w_j \leq L+1$. Consequently, replacing the constraint $\sum_{k=0}^{w_j} x_{j,k} \leq P_j$ by $\sum_{k=0}^{L+1} x_{j,k} = P_j$ would yield the same optimal reward. The second set of constraints – called inventory availability constraints – guarantees that assembly could only happen when there are enough component inventories. While an optimal allocation can be computed for a given base stock level S and demand ξ, we still need to determine the optimal base stock levels. Thus, we use the two-stage stochastic integer program $\big(Joint(B)\big)$ where the first stage determines the base stock levels and the second stage maximizes the expectation of the component allocations:

$$\begin{aligned} \max \quad & \mathrm{E}[Alloc(S,\xi)] & & \big(Joint(B)\big) \\ & \sum_{i=1}^{n} (c_i\, S_i) \leq B & & \\ & S_i \in \mathbb{Z}_+ & & i = 1, \ldots, n \end{aligned}$$

We recall in Section 2.2 the sample average approximation method used to solve $\big(Joint(B)\big)$.

2.2. Sample average approximation method

The sample average approximation (SAA) method, see Kleywegt et al. [9], consists of the following steps:

(i) generate M independent samples for $l = 1, \ldots, M$ with N realizations for each sample. The vector $\xi_l^N = (\xi(\omega_l^1), \xi(\omega_l^2), \ldots, \xi(\omega_l^N))$ represents the N realizations of the lth sample,

(ii) solve the optimization problem ($INLP$) for each sample, which is the associated deterministic version of $\big(Joint(B)\big)$. where the objective function is set to $\frac{1}{N} \sum_{h=1}^{N} Alloc(S, \xi(\omega_l^h))$ as described below. Note that ($INLP$) is non-linear not only due to the integrality constraints but also due to the right hand side of the inventory availability constraints. Let $\hat{S}_l$ denote the optimal base stock levels for $(INLP)$ and $\hat{G}(\hat{S}_l)$ denote its optimal objective value.

$$\max \quad \frac{1}{N}\sum_{h=1}^{N}\sum_{j=1}^{m}\sum_{k=0}^{w_j}(r_{j,k}\, x_{j,k}^h) \qquad (INLP)$$
$$\sum_{k=0}^{w_j} x_{j,k}^h \le P_j^h \qquad j=1,\ldots,m, \quad h=1,\ldots,N$$
$$\sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j}\, x_{j,\mu}^h) \le O_{i,k}^h \qquad i=1,\ldots,n, \quad k=0,\ldots,L+1, \quad h=1,\ldots,N$$
$$\sum_{i=1}^{n}(c_i\, S_i) \le B$$
$$S_i \in \mathbb{Z}_+ \qquad i=1,\ldots,n$$
$$x_{j,k}^h \in \mathbb{Z}_+ \qquad j=1,\ldots,m, \quad k=0,\ldots,L+1, \quad h=1,\ldots,N$$

(iii) generate a different sample $\xi^{N'}$ with $N' \gg N$ realizations and compare the performance among all the base stock vectors $\hat{S}_l$ solved in (ii) by solving $\big(Alloc(S,\xi^{N'})\big)$ with $S=\hat{S}_l$. Let $\bar{G}(\hat{S}_l)$ be the new optimal objective value.

(iv) select the optimal base stock vector $\hat{S}^*$ achieving the best performance among all the base stock vectors; that is, $\hat{S}^* = \mathrm{argmax}\{\bar{G}(\hat{S}_l) : l=1,\ldots,M\}$.

Let $\hat{G}_M = \frac{1}{M}\sum_{l=1}^{M} \hat{G}(\hat{S}_l)$, $\bar{G}_{N'} = \bar{G}(\hat{S}^*)$, and G^* be the optimal objective value of $\big(Joint(B)\big)$. Since $\bar{G}_{N'} \le G^* \le \hat{G}_M$ under certain conditions for N, M, N', see Birge and Louveaux [3], $\bar{G}_{N'}$ and $\hat{G}_M$ are, respectively, a lower and an upper bound for G^*. For more details concerning the statistical testing of optimality for the SAA method, and the selection of N, M, and N', see Kleywegt et al. [9]. Note that $O_{i,k} = (S_i - D_i^{L_i-k})^+$ is a piecewise linear function; and we use the standard Big-M method to check whether $(S_i - D_i^{L_i-k})$ is positive.

3. Theoretical results for two-product ATO systems

A few additional notations are required in the remainder of the paper. Let $(BOM_\circ^N)$, $(BOM_\bullet^N)$ and $(BOM_{\color{green}\bullet}^N)$ denote, respectively, non-commonality, full commonality, and partial commonality configurations. Let $x_j^{\circ h}$, $x_j^{\bullet h}$ and $x_j^{{\color{green}\bullet} h}$ denote the number of product j assembled at realization h for, respectively, $(BOM_\circ^N)$, $(BOM_\bullet^N)$ and $(BOM_{\color{green}\bullet}^N)$. Let $S_{j\_i}^\circ$ and $S_{j\_i}^{\color{green}\bullet}$ denote, respectively, the base stock levels of dedicated component i for product j for $(BOM_\circ^N)$ and $(BOM_{\color{green}\bullet}^N)$. Let $S_{i'}^{\color{green}\bullet}$ and $S_{i'}^\bullet$ denote, respectively, the base stock levels of common component i' for $(BOM_{\color{green}\bullet}^N)$ and $(BOM_\bullet^N)$. Finally, let c_{j_i} denote the cost of component i for product j.

3.1. Two-product system with full overlap

In the full overlap configuration, product 1 and product 2 use exactly the same set of components. To simplify the analysis, all the product time windows are set to 0 and BOMs are set to 1. In other words, each unit product only contains one unit component, and the reward can be collected only if the assembly happens in the same period of the arrival of the demand.

3.1.1. Non-commonality configuration $(BOM_\circ^N)$

The non-commonality configuration consists of two products, each comprising n different components, as shown in Table 2 where C_i^j denotes dedicated component i used to assemble product j.

Table 2
BOM: non-commonality configuration with full overlap.

	C_1^1	C_1^2	C_2^1	C_2^2	...	C_n^1	C_n^2
P_1	1	0	1	0	...	1	0
P_2	0	1	0	1	...	0	1

The corresponding SAA formulation $(BOM_\circ^N)$ is as follows:

$$\max \quad \frac{1}{N}\sum_{h=1}^{N}(r_1\, x_1^{\circ h} + r_2\, x_2^{\circ h}) \qquad (BOM_\circ^N)$$
$$x_1^{\circ h} \le (S_{1\_i}^\circ - D_1^h)^+ \qquad i=1,\ldots,n, \quad h=1,\ldots,N$$
$$x_2^{\circ h} \le (S_{2\_i}^\circ - D_2^h)^+ \qquad i=1,\ldots,n, \quad h=1,\ldots,N$$
$$x_1^{\circ h} \le P_1^h, \quad x_2^{\circ h} \le P_2^h \qquad h=1,\ldots,N$$

$$\sum_{i=1}^{n}\left(c_{1\_i}\,S_{1\_i}^{\circ}+c_{2\_i}\,S_{2\_i}^{\circ}\right)\le B$$

$$x_1^{\circ h},x_2^{\circ h},S_{1\_i}^{\circ},S_{2\_i}^{\circ}\in\mathbb{Z}_+ \qquad i=1,\dots,n,\quad h=1,\dots,N$$

3.1.2. Full commonality configuration ($BOM_{\bullet}^{N}$)

In the full commonality configuration, components C_i^1 and C_i^2 in ($BOM_{\circ}^{N}$) are replaced by a common component C_i where $i=1,\dots,n$. Therefore there are n common components in total, see Table 3.

