Small Primitive Zonotopes

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Abstract We study a family of lattice polytopes, called *primitive zonotopes*, describe instances with small parameters, and discuss connections to the largest diameter of lattice polytopes and to the computational complexity of multicriteria matroid optimization. Complexity results and open questions are also presented.

Keywords Lattice polytopes · Primitive integer vectors · Matroid optimization · Diameter

1 Introduction

Recent results dealing with the combinatorial, geometric, and algorithmic aspects of linear optimization include Santos' counterexample [27] to the Hirsch conjecture, and Allamigeon, Benchimol, Gaubert, and Joswig's counterexample [2] to a continuous analogue of the polynomial Hirsch conjecture. Borgwardt, De Loera, and Finhold [4] showed that the Hirsch bound holds for transportation polytopes. Kalai and Kleitman's upper bound [18] for the diameter of polytopes was strengthened by Todd [32] and by Sukegawa [30].

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M. D. E. Conder et al. (eds.), *Discrete Geometry and Symmetry*,
Springer Proceedings in Mathematics & Statistics 234,
https://doi.org/10.1007/978-3-319-78434-2_5

Focusing on lattice polytopes; that is bounded polytopes whose vertices are integer-valued, Del Pia and Michini [7] strengthened Kleinschmidt and Onn's upper bound [19] for the diameter of lattice polytopes. Multicriteria matroid optimization is a generalization of standard linear matroid optimization introduced by Onn and Rothblum [26] where each basis is evaluated according to several, rather than one, criteria, and these values are traded-in by a convex function.

The article pursues the study of the *primitive zonotopes* initiated in [10] and is organized as follows. After recalling their definition and providing some of their combinatorial properties, we highlight in Sect. 2 connections to convex multicriteria matroid optimization, and to the diameter of lattice polytopes. In particular, we strengthen the bounds on the maximum number $m(d, 1)$ of greedily solvable linear single criterion counterparts needed to solve any d-criteria 1-bounded instance. Section 3 focuses on primitive zonotopes of small dimension d, norm q, and order p. The diameter, grid embedding size, and number of vertices are given for values of (d, q, p) yielding computationally tractable primitive zonotopes. Complexity results and open questions are discussed in Sect. 4. In particular, we show that linear optimization and separation over primitive zonotopes can be done in polynomial time, as well as deciding whether a given point, respectively a pair of points, is a vertex, respectively an edge. Proofs for Sects. 2.2 and 3 are given in Sect. 5.

2 Primitive Zonotopes

2.1 Zonotopes Generated by Short Primitive Vectors

The convex hull of integer-valued points is called a *lattice polytope* and, if all the vertices are drawn from $\{0, 1, \ldots, k\}^d$, is refereed to as a *lattice* (d, k)*-polytope*. For simplicity, we only consider full dimensional lattice (d, k)-polytopes. Given a finite set G of vectors, also called the generators, the *zonotope* generated by G is the convex hull of all signed sums of the elements of G. We consider zonotopes generated by short integer vectors in order to keep the grid embedding size relatively small. In addition, we restrict to integer vectors which are pairwise linearly independent in order to maximize the diameter. Thus, for $q = \infty$ or a positive integer, and d, p positive integers, we consider the *primitive zonotope* $Z_q(d, p)$ defined as the zonotope generated by the primitive integer vectors of q-norm at most p:

$$
\begin{aligned}
Z_q(d, p) &= \sum [-1, 1]\{v \in \mathbb{Z}^d \ : \ \|v\|_q \leq p\, ,\ \gcd(v) = 1\, ,\ v \succ 0\} \\
&= \operatorname{conv}\left(\sum \{\lambda_v v \ : \ v \in \mathbb{Z}^d\, , \|v\|_q \leq p\, ,\ \gcd(v) = 1\, ,\ v \succ 0\} \ : \ \lambda_v = \pm 1\right)
\end{aligned}
$$

where $\gcd(v)$ is the largest integer dividing all entries of v, and $\succ$ the lexicographic order on $\mathbb{R}^d$, i.e. $v \succ 0$ if the first nonzero coordinate of v is positive. Similarly, we consider the Minkowski sum $H_q(d, p)$ of the generators of $Z_q(d, p)$:

$$H_q(d, p) = \sum [0, 1]\{v \in \mathbb{Z}^d \ : \ \|v\|_q \leq p\ , \ \gcd(v) = 1\ , \ v \succ 0\}.$$

In other words, $H_q(d, p)$ is, up to translation, the image of $Z_q(d, p)$ by a homothety of factor $1/2$. We also consider the *positive primitive zonotope* $Z_q^+(d, p)$ defined as the zonotope generated by the primitive integer vectors of q-norm at most p with nonnegative coordinates:

$$Z_q^+(d, p) = \sum [-1, 1]\{v \in \mathbb{Z}_+^d \ : \ \|v\|_q \leq p\ , \ \gcd(v) = 1\}$$

where $\mathbb{Z}_+ = \{0, 1, \dots\}$. Similarly, we consider the Minkowski sum of the generators of $Z_q^+(d, p)$:

$$H_q^+(d, p) = \sum [0, 1]\{v \in \mathbb{Z}_+^d \ : \ \|v\|_q \leq p\ , \ \gcd(v) = 1\}.$$

We illustrate the primitive zonotopes with a few examples:

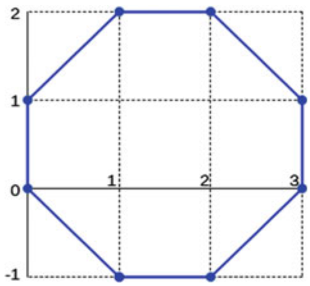
(i) For finite q, $Z_q(d, 1)$ is generated by the d unit vectors and forms the $\{-1, 1\}^d$-cube. $H_q(d, 1)$ is the $\{0, 1\}^d$-cube.

(ii) $Z_1(d, 2)$ is the permutahedron of type B_d and thus, $H_1(d, 2)$ is, up to translation, a lattice $(d, 2d-1)$-polytope with $2^d d!$ vertices and diameter d^2. For example, $Z_1(2, 2)$ is generated by $\{(0, 1), (1, 0), (1, 1), (1, -1)\}$ and forms the octagon whose vertices are $\{(-3, -1), (-3, 1), (-1, 3), (1, 3), (3, 1), (3, -1), (1, -3), (-1, -3)\}$. $H_1(2, 2)$ is, up to translation, a lattice $(2, 3)$-polygon, see Fig. 1. $Z_1(3, 2)$ is congruent to the truncated cuboctahedron, see Fig. 2 for an illustration, which is also called the great rhombicuboctahedron and is the Minkowski sum of an octahedron and a cuboctahedron, see for instance Eppstein [12]. $H_1(3, 2)$ is, up to translation, a lattice $(3, 5)$-polytope with diameter 9 and 48 vertices.

(iii) $H_1^+(d, 2)$ is the Minkowski sum of the permutahedron with the $\{0, 1\}^d$-cube. Thus, $H_1^+(d, 2)$ is a lattice (d, d)-polytope with diameter $\binom{d+1}{2}$.

(iv) $Z_\infty(3, 1)$ is congruent to the truncated small rhombicuboctahedron, see Fig. 3 for an illustration, which is the Minkowski sum of a cube, a truncated octahedron,

Fig. 1 $H_1(2, 2)$



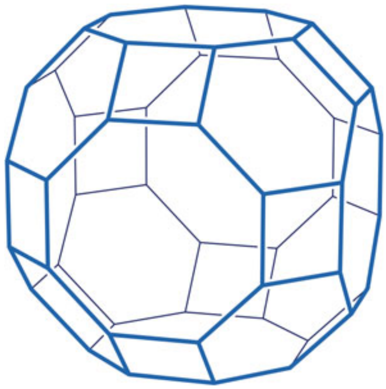


Fig. 2 $Z_1(3, 2)$ is congruent to the truncated cuboctahedron

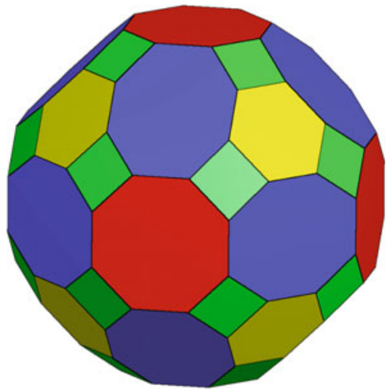


Fig. 3 $Z_\infty(3, 1)$ is congruent to the truncated small rhombicuboctahedron

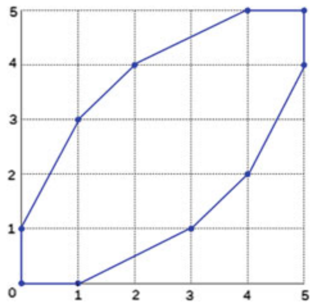


Fig. 4 $H^+_\infty(2, 2)$

and a rhombic dodecahedron, see for instance Eppstein [12]. $H_\infty(3, 1)$ is, up to translation, a lattice (3, 9)-polytope with diameter 13 and 96 vertices.

(v) $Z^+_\infty(2, 2)$ is generated by $\{(0, 1), (1, 0), (1, 1), (1, 2), (2, 1)\}$ and forms the decagon whose vertices are $\{(-5, -5), (-5, -3), (-3, -5), (-3, 1), (-1, 3), (1, -3), (3, -1), (3, 5), (5, 3), (5, 5)\}$. $H^+_\infty(2, 2)$ is a lattice (2, 5)-polygon, see Fig. 4.

2.2 Combinatorial Properties of the Primitive Zonotopes

We recall properties concerning $Z_q(d, p)$ and $Z_q^+(d, p)$, and in particular their symmetry group, diameter, and vertices. $Z_1(d, 2)$ is the permutahedron of type B_d as its generators form the root system of type B_d, see [17]. Thus, $Z_1(d, 2)$ has $2^d d!$ vertices and its symmetry group is B_d. The properties listed in this section are extensions to $Z_q(d, p)$ of known properties of $Z_1(d, 2)$ whose proofs are given in Sect. 5.1. We refer to Fukuda [14], Grünbaum [16], and Ziegler [33] for polytopes and, in particular, zonotopes.