Table 3
BOM: full commonality configuration with full overlap.

	C_1	C_2	C_3	...	C_n
P_1	1	1	1	...	1
P_2	1	1	1	...	1

The corresponding SAA formulation ($BOM_{\bullet}^{N}$) is as follows:

$$\max\quad \frac{1}{N}\sum_{h=1}^{N}(r_1\,x_1^{\bullet h}+r_2\,x_2^{\bullet h}) \qquad \left(BOM_{\bullet}^{N}\right)$$

$$x_1^{\bullet h}+x_2^{\bullet h}\le(S_{i'}^{\bullet}-D_1^h-D_2^h)^+ \qquad i'=1,\dots,n,\quad h=1,\dots,N$$

$$x_1^{\bullet h}\le P_1^h,\quad x_2^{\bullet h}\le P_2^h \qquad h=1,\dots,N$$

$$\sum_{i'=1}^{n}c_{i'}\,S_{i'}^{\bullet}\le B$$

$$x_1^{\bullet h},x_2^{\bullet h},S_{i'}^{\bullet}\in\mathbb{Z}_+ \qquad i'=1,\dots,n,\quad h=1,\dots,N$$

3.1.3. Partial commonality configuration ($BOM_{{\color{green}\bullet}}^{N}$)

In a partial commonality configuration, let I be a nonempty and strict subset of $\{1,2,\dots,n\}$ such that components C_i^1 and C_i^2 in ($BOM_{\circ}^{N}$) are replaced by a common component C_i for $i\in I$. Without loss of generality, we can assume that $1\notin I$ and $n\in I$, see Table 4 where $d=n-|I|$ is the number of dedicated components.

Table 4
BOM: partial commonality configuration.

	C_1^1	C_1^2	...	C_d^1	C_d^2	C_{d+1}	C_{d+2}	...	C_{n-1}	C_n
P_1	1	0	...	1	0	1	1	...	1	1
P_2	0	1	...	0	1	1	1	...	1	1

The corresponding SAA formulation ($BOM_{{\color{green}\bullet}}^{N}$) is as follows:

$$\max\quad \frac{1}{N}\sum_{h=1}^{N}(r_1\,x_1^{{\color{green}\bullet} h}+r_2\,x_2^{{\color{green}\bullet} h}) \qquad \left(BOM_{{\color{green}\bullet}}^{N}\right)$$

$$x_1^{{\color{green}\bullet} h}\le(S_{1\_i}^{{\color{green}\bullet}}-D_1^h)^+ \qquad i=1,\dots,d,\quad h=1,\dots,N$$

$$x_2^{{\color{green}\bullet} h}\le(S_{2\_i}^{{\color{green}\bullet}}-D_2^h)^+ \qquad i=1,\dots,d,\quad h=1,\dots,N$$

$$x_1^{{\color{green}\bullet} h}+x_2^{{\color{green}\bullet} h}\le(S_{i'}^{{\color{green}\bullet}}-D_1^h-D_2^h)^+ \qquad i'=d+1,\dots,n,\quad h=1,\dots,N$$

$$x_1^{{\color{green}\bullet} h}\le P_1^h,\quad x_2^{{\color{green}\bullet} h}\le P_2^h \qquad h=1,\dots,N$$

$$\sum_{i=1}^{d}\left(c_{1\_i}\,S_{1\_i}^{{\color{green}\bullet}}+c_{2\_i}\,S_{2\_i}^{{\color{green}\bullet}}\right)+\sum_{i'=d+1}^{n}c_{i'}\,S_{i'}^{{\color{green}\bullet}}\le B$$

$$x_1^{{\color{green}\bullet} h},x_2^{{\color{green}\bullet} h},S_{1\_i}^{{\color{green}\bullet}},S_{2\_i}^{{\color{green}\bullet}},S_{i'}^{{\color{green}\bullet}}\in\mathbb{Z}_+ \qquad i=1,\dots,n,\quad i'=d+1,\dots,n,\quad h=1,\dots,N$$

3.2. Two-product system with partial overlap

In a partial overlap configuration, some components are used only for product 1 or product 2 by design, therefore these components are not allowed to be replaced by common components.

3.2.1. Non-commonality configuration ($BOM^N_\circ$)

The non-commonality configuration consists of two products, product 1 comprising n_1 different components and product 2 comprising n_2 different components, see Table 5.

Table 5
BOM: non-commonality configuration with partial overlap.

	C^1_{n+1}	...	$C^1_{n_1}$	C^1_1	C^2_1	...	C^1_n	C^2_n	C^2_{n+1}	...	$C^2_{n_2}$
P_1	1	...	1	1	0	...	1	0	0	...	0
P_2	0	...	0	0	1	...	0	1	1	...	1

Let $B^\circ_1 = \sum_{i_1=n+1}^{n_1} c_{1\_i_1} S^\circ_{1\_i_1}$, and $B^\circ_2 = \sum_{i_2=n+1}^{n_2} c_{2\_i_2} S^\circ_{2\_i_2}$. Then the corresponding SAA formulation ($BOM^N_\circ$) is as follows:

$$
\begin{aligned}
\max \quad & \frac{1}{N}\sum_{h=1}^{N}(r_1\, x_1^{\circ h} + r_2\, x_2^{\circ h}) && \left(BOM^N_\circ\right)\\
& x_1^{\circ h} \le (S^\circ_{1\_i_1} - D_1^h)^+ && i_1 = n+1,\dots,n_1, \quad h = 1,\dots,N\\
& x_2^{\circ h} \le (S^\circ_{2\_i_2} - D_2^h)^+ && i_2 = n+1,\dots,n_2, \quad h = 1,\dots,N\\
& x_1^{\circ h} \le (S^\circ_{1\_i} - D_1^h)^+ && i = 1,\dots,n, \quad h = 1,\dots,N\\
& x_2^{\circ h} \le (S^\circ_{2\_i} - D_2^h)^+ && i = 1,\dots,n, \quad h = 1,\dots,N\\
& x_1^{\circ h} \le P_1^h, \quad x_2^{\circ h} \le P_2^h && h = 1,\dots,N\\
& \sum_{i=1}^{n}\left(c_{1\_i}\, S^\circ_{1\_i} + c_{2\_i}\, S^\circ_{2\_i}\right) + B^\circ_1 + B^\circ_2 \le B\\
& x_1^{\circ h}, x_2^{\circ h}, S^\circ_{1\_i}, S^\circ_{2\_i}, \in \mathbb{Z}_+ && i = 1,\dots,n, \quad h = 1,\dots,N\\
& S^\circ_{1\_i_1}, S^\circ_{2\_i_2}, \in \mathbb{Z}_+ && i_1 = n+1,\dots,n_1, \quad i_2 = n+1,\dots,n_2
\end{aligned}
$$

3.2.2. Full commonality configuration ($BOM^N_\bullet$)

In the full commonality configuration, component C^1_i and C^2_i in ($BOM^N_\circ$) are replaced by a common component C_i where $i = 1,\dots,n$. Therefore there are n common components in total, see Table 6.

Table 6
BOM: full commonality configuration with partial overlap.

	C^1_{n+1}	...	$C^1_{n_1}$	C_1	C_2	C_3	...	C_n	C^2_{n+1}	...	$C^2_{n_2}$
P_1	1	...	1	1	1	1	...	1	0	...	0
P_2	0	...	0	1	1	1	...	1	1	...	1

Let $B^\bullet_1 = \sum_{i_1=n+1}^{n_1} c_{1\_i_1} S^\bullet_{1\_i_1}$, and $B^\bullet_2 = \sum_{i_2=n+1}^{n_2} c_{2\_i_2} S^\bullet_{2\_i_2}$. Then the corresponding SAA formulation ($BOM^N_\bullet$) is as follows:

$$
\begin{aligned}
\max \quad & \frac{1}{N}\sum_{h=1}^{N}(r_1\, x_1^{\bullet h} + r_2\, x_2^{\bullet h}) && \left(BOM^N_\bullet\right)\\
& x_1^{\bullet h} \le (S^\bullet_{1\_i_1} - D_1^h)^+ && i_1 = n+1,\dots,n_1, \quad h = 1,\dots,N\\
& x_2^{\bullet h} \le (S^\bullet_{2\_i_2} - D_2^h)^+ && i_2 = n+1,\dots,n_2, \quad h = 1,\dots,N\\
& x_1^{\bullet h} + x_2^{\bullet h} \le (S^\bullet_{i'} - D_1^h - D_2^h)^+ && i' = 1,\dots,n, \quad h = 1,\dots,N\\
& x_1^{\bullet h} \le P_1^h, \quad x_2^{\bullet h} \le P_2^h && h = 1,\dots,N\\
& \sum_{i'=1}^{n} c_{i'}\, S^\bullet_{i'} + B^\bullet_1 + B^\bullet_2 \le B\\
& x_1^{\bullet h}, x_2^{\bullet h}, S^\bullet_{i'} \in \mathbb{Z}_+ && i' = 1,\dots,n, \quad h = 1,\dots,N\\
& S^\bullet_{1\_i_1}, S^\bullet_{2\_i_2}, \in \mathbb{Z}_+ && i_1 = n+1,\dots,n_1, \quad i_2 = n+1,\dots,n_2
\end{aligned}
$$

3.2.3. Partial commonality configuration ($BOM^N_\bullet$)

In a partial commonality configuration, let I be a nonempty and strict subset of $\{1, 2, \dots, n\}$ such that components C^1_i and C^2_i in ($BOM^N_\circ$) are replaced by a common component C_i for $i \in I$. Without loss of generality, we can assume that $1 \notin I$ and $n \in I$, see Table 7 where $d = n - |I|$ is the number of dedicated components.

Table 7
BOM: partial commonality configuration with partial overlap.

	C^1_{n+1}	...	$C^1_{n_1}$	C^1_1	C^2_1	...	C^1_d	C^2_d	C_{d+1}	...	C_n	C^2_{n+1}	...	$C^2_{n_2}$
P_1	1	...	1	1	0	...	1	0	1	...	1	0	...	0
P_2	0	...	0	0	1	...	0	1	1	...	1	1	...	1

Let $B_1^{\color{green}\bullet} = \sum_{i_1=n+1}^{n_1} c_{1\_i_1} S^{\color{green}\bullet}_{1\_i_1}$, and $B_2^{\color{green}\bullet} = \sum_{i_2=n+1}^{n_2} c_{2\_i_2} S^{\color{green}\bullet}_{2\_i_2}$. Then the corresponding SAA formulation $(BOM^N_{\color{green}\bullet})$ is as follows:

$$
\begin{array}{rlr}
\max & \frac{1}{N}\sum_{h=1}^{N}(r_1 x_1^{{\color{green}\bullet}h} + r_2 x_2^{{\color{green}\bullet}h}) & (BOM^N_{\color{green}\bullet}) \\
& x_1^{{\color{green}\bullet}h} \le (S^{\color{green}\bullet}_{1\_i_1} - D^h_1)^+ & i_1 = n+1,\dots,n_1, \quad h = 1,\dots,N \\
& x_2^{{\color{green}\bullet}h} \le (S^{\color{green}\bullet}_{2\_i_2} - D^h_2)^+ & i_2 = n+1,\dots,n_2, \quad h = 1,\dots,N \\
& x_1^{{\color{green}\bullet}h} \le (S^{\color{green}\bullet}_{1\_i} - D^h_1)^+ & i = 1,\dots,d, \quad h = 1,\dots,N \\
& x_2^{{\color{green}\bullet}h} \le (S^{\color{green}\bullet}_{2\_i} - D^h_2)^+ & i = 1,\dots,d, \quad h = 1,\dots,N \\
& x_1^{{\color{green}\bullet}h} + x_2^{{\color{green}\bullet}h} \le (S^{\color{green}\bullet}_{i'} - D^h_1 - D^h_2)^+ & i' = d+1,\dots,n, \quad h = 1,\dots,N \\
& x_1^{{\color{green}\bullet}h} \le P^h_1, \quad x_2^{{\color{green}\bullet}h} \le P^h_2 & h = 1,\dots,N \\
& \sum_{i=1}^{d}\left(c_{1\_i} S^{\color{green}\bullet}_{1\_i} + c_{2\_i} S^{\color{green}\bullet}_{2\_i}\right) + \sum_{i'=d+1}^{n} c_{i'} S^{\color{green}\bullet}_{i'} + B^{\color{green}\bullet}_1 + B^{\color{green}\bullet}_2 \le B & \\
& x_1^{{\color{green}\bullet}h}, x_2^{{\color{green}\bullet}h} \in \mathbb{Z}_+ & h = 1,\dots,N \\
& S^{\color{green}\bullet}_{1\_i}, S^{\color{green}\bullet}_{2\_i}, S^{\color{green}\bullet}_{i'} \in \mathbb{Z}_+ & i = 1,\dots,n, \quad i' = d+1,\dots,n \\
& S^{\color{green}\bullet}_{1\_i_1}, S^{\color{green}\bullet}_{2\_i_2}, \in \mathbb{Z}_+ & i_1 = n+1,\dots,n_1, \quad i_2 = n+1,\dots,n_2
\end{array}
$$

3.3. *Main theorem and examples contrasting and comparing $(BOM^N_\circ)$ and $(BOM^N_\bullet)$*

Before stating Theorem 1 in Section 3.3.4, we provide some intuition via simple examples illustrating that a feasible allocation for partial commonality can be infeasible for full commonality or non-commonality, and that non-commonality can be beneficial over full commonality under some conditions.