Property 2.1

(i) *$Z_q(d, p)$ is invariant under the symmetries induced by coordinate permutations and the reflections induced by sign flips.*
(ii) *The sum $\sigma_q(d, p)$ of all the generators of $Z_q(d, p)$ is a vertex of both $Z_q(d, p)$ and $H_q(d, p)$. The origin is a vertex of $H_q(d, p)$, and $-\sigma_q(d, p)$ is a vertex of $Z_q(d, p)$.*
(iii) *The coordinates of the vertices of $Z_q(d, p)$ are odd. Thus, the number of vertices of $Z_q(d, p)$ is a multiple of 2^d.*
(iv) *$H_q(d, p)$ is, up to translation, a lattice (d, k)-polytope where k is the sum of the first coordinates of all generators of $Z_q(d, p)$*
(v) *The diameter of $Z_q(d, p)$, respectively $Z_q^+(d, p)$, is equal to the number of its generators.*

Property 2.2

(i) *$Z_q^+(d, p)$ is centrally symmetric and invariant under the symmetries induced by coordinate permutations.*
(ii) *The sum $\sigma_q^+(d, p)$ of all the generators of $Z_q^+(d, p)$ is a vertex of both $Z_q^+(d, p)$ and $H_q^+(d, p)$. The origin is a vertex of $H_q^+(d, p)$, and $-\sigma_q^+(d, p)$ is a vertex of $Z_q^+(d, p)$.*

A vertex v of $Z_q(d, p)$ is called *canonical* if $v_1 \geq \cdots \geq v_d > 0$. Property 2.1 item (i) implies that the vertices of $Z_q(d, p)$ are all the coordinate permutations and sign flips of its canonical vertices.

Property 2.3

(i) *A canonical vertex v of $Z_q(d, p)$ is the unique maximizer of $\{\max c^T x : x \in Z_q(d, p)\}$ for some vector c satisfying $c_1 > c_2 > \cdots > c_d > 0$.*
(ii) *$Z_1(d, 2)$ has $2^d d!$ vertices corresponding to all coordinate permutations and sign flips of the unique canonical vertex $\sigma_1(d, 2) = (2d - 1, 2d - 3, \ldots, 1)$.*
(iii) *For $q = \infty$ or $p \neq 1$, $Z_q(d, p)$ has at least $2^d d!$ vertices including all coordinate permutations and sign flips of the canonical vertex $\sigma_q(d, p)$.*
(iv) *$Z_\infty^+(d, 1)$ has at least $2 + 2d!$ vertices including the $2d!$ permutations of $\pm\sigma(d)$ where $\sigma(d)$ is a vertex with pairwise distinct coordinates, and the 2 vertices $\pm\sigma_\infty^+(d, 1)$.*

2.3 Primitive Zonotopes as Lattice Polytopes with Large Diameter

Let $\delta(d,k)$ be the maximum possible edge-diameter over all lattice (d,k)-polytopes. Finding lattice polygons with the largest diameter; that is, to determine $\delta(2,k)$, was investigated independently in the early nineties by Thiele [31], Balog and Bárány [3], and Acketa and Žunić [1]. This question can be found in Ziegler's book [33] as Exercise 4.15. The answer is summarized in Proposition 2.4, with the role of primitive zonotopes highlighted.

Proposition 2.4 *$\delta(2,k)$ is achieved, up to translation, by the Minkowski sum of a subset of the generators of $H_1(2,p)$ for a proper p. In particular, for $k = \sum_{1\le j\le p} j\phi(j)$ for some p, $\delta(2,k)$ is uniquely achieved, up to translation, by $H_1(2,p)$.*

In general dimension, Naddef [24] showed in 1989 that $\delta(d,1) = d$, Kleinschmidt and Onn [19] generalized this result in 1992 showing that $\delta(d,k) \le kd$, before Del Pia and Michini [7] strengthened the upper bound to $\delta(d,k) \le kd - \lceil d/2 \rceil$ for $k \ge 2$, and showed that $\delta(d,2) = \lfloor 3d/2 \rfloor$. Deza and Pournin [11] further strengthened the upper bound to $kd - \lceil 2d/3 \rceil - (k-3)$ for $k \ge 3$ and showed that $\delta(4,3) = 8$. The quantities $\delta(3,4) = 7$ and $\delta(3,5) = 9$, respectively $\delta(3,6) = 10$ and $\delta(5,3) = 10$, were computationally determined in [5], respectively [8]. Concerning the lower bound, Deza, Manoussakis, and Onn [10] showed that $\delta(d,k) \ge \lfloor (k+1)d/2 \rfloor$ for $k < 2d$. These bounds are summarized in Proposition 2.5, and Conjecture 2.6 given in [10] is recalled.

Proposition 2.5

(i) $\delta(d,k) = \lfloor (k+1)d/2 \rfloor$ *for* $(d,k) = (d,1), (d,2), (2,3), (3,3), (4,3), (5,3), (3,4), (3,5)$, *and* $(3,6)$.
(ii) $2d \le \delta(d,3) \le \lfloor 7d/3 \rfloor - 1$ *for* $d \not\equiv 2 \mod 3$, *and* $\delta(d,3) \le \lfloor 7d/3 \rfloor$ *otherwise,*
(iii) $\delta(d,k) \ge \lfloor (k+1)d/2 \rfloor$ *for* $k < 2d$,
(iv) $\delta(d,k) \le kd - \lceil 2d/3 \rceil - (k-2)$ *for* $k \ge 4$

Conjecture 2.6 $\delta(d,k)$ is achieved, up to translation, by a Minkowski sum of lattice vectors. In particular, $\delta(d,k) \le \lfloor (k+1)d/2 \rfloor$ for any d and k, and $\delta(d,k) = \lfloor (k+1)d/2 \rfloor$ when $k < 2d$.