3.3.1. *An allocation feasible for partial commonality but infeasible for full commonality*

Due to the plus sign in the $(BOM^N_{\color{green}\bullet})$ and $(BOM^N_\bullet)$ formulations, $x_1^{{\color{green}\bullet}h} \le (S^{\color{green}\bullet}_{1\_i} - D^h_1)^+$ and $x_2^{{\color{green}\bullet}h} \le (S^{\color{green}\bullet}_{2\_i} - D^h_2)^+$ do not always imply that $x_1^{\bullet h} + x_2^{\bullet h} \le (S^\bullet_{i'} - D^h_1 - D^h_2)^+$. Assume that $S^{\color{green}\bullet}_{i'} > S^{\color{green}\bullet}_{1\_i} + S^{\color{green}\bullet}_{2\_i}$ in the $(BOM^N_{\color{green}\bullet})$ formulation and consider the following example:
Partial commonality: Let $S^{\color{green}\bullet}_{1\_i} - D^h_1 > 0$, $S^{\color{green}\bullet}_{2\_i} - D^h_2 \le 0$ and $S^{\color{green}\bullet}_{i'} - D^h_1 - D^h_2 > 0$; then $x_1^{{\color{green}\bullet}h} > 0$ and $x_2^{{\color{green}\bullet}h} = 0$ forms a feasible allocation for partial commonality.
Full commonality: Let $S^\bullet_{i'} = S^{\color{green}\bullet}_{1\_i} + S^{\color{green}\bullet}_{2\_i} < S^{\color{green}\bullet}_{i'}$ and then it is possible to have $S^\bullet_{i'} - D^h_1 - D^h_2 \le 0$ for $i' = 1,\dots,d$. Therefore, $x_1^{\bullet h} = 0$ and $x_2^{\bullet h} = 0$ is the only feasible allocation for full commonality.

3.3.2. *Non-commonality can be beneficial over full commonality*

Consider an ATO system consisting of 2 components shared by 2 products, and assume that $B = 10$, $c_1 = c_2 = r_1 = r_2 = 1$, $N = 2$, $D^1_1 = 1$, $D^1_2 = 4$, $P^1_1 = P^1_2 = 1$, $D^2_1 = 2$, $D^2_2 = 3$, and $P^2_1 = P^2_2 = 1$.
Full commonality: For both realizations, 5 units C_1 and 5 units C_2 are used to fulfill previous orders and, at the current period, there is no component available for further assembly. Therefore, $x_1^{\bullet h} = x_2^{\bullet h} = 0$ and the optimal value is 0.
Non-commonality: Let $S^\circ_{1\_1} = S^\circ_{1\_2} = 2$ and $S^\circ_{2\_1} = S^\circ_{2\_2} = 3$. For the first realization, 1 unit C^1_1, 1 unit C^1_2 and all 3 units C^2_1 and 1 unit C^2_2 are used to fulfill previous orders. At the current period, there are 1 unit C^1_1 and 1 unit C^1_2 still available. Thus, $x_1^{\circ 1} = 1$. For the second realization, all components are used to fulfill previous orders. Thus, $x_1^{\circ 2} = 0 = x_2^{\circ 2} = 0$ and the objective value is 1.

3.3.3. *An allocation feasible for partial commonality but infeasible for non-commonality*

Assume that $S^{\color{green}\bullet}_{i'} < S^{\color{green}\bullet}_{1\_i} + S^{\color{green}\bullet}_{2\_i}$ in the $(BOM^N_{\color{green}\bullet})$ formulation and consider the following example:
Partial commonality: Let $S^{\color{green}\bullet}_{1\_i} - D^h_1 > 0$, $S^{\color{green}\bullet}_{2\_i} - D^h_2 > 0$ and $S^{\color{green}\bullet}_{i'} - D^h_1 - D^h_2 > 0$; then $x_1^{{\color{green}\bullet}h} > 0$ and $x_2^{{\color{green}\bullet}h} > 0$ forms a feasible allocation for partial commonality.
Non-commonality: Let $S^\circ_{1\_i} + S^\circ_{2\_i} = S^{\color{green}\bullet}_{i'} < S^{\color{green}\bullet}_{1\_i} + S^{\color{green}\bullet}_{2\_i}$ and then it is possible to have $S^\circ_{1\_i} - D^h_1 \le 0$ for $i = d+1,\dots,n$. Therefore $x_1^{\circ h} = 0$ and $x_2^{\circ h} > 0$ is a feasible allocation for full commonality.

All plus signs in the $(BOM_{\color{green}\bullet}^N)$ formulation can be removed for this example. Thus, any feasible allocation for partial commonality is feasible for full commonality; that is, full commonality performs at least as well as non-commonality for such instances.

3.3.4. Main theorem

The existence of partial commonality structure makes possible ATO systems more challenging and significantly increases the number of possible BOMs. Theorem 1 states that an optimal BOM can be found by assuming either the full commonality or the non-commonality configuration. Consequently, a search through possibly exponential number of BOMs can be avoided.

Theorem 1. *Given a budget B, let $x_1^{{\color{green}\bullet}h*}$ and $x_2^{{\color{green}\bullet}h*}$ denote the optimal solutions of $(BOM_{\color{green}\bullet}^N)$ for $h = 1, \ldots, N$. Then, $x_1^{{\color{green}\bullet}h*}$ and $x_2^{{\color{green}\bullet}h*}$ are feasible solutions in either $(BOM_\bullet^N)$ or $(BOM_\circ^N)$.*

4. Proof of Theorem 1

4.1. Two-product system with full overlap

Let $x_{j,h}$, $y_{j,h}$ and $z_{j,h}$ denote, respectively, a feasible solution for product j in realization h for $(BOM_\circ^N)$, $(BOM_{\color{green}\bullet}^N)$ and $(BOM_\bullet^N)$. In $(BOM_{\color{green}\bullet}^N)$, due to the symmetry of the structure, we can assume, at optimality, that the base stock levels of the dedicated components for product 1 are equally distributed; that is, $S_{1\_i_\alpha}^{\color{green}\bullet}{}^* = S_{1\_i_\beta}^{\color{green}\bullet}{}^*$, where $1 \le i_\alpha \le i_\beta \le d$. This is also true for the dedicated components for product 2 and shared components. The base stock levels are independent of the component indexes i and i', and therefore we use the following additional notations in Section 4. Let Y_j and Y denote, respectively, the base stock levels of any dedicated component for product j and any shared component. Recall that a superscripted $*$ indicates an optimal solution. Let Y_j^* denote an optimal base stock level of any dedicated component for product j; that is, $S_{1\_i}^{\color{green}\bullet}{}^* = Y_1^*$ and $S_{2\_i}^{\color{green}\bullet}{}^* = Y_2^*$ for all i. Finally, let Y^* denote an optimal base stock level of any shared component; that is, $S_{C\_i}^{\color{green}\bullet}{}^* = Y^*$ for all i'.

We have the following assumptions:

1. While proving $y_{1,h}^*$ and $y_{2,h}^*$ are feasible in $(BOM_\circ^N)$, let $S_{1\_i}^\circ = Y_1^*$ and $S_{2\_i}^\circ = Y_2^*$ when $i = 1, \ldots, d$; $S_{1\_i}^\circ + S_{2\_i}^\circ = Y^*$, $S_{1\_i_\alpha}^\circ = S_{1\_i_\beta}^\circ$ and $S_{2\_i_\alpha}^\circ = S_{2\_i_\beta}^\circ$ when $i, i_\alpha, i_\beta = d+1, \ldots, n$. To simplify the notation, let X_j and U_j denote, respectively, the base stock levels of dedicated components for product j for $(BOM_\circ^N)$ when $i = 1, \ldots, d$ and $i = d+1, \ldots, n$; that is, $X_j = Y_j^*$ and $U_1 + U_2 = Y^*$.
2. While proving $y_{1,h}^*$ and $y_{2,h}^*$ are feasible in $(BOM_\bullet^N)$, let $S_{i'}^\bullet = Y_1^* + Y_2^*$ when $i' = 1, \ldots, d$; and $S_{i'}^\bullet = Y^*$ when $i' = d+1, \ldots, n$. To simplify the notation, let Z and V denote, respectively, the base stock levels of shared components for $(BOM_\bullet^N)$ when $i' = 1, \ldots, d$ and $i' = d+1, \ldots, n$; that is, $Z = Y_1^* + Y_2^*$ and $V = Y^*$.
3. The cost of a shared component is equal to the cost of the dedicated component it replaces. In the full overlap configuration, all components are potential shared components; that is, $c_{1\_i} = c_{2\_i} = c_{i'}$ for all indexes i and i'.

4.1.1. Case $N = 1$

We first consider the case $N = 1$; that is, only one realization is used in the SAA method. The associated formulations $(BOM_\circ^1)$, $(BOM_\bullet^1)$ and $(BOM_{\color{green}\bullet}^1)$ correspond to a deterministic demand where P_1^1 and P_2^1 represent the demands in the current period for, respectively, product 1 and 2, and D_1^1 and D_2^1 represent the overall demands from all previous periods.

$$
\begin{aligned}
\max \quad & r_1\, x_{1,1} + r_2\, x_{2,1} && \big(BOM_\circ^1\big)\\
& x_{1,1} \le (X_1 - D_1^1)^+ \\
& x_{1,1} \le (U_1 - D_1^1)^+ \\
& x_{2,1} \le (X_2 - D_2^1)^+ \\
& x_{2,1} \le (U_2 - D_2^1)^+ \\
& x_{1,1} \le P_1^1, \quad x_{2,1} \le P_2^1 \\
& X_1 \sum_{i=1}^{d} c_{1\_i} + X_2 \sum_{i=1}^{d} c_{2\_i} + U_1 \sum_{i=d+1}^{n} c_{1\_i} + U_2 \sum_{i=d+1}^{n} c_{2\_i} \le B \\
& x_{1,1}, x_{2,1}, X_1, X_2, U_1, U_2 \in \mathbb{Z}_+
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & r_1\, z_{1,1} + r_2\, z_{2,1} && \big(BOM_\bullet^1\big)\\
& z_{1,1} + z_{2,1} \le (Z - D_1^1 - D_2^1)^+ \\
& z_{1,1} + z_{2,1} \le (V - D_1^1 - D_2^1)^+
\end{aligned}
$$

$$
\begin{aligned}
& z_{1,1} \le P_1^1, \quad z_{2,1} \le P_2^1 \\
& Z \sum_{i'=1}^{d} c_{i'} + V \sum_{i'=d+1}^{n} c_{i'} \le B \\
& z_{1,1}, z_{2,1}, Z, V \in \mathbb{Z}_+
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & r_1 y_{1,1} + r_2 y_{2,1} && (BOM_{\bullet}^1) \\
& y_{1,1} \le (Y_1 - D_1^1)^+ \\
& y_{2,1} \le (Y_2 - D_2^1)^+ \\
& y_{1,1} + y_{2,1} \le (Y - D_1^1 - D_2^1)^+ \\
& y_{1,1} \le P_1^1, \quad y_{2,1} \le P_2^1 \\
& Y_1 \sum_{i=1}^{d} c_{1\_i} + Y_2 \sum_{i=1}^{d} c_{2\_i} + Y \sum_{i'=d+1}^{n} c_{i'} \le B \\
& y_{1,1}, y_{2,1}, Y_1, Y_2, Y \in \mathbb{Z}_+
\end{aligned}
$$

First of all, we want to prove that with the constraint $Y_1^* \sum_{i=1}^{d} c_{1\_i} + Y_2^* \sum_{i=1}^{d} c_{2\_i} + Y^* \sum_{i'=d+1}^{n} c_{i'} \le B$, either the constraint $X_1 \sum_{i=1}^{d} c_{1\_i} + X_2 \sum_{i=1}^{d} c_{2\_i} + U_1 \sum_{i=d+1}^{n} c_{1\_i} + U_2 \sum_{i=d+1}^{n} c_{2\_i} \le B$ or the constraint $Z \sum_{i'=1}^{d} c_{i'} + V \sum_{i'=d+1}^{n} c_{i'} \le B$ holds. The former can be proved by substituting assumptions 1 and 3, while the latter can be proved by substituting assumptions 2 and 3.

Then, to show that $y_{1,1}^*$ and $y_{2,1}^*$ are feasible for either $(BOM_{\bullet}^1)$ or $(BOM_{\circ}^1)$, we consider the following three cases.

Case 1: Reward from both product 1 and 2 are 0, i.e. $y_{1,1}^* = 0$ and $y_{2,1}^* = 0$ and the point $y_{1,1}^* = 0$ and $y_{2,1}^* = 0$ is a feasible solution for either $(BOM_{\bullet}^1)$ or $(BOM_{\circ}^1)$.

Take $(BOM_{\bullet}^1)$ as an example:

- $y_{1,1}^* + y_{2,1}^* = 0 \le (Z - D_1^1 - D_2^1)^+$, this is always true by the definition of $+$.
- $y_{1,1}^* + y_{2,1}^* = 0 \le (V - D_1^1 - D_2^1)^+$, this is always true by the definition of $+$.
- $y_{1,1}^* = 0 \le P_1^1, \quad y_{2,1}^* = 0 \le P_2^1$, this is always true because P_1^1 and P_2^1 are both nonnegative.
- $y_{1,1}^*, y_{2,1}^* \in \mathbb{Z}_+$, this is always true because 0 is a nonnegative integer.

Note: If the optimal solution $y_{j,h}^*$ is zero, then the point $y_{j,h}^* = 0$ is feasible for either $(BOM_{\bullet}^N)$ or $(BOM_{\circ}^N)$.

Case 2: We get some reward from exactly one of the products.

Case 2.1: Getting reward only from product 1, i.e. $y_{1,1}^* > 0$, and $y_{2,1}^* = 0$. We want to show that the point $y_{1,1}^* > 0$, and $y_{2,1}^* = 0$ is a feasible solution for $BOM_{\circ}^1$.