Note that Conjecture 2.6 holds for all known values of $\delta(d,k)$ given in Table 1, and hypothesizes, in particular, that $\delta(d,3) = 2d$.

Soprunov and Soprunova [29] considered the Minkowski length of a lattice polytope P; that is, the largest number of lattice segments whose Minkowski sum is contained in P. For example, the Minkowski length of the $\{0,k\}^d$-cube is kd. We consider a variant of the Minkowski length and the special case when P is the $\{0,k\}^d$-cube. Let $L(d,k)$ denote the largest number of pairwise linearly independent lattice segments

Table 1 Largest diameter $\delta(d, k)$ over all lattice (d, k)-polytopes

		k									
	$\delta(d,k)$	1	2	3	4	5	6	7	8	9	10
d	1	1	1	1	1	1	1	1	1	1	1
	2	2	3	4	4	5	6	6	7	8	8
	3	3	4	6	7	9	10				
	4	4	6	8							
	5	5	7	10							
	⋮	⋮	⋮								
	d	d	$\lfloor 3d/2 \rfloor$								

whose Minkowski sum is contained in the $\{0, k\}^d$-cube. One can check that the generators of $H_1(d, 2)$ form the largest, and unique, set of primitive lattice vectors which Minkowski sum fits within the $\{0, k\}^d$-cube for $k = 2d - 1$; that is, for k being the sum of the first coordinates of the d^2 generators of $H_1(d, 2)$. Thus, $L(d, 2d - 1) = \delta(H_1(d, 2)) = d^2$. Similarly, $L(2, k) = \delta(2, k)$ for all k, and $L(d, k) = \lfloor (k + 1)d/2 \rfloor$ for $k \le 2d - 1$.

2.4 Primitive Zonotopes and Convex Matroid Optimization

We consider the convex multicriteria matroid optimization framework of Melamed, Onn and Rothblum, see [22, 25, 26]. Call $S \subset \{0, 1\}^n$ a *matroid* if it is the set of the indicators of bases of a matroid over $\{1, \ldots, n\}$. For instance, S can be the set of indicators of spanning trees in a connected graph with n edges. For a $d \times n$ matrix W, let $WS = \{Wx : x \in S\}$, and let $\operatorname{conv}(WS) = W\operatorname{conv}(S)$ be the projection to $\mathbb{R}^d$ of $\operatorname{conv}(S)$ by W. Given a convex function $f : \mathbb{R}^d \to \mathbb{R}$, *convex matroid optimization* deals with maximizing the composite function $f(Wx)$ over S; that is, $\max\ \{f(Wx)\ :\ x \in S\}$, and is concerned with $\operatorname{conv}(WS)$; that is, the projection of the set of the feasible points. The maximization problem can be interpreted as a problem of multicriteria optimization, where each row of W gives a linear criterion $W_i x$ and f compromises these criteria. Thus, W is called the *criteria* matrix or *weight* matrix. The projection polytope $\operatorname{conv}(WS)$ and its vertices play a key role in solving the maximization problem as, for any convex function f, there is an optimal solution x whose projection $y = Wx$ is a vertex of $\operatorname{conv}(WS)$. In particular, the enumeration of all vertices of $\operatorname{conv}(WS)$ enables to compute the optimal objective value by picking a vertex attaining the optimal value $f(y) = f(Wx)$. Thus, it suffices that f is presented by a *comparison oracle* that, queried on vectors $y, z \in \mathbb{R}^d$, asserts whether or not $f(y) < f(z)$. *Coarse* criteria matrices; that is, W whose entries are small or in $\{0, 1, \ldots, p\}$, are of particular interest. In multicriteria combinatorial optimization, this case corresponds to the weight $W_{i,j}$ attributed to element j of the ground set $\{1, \ldots, n\}$ under criterion i being small or in $\{0, 1, \ldots, p\}$ for all i, j. In the remainder, we only consider $\{0, 1, \ldots, p\}$-valued W.

Let $m(d, p)$ denote the number of vertices of $H_\infty(d, p)$. Theorem 2.7, given in [10], settles the computational complexity of the multicriteria optimization problem by showing that the maximum number of vertices of the projection polytope $\mathrm{conv}(WS)$ of any matroid S on n elements and any d-criteria p-bounded utility matrix; that is, $W \in \{0, 1, \ldots, p\}^{d\times n}$, is equal to $m(d, p)$, and hence is in particular *independent* of n, S, and W.

Theorem 2.7 *Let d, p be any positive integers. Then, for any positive integer n, any matroid $S \subset \{0, 1\}^n$, and any d-criteria p-bounded utility matrix W, the primitive zonotope $H_\infty(d, p)$ refines* $\mathrm{conv}(WS)$. *Moreover, $H_\infty(d, p)$ is a translation of* $\mathrm{conv}(WS)$ *for some matroid S and d-criteria p-bounded utility matrix W. Thus, the maximum number of vertices of* $\mathrm{conv}(WS)$ *for any n, any matroid $S \subset \{0, 1\}^n$, and any d-criteria p-bounded utility matrix W, equals the number $m(d, p)$ of vertices of $H_\infty(d, p)$, and hence is independent of n, S, and W. Also, for any fixed d and convex $f : \mathbb{R}^d \to \mathbb{R}$, the multicriteria matroid optimization problem can be solved using a number of arithmetic operations and queries to the oracles of S and f which is polynomial in n and p using $m(d, p)$ greedily solvable linear matroid optimization counterparts.*

Theorem 2.8 *The number $m(d, 1)$ of vertices of $H_\infty(d, 1)$ satisfies*

$$2^d d! \le m(d, 1) \le 2 \sum_{0\le i\le d-1} \binom{(3^d-3)/2}{i} - 2\binom{(3^{d-1}-3)/2}{d-1}.$$

Proof The first inequality restates item (iii) of Property 2.3 where (q, d, p) is set to $(\infty, d, 1)$. The second inequality is obtained by exploiting the structure of the generators of $H_\infty(d, 1)$. One can check that $H_\infty(d, 1)$ has $(3^d - 1)/2$ generators and that removing the first zero of the generators of $H_\infty(d, 1)$ starting with zero yields exactly the $(3^{d-1} - 1)/2$ generators of $H_\infty(d - 1, 1)$. We recall that the number of vertices $f_0(Z)$ of a d-dimensional zonotope Z generated by m generators is bounded by $\bar{f}(d, m) = 2 \sum_{0\le i\le d-1} \binom{m-1}{i}$. By duality, the number $f_0(Z)$ of vertices of a zonotope Z is equal to the number $f_{d-1}(\mathcal{A})$ of cells of the associate hyperplane arrangement $\mathcal{A}$ where each generator m^j of Z corresponds to an hyperplane h^j of $\mathcal{A}$. The inequality $f_0(Z) \le \bar{f}(d, m)$ is based on the inequality $f_{d-1}(\mathcal{A}) \le f_{d-1}(\mathcal{A} \setminus h^j) + f_{d-1}(\mathcal{A} \cap h^j)$ for any hyperplane h^j of $\mathcal{A}$ where $\mathcal{A} \setminus h^j$ denotes the arrangement obtained by removing h^j from $\mathcal{A}$, and $\mathcal{A} \cap h^j$ denotes the arrangement obtained by intersecting $\mathcal{A}$ with h^j. This last inequality and the duality between zonotopes and hyperplane arrangements are detailed, for example, in [14]. Recursively applying this inequality to the arrangement $\mathcal{A}_\infty(d, 1)$ associated to $H_\infty(d, 1)$ till the remaining $(3^{d-1} - 1)/2$ hyperplanes form a $(d - 1)$-dimensional arrangement equivalent to $\mathcal{A}_\infty(d - 1, 1)$ yields: $f_{d-1}(\mathcal{A}_\infty(d, 1)) \le \bar{f}(d, (3^d - 1)/2) - \big(\bar{f}(d, (3^{d-1} - 1)/2) - \bar{f}(d - 1, (3^{d-1} - 1)/2)\big)$ which completes the proof since $f_{d-1}(\mathcal{A}_\infty(d, 1)) = f_0(H_\infty(d, 1))$ and $\bar{f}(d, m) - \bar{f}(d - 1, m) = 2\binom{m-1}{d}$. In other words, the inequality is based on the inductive build-up of $H_\infty(d, 1)$ starting with the $(3^{d-1}-3)/2$ generators with zero as first coordinate, and noticing that these $(3^{d-1}-3)/2$ generators belong to a lower dimensional space. □

3 Small Primitive Zonotopes $H_q(d,p)$ and $H_q^+(d,p)$

In this section we provide the number of vertices, the diameter; that is, the number of generators, and the grid embedding size for $H_q(d,p)$ and $H_q^+(d,p)$ for small d and p, and $q=1,2$, and ∞. We recall that, up to translation, $Z_q(d,p)$, respectively $Z_q^+(d,p)$, is the image of $H_q(d,p)$, respectively $H_q^+(d,p)$, by a homothety of factor 2. Thus $Z_q(d,p)$ and $H_q(d,p)$, respectively $Z_q^+(d,p)$ and $H_q^+(d,p)$, have the same number of vertices and the same diameter, while the grid embedding size of the $Z_q(d,p)$, respectively $Z_q^+(d,p)$, is twice the one of $H_q(d,p)$, respectively $H_q^+(d,p)$. Since both $H_q(d,1)$ and $H_q^+(d,1)$ are equal to the $\{0,1\}^d$-cube for finite q, both are omitted from the tables provided in this section. The vertex enumeration was performed using standard algorithms described, for instance, in [14]. The Euler totient function counting positive integers less than or equal to j and relatively prime with j is denoted by $\phi(j)$. Note that $\phi(1)$ is set to 1.

Enumerative questions concerning $H_q(d,p)$ and $H_q^+(d,p)$ have been studied in various settings. We list a few instances, and the associated OEI sequences, see [28] for details and references therein.