$y_{2,1}^* = 0$ is a feasible solution of $(BOM_{\circ}^1)$. Since $y_{1,1}^*$ is an optimal solution of $(BOM_{\bullet}^1)$, the following inequalities are valid:

$$
\begin{aligned}
& y_{1,1}^* \le (Y_1^* - D_1^1)^+ \\
& y_{1,1}^* \le (Y^* - D_1^1 - D_2^1)^+
\end{aligned}
$$

To prove $y_{1,1}^*$ is feasible in $(BOM_{\circ}^1)$, we need to show that $y_{1,1}^* \le (X_1 - D_1^1)^+$ and $y_{1,1}^* \le (U_1 - D_1^1)^+$. Let $U_2 = 0$; that is, all the budget spent on the shared components is used to buy dedicated components for product 1.

$$
\begin{aligned}
& y_{1,1}^* \le (Y_1^* - D_1^1)^+ = (X_1 - D_1^1)^+ && \langle substitution \rangle \\
& y_{1,1}^* \le (Y^* - D_1^1 - D_2^1)^+ = (U_1 - D_1^1 - D_2^1)^+ \le (U_1 - D_1^1)^+ && \langle recall\ D_2^1 \ge 0 \rangle
\end{aligned}
$$

Case 2.2: Getting reward only from product 2, i.e. $y_{1,1}^* = 0$, and $y_{2,1}^* > 0$. We want to show that the point $y_{1,1}^* = 0$, and $y_{2,1}^* > 0$ is a feasible solution for $(BOM_{\circ}^1)$. The proof is the same as for Case 2.1 considering $U_1 = 0$.

Case 3: We get reward from both products 1 and 2, i.e. $y_{1,1}^* > 0$ and $y_{2,1}^* > 0$. We want to show that the point $y_{1,1}^* > 0$ and $y_{2,1}^* > 0$ is a feasible solution for $(BOM_{\bullet}^1)$.

Since $y_{1,1}^*$ and $y_{2,1}^*$ is an optimal solution of $(BOM_{\bullet}^1)$, the following inequalities hold:

$$
\begin{aligned}
& y_{1,1}^* \le (Y_1^* - D_1^1)^+ \\
& y_{2,1}^* \le (Y_2^* - D_1^2)^+ \\
& y_{1,1}^* + y_{2,1}^* \le (Y^* - D_1^1 - D_2^1)^+
\end{aligned}
$$

To prove $y^*_{1,1}$ and $y^*_{2,1}$ is feasible in $(BOM^1_\bullet)$, we need to show that $y^*_{1,1} + y^*_{2,1} \leq (Z - D^1_1 - D^1_2)^+$ and $y^*_{1,1} + y^*_{2,1} \leq (V - D^1_1 - D^1_2)^+$.

Since $y^*_{1,1} > 0$ and $y^*_{2,1} > 0$, all the plus signs can be removed.

$$\begin{array}{rrcl} & y^*_{1,1} \leq Y^*_1 - D^1_1 \quad \text{and} & & y^*_{2,1} \leq Y^*_2 - D^2_1 \\ \Longrightarrow & y^*_{1,1} + y^*_{2,1} & \leq & Y^*_1 + Y^*_2 - D^1_1 - D^1_2 \\ \Longrightarrow & & = & Z - D^1_1 - D^1_2, \end{array}$$

and

$$y^*_{1,1} + y^*_{2,1} \leq (Y^* - D^1_1 - D^1_2)^+ = (V - D^1_1 - D^1_2)^+. \qquad \langle \textit{substitution} \rangle$$

4.1.2. General case

We assume for N realizations, each with probability $1/N$. Without loss of generality, we omit this constant term in the objectives. In the associated formulations $(BOM^N_\circ)$, $(BOM^N_\bullet)$ and $(BOM^N_{\color{green}\bullet})$ below, superscripts are used to distinguish different realizations. For example, $x_{1,h}, x_{2,h}, D^h_1, D^h_2, P^h_1$, and P^h_2 refer to the h-th realization.

$$\begin{array}{rll} \max & \displaystyle\sum_{h=1}^{N} \big(r_1\, x_{1,h} + r_2\, x_{2,h}\big) & \big(BOM^N_\circ\big) \\ & x_{1,h} \leq (X_1 - D^h_1)^+ & h = 1, \ldots, N \\ & x_{1,h} \leq (U_1 - D^h_1)^+ & h = 1, \ldots, N \\ & x_{2,h} \leq (X_2 - D^h_2)^+ & h = 1, \ldots, N \\ & x_{2,h} \leq (U_2 - D^h_2)^+ & h = 1, \ldots, N \\ & x_{1,h} \leq P^h_1, \quad x_{2,h} \leq P^h_2 & h = 1, \ldots, N \\ & X_1 \displaystyle\sum_{i=1}^{d} c_{1\_i} + X_2 \sum_{i=1}^{d} c_{2\_i} + U_1 \sum_{i=d+1}^{n} c_{1\_i} + U_2 \sum_{i=d+1}^{n} c_{2\_i} \leq B & \\ & x_{1,h}, x_{2,h}, X_1, X_2, U_1, U_2 \in \mathbb{Z}_+ & h = 1, \ldots, N \end{array}$$

$$\begin{array}{rll} \max & \displaystyle\sum_{h=1}^{N} \big(r_1\, z_{1,h} + r_2\, z_{2,h}\big) & \big(BOM^N_\bullet\big) \\ & z_{1,h} + z_{2,h} \leq (Z - D^h_1 - D^h_2)^+ & h = 1, \ldots, N \\ & z_{1,h} + z_{2,h} \leq (V - D^h_1 - D^h_2)^+ & h = 1, \ldots, N \\ & z_{1,h} \leq P^h_1, \quad z_{2,h} \leq P^h_2 & h = 1, \ldots, N \\ & Z \displaystyle\sum_{i'=1}^{d} c_{i'} + V \sum_{i'=d+1}^{n} c_{i'} \leq B & \\ & z_{1,h}, z_{2,h}, Z, V \in \mathbb{Z}_+ & h = 1, \ldots, N \end{array}$$

$$\begin{array}{rll} \max & \displaystyle\sum_{h=1}^{N} \big(r_1\, y_{1,h} + r_2\, y_{2,h}\big) & \big(BOM^N_{\color{green}\bullet}\big) \\ & y_{1,h} \leq (Y_1 - D^h_1)^+ & h = 1, \ldots, N \\ & y_{2,h} \leq (Y_2 - D^h_2)^+ & h = 1, \ldots, N \\ & y_{1,h} + y_{2,h} \leq (Y - D^h_1 - D^h_2)^+ & h = 1, \ldots, N \\ & y_{1,h} \leq P^h_1, \quad y_{2,h} \leq P^h_2 & h = 1, \ldots, N \\ & Y_1 \displaystyle\sum_{i=1}^{d} c_{1\_i} + Y_2 \sum_{i=1}^{d} c_{2\_i} + Y \sum_{i'=d+1}^{n} c_{i'} \leq B & \\ & y_{1,h}, y_{2,h}, Y_1, Y_2, Y \in \mathbb{Z}_+ & h = 1, \ldots, N \end{array}$$

For any realization, the optimal assembly decision will fall into one of the four, mutually exclusive, outcomes: $y^*_{1,h} > 0$ and $y^*_{2,h} > 0$; $y^*_{1,h} > 0$ and $y^*_{2,h} = 0$; $y^*_{1,h} = 0$ and $y^*_{2,h} > 0$; and $y^*_{1,h} = 0$ and $y^*_{2,h} = 0$.

Consequently the set of all realizations can be partitioned into four non-overlapping subsets: the subset T^{++} of realizations in which $y^*_{1,h} > 0$ and $y^*_{2,h} > 0$, the subset T^{+0} of realizations in which $y^*_{1,h} > 0$ and $y^*_{2,h} = 0$, the subset T^{0+} of realizations in which $y^*_{1,h} = 0$ and $y^*_{2,h} > 0$, and the subset T^{00} of realizations in which $y^*_{1,h} = 0$ and $y^*_{2,h} = 0$.

According to the definitions of Y^*_1, Y^*_2 and Y^*, the following inequalities are valid. Note that the right hand side of constraints (E_1) to (E_7) are positive, therefore all plus signs can be removed.

$$
\begin{aligned}
y^*_{1,h} &\le (Y^*_1 - D^h_1)^+ && h \in T^{++} && (E_1)\\
y^*_{2,h} &\le (Y^*_2 - D^h_2)^+ && h \in T^{++} && (E_2)\\
y^*_{1,h} + y^*_{2,h} &\le (Y^* - D^h_1 - D^h_2)^+ && h \in T^{++} && (E_3)\\
y^*_{1,h} &\le (Y^*_1 - D^h_1)^+ && h \in T^{+0} && (E_4)\\
y^*_{1,h} &\le (Y^* - D^h_1 - D^h_2)^+ && h \in T^{+0} && (E_5)\\
y^*_{2,h} &\le (Y^*_2 - D^h_2)^+ && h \in T^{0+} && (E_6)\\
y^*_{2,h} &\le (Y^* - D^h_1 - D^h_2)^+ && h \in T^{0+} && (E_7)
\end{aligned}
$$

The T^{00} cases being trivial, we just need to prove that Theorem 1 holds for realizations in $T^{++} \bigcup T^{+0} \bigcup T^{0+}$.

To obtain an optimal solution, we must satisfy:

$$Y^*_1 = \max_{(g,p)\in(T^{++}\times T^{+0})} \left\{D^g_1 + y^*_{1,g}, D^p_1 + y^*_{1,p}\right\},$$

$$Y^*_2 = \max_{(g,p)\in(T^{++}\times T^{0+})} \left\{D^g_2 + y^*_{2,g}, D^p_2 + y^*_{2,p}\right\},$$

$$Y^* = \max_{(g,p,q)\in(T^{++}\times T^{+0}\times T^{0+})} \left\{D^g_1 + D^g_2 + y^*_{1,g} + y^*_{2,g}, D^p_1 + D^p_2 + y^*_{1,p}, D^q_1 + D^q_2 + y^*_{2,q}\right\}.$$

Clearly, either $Y^* \ge Y^*_1 + Y^*_2$ or $Y^* < Y^*_1 + Y^*_2$.

Case 1: If $Y^* \ge Y^*_1 + Y^*_2$, then the point $y^*_{1,h}$ and $y^*_{2,h}$ is feasible in $\left(BOM^N_\circ\right)$. We need to show that

$$
\begin{aligned}
&y^*_{1,h} \le (X_1 - D^h_1)^+ && h \in T^{++} && (F_1)\\
&y^*_{2,h} \le (X_2 - D^h_2)^+ && h \in T^{++} && (F_2)\\
&y^*_{1,h} \le (U_1 - D^h_1)^+ && h \in T^{++} && (F_3)\\
&y^*_{2,h} \le (U_2 - D^h_2)^+ && h \in T^{++} && (F_4)\\
&y^*_{1,h} \le (X_1 - D^h_1)^+ && h \in T^{+0} && (F_5)\\
&y^*_{1,h} \le (U_1 - D^h_1)^+ && h \in T^{+0} && (F_6)\\
&y^*_{2,h} \le (X_2 - D^h_2)^+ && h \in T^{0+} && (F_7)\\
&y^*_{2,h} \le (U_2 - D^h_2)^+ && h \in T^{0+} && (F_8)
\end{aligned}
$$

One can check that $(E_1) \Rightarrow (F_1)$, $(E_2) \Rightarrow (F_2)$, $(E_4) \Rightarrow (F_5)$, and $(E_6) \Rightarrow (F_7)$.

Let $U_2 = Y^*_2$, for (F_3):

$$
\begin{aligned}
&U_1 = Y^* - U_2 \ge Y^*_1 + Y^*_2 - U_2 = Y^*_1,\\
&\text{thus} \quad U_1 \ge Y^*_1 = \max_{(g,p)\in(T^{++}\times T^{+0})} \left\{D^g_1 + y^*_{1,g}, D^p_1 + y^*_{1,p}\right\} \ge D^h_1 + y^*_{1,h}, \quad h \in T^{++}.
\end{aligned}
$$

Therefore $\quad y^*_{1,h} \le (U_1 - D^h_1)^+, \quad h \in T^{++}.$

For (F_4):

$$U_2 = Y^*_2 = \max_{(g,p)\in(T^{++}\times T^{0+})} \left\{D^g_2 + y^*_{2,g}, D^p_2 + y^*_{2,p}\right\} \ge D^h_2 + y^*_{2,h}, \quad h \in T^{++}$$

Therefore $\quad y^*_{2,h} \le (U_2 - D^h_2)^+, \quad h \in T^{++}.$

For (F_6):

$$
\begin{aligned}
U_1 - D^h_1 \ge Y^*_1 - D^h_1 &= \max_{(g,p)\in(T^{++}\times T^{+0})} \left\{D^g_1 + y^*_{1,g}, D^p_1 + y^*_{1,p}\right\} - D^h_1\\
&\ge D^h_1 + y^*_{1,h} - D^h_1 = y^*_{1,h}, \quad h \in T^{+0}
\end{aligned}
$$

Therefore $\quad y^*_{1,h} \le (U_1 - D^h_1)^+, \quad h \in T^{+0}.$

For (F_8):

$$U_2 - D^h_2 = Y^*_2 - D^h_2 = \max_{(g,p)\in(T^{++}\times T^{0+})} \left\{D^g_2 + y^*_{2,g}, D^p_2 + y^*_{2,p}\right\} - D^h_2$$

$$\geq D_2^h + y_{2,h}^* - D_2^h = y_{2,h}^*, \quad h \in T^{0+}$$

Therefore $\quad y_{2,h}^* \leq (U_2 - D_2^h)^+, \quad h \in T^{0+}.$

Case 2: If $Y^* < Y_1^* + Y_2^*$, then the point $y_{1,h}^*$ and $y_{2,h}^*$ is feasible in $\left(BOM_{\bullet}^N\right)$. We need to show that

$$
\begin{aligned}
y_{1,h}^* + y_{2,h}^* &\leq (Z - D_1^h - D_2^h)^+ & h &\in T^{++} & (G_1)\\
y_{1,h}^* + y_{2,h}^* &\leq (V - D_1^h - D_2^h)^+ & h &\in T^{++} & (G_2)\\
y_{1,h}^* &\leq (Z - D_1^h - D_2^h)^+ & h &\in T^{+0} & (G_3)\\
y_{1,h}^* &\leq (V - D_1^h - D_2^h)^+ & h &\in T^{+0} & (G_4)\\
y_{2,h}^* &\leq (Z - D_1^h - D_2^h)^+ & h &\in T^{0+} & (G_5)\\
y_{2,h}^* &\leq (V - D_1^h - D_2^h)^+ & h &\in T^{0+} & (G_6)
\end{aligned}
$$