(i) $f_0(H_\infty^+(d,1))$ corresponds to the OEI sequence A034997 giving the number of generalized retarded functions in quantum field theory. The value of $f_0(H_\infty^+(d,1))$ was determined till $d=8$.
(ii) $f_0(H_\infty(d,1))$, which is the number of regions of hyperplane arrangements with $\{-1,0.1\}$-valued normals in dimension d, corresponds to the OEI sequence A009997 giving $f_0(H_\infty(d,1))/(2^d d!)$. The value of $f_0(H_\infty(d,1))$ was determined till $d=7$.
(iii) $\delta(H_\infty^+(d,p))$ corresponds to the OEI sequence A090030 with further cross-referenced sequences for $d\le 7$ and $p\le 8$.
(iv) $\delta(H_1^+(3,p))$, respectively $\delta(H_2^+(2,p))$, $\delta(H_\infty(d,2))$, $\delta(H_\infty(2,p))/4$, $\delta(H_2(2,p))/2$, $\delta(H_1^+(d,3))$, and $\delta(H_2^+(d,2))$, corresponds to the OEI sequence A048134, respectively A049715, A005059, A002088, A175341, A008778, and A055795.
(v) the grid embedding size of $H_2(d,2)$, respectively $H_\infty(d,2)$ and $H_1^+(d,3)$, corresponds to the OEI sequence A161712, respectively A080961 and A052905.

3.1 Small Primitive Zonotopes $H_q(d,p)$

In Tables 2, 3, and 4, the number of vertices $f_0(H_q(d,p))$ is divided by $2^d d!$ and followed by its diameter $\delta(H_q(d,p))$ and grid embedding size. For instance, the entry

Table 2 Small primitive zonotopes $H_1(d, p)$

	$H_1(d, p)$	p				
		2	3	4	5	6
d	2	1 (4, 3)	2 (8, 9)	3 (12, 17)	5 (20, 37)	6 (24, 49)
	3	1 (9, 5)	7 (25, 21)	26 (49, 53)	102 (97, 133)	227 (145, 229)
	4	1 (16, 7)	40 (56, 37)	531 (136, 117)	6741 (312, 337)	? (560, 709)
	5	1 (25, 9)	339 (105, 57)	? (305, 217)	? (801, 713)	? (1681, 1769)

Table 3 Small primitive zonotopes $H_2(d, p)$

	$H_2(d, p)$	p			
		2	3	4	5
d	2	1 (4, 3)	2 (8, 9)	4 (16, 27)	6 (24, 51)
	3	2 (13, 9)	26 (49, 57)	126 (109, 161)	443 (205, 377)
	4	14 (40, 27)	1427 (192, 193)	? (592, 795)	? (1424, 2411)
	5	273 (105, 65)	? (641, 577)		

Table 4 Small primitive zonotopes $H_\infty(d, p)$

	$H_\infty(d, p)$	p			
		1	2	3	4
d	2	1 (4, 3)	2 (8, 9)	4 (16, 27)	6 (24, 51)
	3	2 (13, 9)	26 (49, 57)	228 (145, 249)	910 (289, 633)
	4	14 (40, 27)	4333 (272, 321)	? (1120, 1923)	? (2928, 6459)
	5	516 (121, 81)			
	6	124,187 (364, 243)			
	7	214,580,603 (1093, 729)			

26(49, 53) for $(q, d, p) = (1, 3, 4)$ in Table 2 indicates that $H_1(3, 4)$ has $26 \times 2^3 3! = 1248$ vertices, diameter 49, and is, up to translation, a lattice (3, 53)-polytope. The rather straightforward proofs are given in Sect. 5.2.

3.1.1 Small Primitive Zonotopes $H_1(d, p)$

Property 3.1

(i) $H_1(d, 1)$ *is the* $\{0, 1\}^d$*-cube,*
(ii) $H_1(d, 2)$ *is, up to translation, a lattice* (d, k)*-polytope with* $k = 2d - 1$*, and diameter* d^2*, and* $2^d d!$ *vertices,*
(iii) $H_1(d, 3)$ *is, up to translation, a lattice* (d, k)*-polytope with* $k = 2d^2 + 2d - 3$*, and diameter* $d(d + 2)(2d - 1)/3$*,*

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(iv) $H_1(d,4)$ *is, up to translation, a lattice* (d,k)*-polytope with* $k = \binom{d-1}{0} + 16\binom{d-1}{1} + 20\binom{d-1}{2} + 8\binom{d-1}{3}$*, and diameter* $d(d^3+2d^2+2d-2)/3$*,*

(v) $H_1(2,p)$ *is, up to translation, a lattice* $(2,k)$*-polygon with* $k = \sum_{1\le j\le p} j\phi(j)$*, and diameter* $2\sum_{1\le j\le p}\phi(j)$*.*

3.1.2 Small Primitive Zonotopes $H_2(d, p)$

Property 3.2

(i) $H_2(d,1)$ *is the* $\{0,1\}^d$*-cube,*

(ii) $H_2(d,2)$ *is, up to translation, a lattice* (d,k)*-polytope with* $k = \sum_{0\le j\le 3} 2^j\binom{d-1}{j}$*, and diameter* $\sum_{0\le j\le 3} 2^j\binom{d}{j+1}$*.*

3.1.3 Small Primitive Zonotopes $H_\infty(d, p)$

Property 3.3

(i) $H_\infty(d,1)$ *is, up to translation, a lattice* (d,k)*-polytope with* $k = 3^{d-1}$*, and diameter* $(3^d-1)/2$*,*

(ii) $H_\infty(d,2)$ *is, up to translation, a lattice* (d,k)*-polytope with* $k = 3\times 5^{d-1} - 2\times 3^{d-1}$*, and diameter* $(5^d-3^d)/2$*,*

(iii) $H_\infty(2,p)$ *is, up to translation, a lattice* $(2,k)$*-polygon with diameter* $4\sum_{1\le j\le p}\phi(j)$*.*

3.2 Small Positive Primitive Zonotopes $H_q^+(d, p)$

In Tables 5, 6, and 7, the number of vertices $f_0(H_q^+(d,p))$ is followed by its diameter $\delta(H_q^+(d,p))$ and grid embedding size. For instance, the entry 1082(15, 5) for $(q,d,p) = (1,5,2)$ in Table 5 indicates that $H_1^+(5,1)$ has 1082 vertices, diameter 15, and is a lattice (5, 5)-polytope.