One can check that $(E_3) \Rightarrow (G_2)$, $(E_5) \Rightarrow (G_4)$, and $(E_7) \Rightarrow (G_6)$.
(E_1) and $(E_2) \Rightarrow (G_1)$: Since $y_{1,h}^* > 0$ and $y_{2,h}^* > 0$, where $h \in T^{++}$, all the plus signs can be removed.

$$
\begin{aligned}
& 0 < y_{1,h}^* \leq Y_1^* - D_1^h \quad \text{and} \quad 0 < y_{2,h}^* \leq Y_2^* - D_2^h\\
\Longrightarrow \quad & 0 < y_{1,h}^* + y_{2,h}^* \leq Y_1^* + Y_2^* - D_1^h - D_2^h\\
\Longrightarrow \quad & \qquad\qquad\qquad = Z - D_1^h - D_2^h, \quad h \in T^{++}
\end{aligned}
$$

Thus, $y_{1,h}^* + y_{2,h}^* \leq (Z - D_1^h - D_2^h)^+, \quad h \in T^{++}.$
For (G_3):

$$
\begin{aligned}
Z &= Y_1^* + Y_2^* > Y^*\\
&= \max_{(g,p,q)\in(T^{++}\times T^{+0}\times T^{0+})} \left\{D_1^g + D_2^g + y_{1,g}^* + y_{2,g}^*, D_1^p + D_2^p + y_{1,p}^*, D_1^q + D_2^q + y_{2,q}^*\right\}\\
&\geq D_1^h + D_2^h + y_{1,h}^*, \quad h \in T^{+0}.
\end{aligned}
$$

Therefore $\quad y_{1,h}^* < (Z - D_1^h - D_2^h)^+ \leq (Z - D_1^h - D_2^h)^+, \quad h \in T^{+0}.$
For (G_5):

$$
\begin{aligned}
Z &= Y_1^* + Y_2^* > Y^*\\
&= \max_{(g,p,q)\in(T^{++}\times T^{+0}\times T^{0+})} \left\{D_1^g + D_2^g + y_{1,g}^* + y_{2,g}^*, D_1^p + D_2^p + y_{1,p}^*, D_1^q + D_2^q + y_{2,q}^*\right\}\\
&\geq D_1^h + D_2^h + y_{2,h}^*, \quad h \in T^{0+}.
\end{aligned}
$$

Therefore $\quad y_{2,h}^* < (Z - D_1^h - D_2^h)^+ \leq (Z - D_1^h - D_2^h)^+, \quad h \in T^{0+}.$

4.2. Two-product system with partial overlap

Given that $x_1^{{\color{green}\bullet}h*} \leq (S_{1\_i_1}^{\color{green}\bullet} - D_1^h)^+$ and $x_2^{{\color{green}\bullet}h*} \leq (S_{2\_i_2}^{\color{green}\bullet} - D_2^h)^+$, where $i_1 = n+1, \ldots, n_1$, $i_2 = n+1, \ldots, n_2$, $h = 1, \ldots, N$, we want to prove that either the constrains $x_1^{{\color{green}\bullet}h*} \leq (S_{1\_i_1}^{\circ} - D_1^h)^+$ and $x_2^{{\color{green}\bullet}h*} \leq (S_{2\_i_2}^{\circ} - D_2^h)^+$, or the constraints $x_1^{{\color{green}\bullet}h*} \leq (S_{1\_i_1}^{\bullet} - D_1^h)^+$ and $x_2^{{\color{green}\bullet}h*} \leq (S_{2\_i_2}^{\bullet} - D_2^h)^+$ hold. Obviously, if we set $S_{1\_i_1}^{\color{green}\bullet} = S_{1\_i_1}^{\circ} = S_{1\_i_1}^{\bullet}$ and $S_{2\_i_2}^{\color{green}\bullet} = S_{2\_i_2}^{\circ} = S_{2\_i_2}^{\bullet}$, then the optimal solutions of $\left(BOM_{\color{green}\bullet}^N\right)$, i.e., $x_1^{{\color{green}\bullet}h*}$ and $x_2^{{\color{green}\bullet}h*}$, trivially satisfy these constraints in both $\left(BOM_{\circ}^N\right)$ and $\left(BOM_{\bullet}^N\right)$. Excluding the above constraints, the remaining part is exactly the same as the full overlap configuration, whose result is already proved.

5. Conclusion and future work

We show that for two-product periodic ATO systems either full component commonality or non-component commonality performs at least as well as any partial component commonality formulation. Consequently, the size of the optimal BOM search space is cut down from an exponential in n to just 2. A possible future direction is to extend this result to multi-product periodic-review ATO systems. While deriving the same theoretical results may be challenging, one may consider a computational approach. Another future direction could be to apply component commonality considering inventory allocation and component design jointly.

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References

[1] Narendra Agrawal, Morris A. Cohen, Optimal material control in an assembly system with component commonality, Nav. Res. Logist. 48 (5) (2001) 409–429.
[2] Yalçin Akçay, Susan H. Xu, Joint inventory replenishment and component allocation optimization in an assemble-to-order system, Manage. Sci. 50 (1) (2004) 99–116.
[3] John R. Birge, François Louveaux, Introduction to Stochastic Programming, Springer, 2011.
[4] Antoine Deza, Kai Huang, Hongfeng Liang, Xiao Jiao Wang, On component commonality for periodic review assemble-to-order systems, Ann. Oper. Res. 265 (1) (2018) 29–46.
[5] Mustafa K. Doğru, Martin I. Reiman, Qiong Wang, A stochastic programming based inventory policy for assemble-to-order systems with application to the W model, Oper. Res. 58 (4-part-1) (2010) 849–864.
[6] Kai Huang, Benchmarking non-first-come-first-served component allocation in an assemble-to-order system, Ann. Oper. Res. 223 (1) (2014) 217–237.
[7] Kai Huang, Ton de Kok, Optimal FCFS allocation rules for periodic-review assemble-to-order systems, Nav. Res. Logist. 62 (2) (2015) 158–169.
[8] Willem van Jaarsveld, Alan Scheller-Wolf, Optimization of industrial-scale assemble-to-order systems, INFORMS J. Comput. 27 (3) (2015) 544–560.
[9] Anton J. Kleywegt, Alexander Shapiro, Tito Homem-de Mello, The sample average approximation method for stochastic discrete optimization, SIAM J. Optim. 12 (2) (2002) 479–502.
[10] Hongfeng Liang, Novel stochastic programming formulations for assemble-to-order systems, (Ph.D. thesis), McMaster University, 2017.
[11] Alex X. Zhang, Demand fulfillment rates in an assemble-to-order system with multiple products and dependent demands, Prod. Oper. Manage. 6 (3) (1997) 309–324.