3.2.1 Small Positive Primitive Zonotopes $H_1^+(d, p)$

Property 3.4

(i) $H_1^+(d,1)$ *is the* $\{0,1\}^d$*-cube,*

(ii) $H_1^+(d,2)$ *is a lattice* (d,k)*-polytope with* $k = d$*, and diameter* $\binom{d+1}{2}$*,*

Table 5 Small positive primitive zonotopes $H_1^+(d, p)$

	$H_1^+(d, p)$	p 2	3	4	5	6
d	2	6 (3, 2)	10 (5, 5)	14 (7, 9)	22 (11, 19)	26 (13, 25)
	3	26 (6, 3)	110 (13, 10)	314 (22, 22)	1022 (40, 52)	1970 (55, 82)
	4	150 (10, 4)	2194 (26, 16)	17,534 (51, 41)	145,198 (103, 106)	593,402 (161, 193)
	5	1082 (15, 5)	71,582 (45, 23)	2,062,682 (100, 67)	? (221, 188)	? (386, 386)
	6	9366 (21, 6)	? (71, 31)	?(176, 106)		

Table 6 Small positive primitive zonotopes $H_2^+(d, p)$

	$H_2^+(d, p)$	p 2	3	4	5
d	2	6 (3, 2)	10 (5, 5)	18 (9, 14)	26 (13, 26)
	3	32 (7, 4)	212 (19, 19)	1010 (40, 54)	3074 (70, 120)
	4	370 (15, 8)	19,438 (55, 49)	362,962 (141, 170)	3,497,862 (299, 462)
	5	10,922 (30, 15)	? (136, 108)	? (441, 487)	

Table 7 Small positive primitive zonotopes $H_\infty^+(d, p)$

	$H_\infty^+(d, p)$	p 1	2	3	4
d	2	6 (3, 2)	10 (5, 5)	18 (9, 14)	26 (13, 26)
	3	32 (7, 4)	212 (19, 19)	1418 (49, 76)	4916 (91, 184)
	4	370 (15, 8)	27,778 (65, 65)	1,275,842 (225, 344)	? (529, 1064)
	5	11,292 (31, 16)	? (211, 211)	? (961, 1456)	? (2851, 5716)
	6	1,066,044 (63, 32)			
	7	347,326,352 (127, 64)			
	8	419,172,756,930 (255, 128)			

(iii) $H_1^+(d, 3)$ *is a lattice* (d, k)*-polytope with* $k = (d^2 + 5d - 4)/2$ *and diameter* $d(d^2 + 6d - 1)/6$.

(iv) $H_1^+(2, p)$ *is a lattice* $(2, k)$*-polygon with* $k = 1 + \sum_{2 \le j \le p} j\phi(j)/2$, *and diameter* $1 + \sum_{1 \le j \le p} \phi(j)$.

3.2.2 Small Positive Primitive Zonotopes $H_2^+(d, p)$

Property 3.5

(i) $H_2^+(d, 1)$ *is the* $\{0, 1\}^d$*-cube,*

(ii) $H_2^+(d, 2)$ *is a* (d, k) *polytope with* $k = \binom{d}{1} + \binom{d}{3}$, *and diameter* $\binom{d+1}{2} + \binom{d+1}{4}$.

3.2.3 Small Positive Primitive Zonotopes $H_\infty^+(d, p)$

Property 3.6

(i) $H_\infty^+(d, 1)$ *is, a lattice* (d, k)*-polytope with* $k = 2^{d-1}$*, and diameter* $2^d - 1$*,*
(ii) $H_\infty^+(d, 2)$ *is a lattice* (d, k)*-polytope with* $k = 3^d - 2^d$*, and diameter* $3^d - 2^d$*,*
(iii) $H_\infty^+(2, p)$ *is a lattice* $(2, k)$*-polygon with diameter* $1 + 2 \sum\limits_{1 \le j \le p} \phi(j)$*.*

4 Complexity Issues

We discuss a few complexity issues related to primitive zonotopes. While we mainly focus on $Z_q(d, p)$, the discussion and results, such as Propositions 4.1 and 4.2, can be adapted to $Z_q^+(d, p)$. As $H_q(d, p)$, respectively $H_q^+(d, p)$, is the translation of the image by a homothety of $Z_q(d, p)$, respectively $Z_q^+(d, p)$, the complexity results are the same.

4.1 Complexity Properties

Proposition 4.1 *For fixed positive integers* p *and* q*, linear optimization over* $Z_q(d, p)$ *is polynomial-time solvable, even in variable dimension* d*.*

Proof Since the q-norm of a generator of $Z_q(d, p)$ is bounded by p, it has at most p^q nonzero entries – which is attained for the vector of all ones and $d = p^q$. Thus, the number of generators of $Z_q(d, p)$ is bounded by $\binom{d}{p^q}(2p)^{p^q} = O\left(d^{p^q}\right)$. Hence, one can explicitly write all the generators of $Z_q(d, p)$ in polynomial time. Consequently, one can compute in polynomial time the following signed sum of generators of $Z_q(d, p)$ for any given rational $c \in \mathbb{R}^d$: $v^* = \sum\limits_{v \in G_q(d,p)} \text{sign}(c^T v)v$ where $G_q(d, p)$ denotes the set of generators of $Z_q(d, p)$. Note that $\text{sign}(0)$ is set to 0. Then, one can show that v^* is a maximizer of $\{\max c^T x : x \in Z_q(d, p)\}$. □

The algorithmic theory developed by Grötschel, Lovász, and Schrijver [15] shows that polynomial-time solvability for linear optimization over a polytope implies polynomial-time solvability for other questions. In particular, Proposition 4.1 implies Proposition 4.2.

Proposition 4.2 *For fixed positive integers* p *and* q*, the following problems are polynomial-time solvable.*

(i) Extremality*: Given* $v \in \mathbb{Z}^d$*, decide if* v *is a vertex of* $Z_q(d, p)$*,*
(ii) Adjacency*: Given* $v^1, v^2 \in \mathbb{Z}^d$*, decide if* $[v^1, v^2]$ *is an edge of* $Z_q(d, p)$*;*
(iii) Separation*: Given rational* $y \in \mathbb{R}^d$*, either assert* $y \in Z_q(d, p)$*, or find* $h \in \mathbb{Z}^d$ *separating* y *from* $Z_q(d, p)$*; that is, satisfying* $h^T y > h^T x$ *for all* $x \in Z_q(d, p)$*.*

4.2 Open Problems

A natural open problem is to find direct algorithms to solve, over both $Z_q(d, p)$ and $Z_q^+(d, p)$, the extremality, adjacency, and separation questions given in Proposition 4.2.

Note that the case $q = \infty$, even for $p = 1$, seems to be significantly harder as the number of nonzero entries in a generator of $Z_\infty(d, p)$ can not bounded by a constant independent of d. Thus, the number of generators of $Z_\infty(d, p)$ is exponential in d. Hence, the complexity of linear optimization, extremality, adjacency, and separation over both $Z_\infty(d, p)$ and $Z_\infty^+(d, p)$, is open. In particular, it is not clear if deciding if a given point is a vertex of $Z_\infty(d, 1)$, or of $Z_\infty^+(d, p)$, is in NP or in coNP.

The remaining open questions deal with a reformulation in term of degree sequence of hypergraphs. The question is presented within the context of $H_q^+(d, p)$ but could be adapted to $H_q(d, p)$. Each subset $H \subseteq \{0, 1\}^d$ can be associated to a hypergraph with ground set $[d]$. The vector $\sum_{h \in H} h$ is called the *degree sequence* of H, and the convex hull of the degree sequences of all hypergraphs with ground set $[d]$ is called the *hypergraph polytope* D_d; and thus $D_d = H_\infty^+(d, 1)$. Considering only k-uniform hypergraphs; that is, subsets $H \subseteq \{0, 1\}^d$ where all vectors in H have k nonzero entries, one obtains the *k-uniform hypergraph polytope* $D_d(k)$ as the convex hull of the degree sequences of all k-uniform hypergraphs. The k-uniform hypergraph polytope, in particular $D_d(2)$ and $D_d(3)$, have been extensively studied, see [6, 13, 20, 23] and references therein. A natural question raised in the literature asks for suitable necessary and sufficient conditions to check whether a vector $h \in D_d(k) \cap \mathbb{Z}^d$ is the degree sequence of some k-uniform hypergraph. A trivial necessary condition is that the sum of the coordinates of h is a multiple of k. For $k = 2$; that is for graphs, the celebrated Erdős-Gallai Theorem [13] shows that the trivial necessary condition is also sufficient. For $k = 3$; that is for 3-uniform hypergraphs, the question was raised by Klivans and Reiner [20]. Liu [21] exhibited counterexamples by constructing *holes* for $d \geq 16$; that is, vectors h in $D_d(3) \cap \mathbb{Z}^d$ such that the sum of the coordinates of h is a multiple of 3, but h is not the degree sequence of a 3-uniform hypergraph. Deza et al. [9] answered a question raised in 1986 by Colbourn, Kocay, and Stinson [6] by showing that deciding whether a given sequence is the degree sequence of a 3-uniform hypergraph is NP-complete.

As there is no trivial congruence necessary condition, we call a vector in $H_q^+(d, p) \cap \mathbb{Z}^d$ a *hole* if it cannot be written as the sum of a subset of the generators of $H_q^+(d, p)$. While the answer to the question "*Does $H_q^+(d, p)$ have holes?*" is likely yes for most p, q, d, it would be interesting to explicitly find such holes and better understand them. A natural follow-up question, provided there are holes, is "*For given fixed positive integers p and q, what is the complexity of deciding if a given vector $h \in H_q^+(d, p) \cap \mathbb{Z}^d$ is a hole, and if not, of writing h as the sum of a subset of generators of $H_q^+(d, p)$?*".

As noted in the proof of Proposition 4.1, there are polynomially many generators for fixed integer p and q. Thus, the above follow-up question is in coNP as, if h is not a hole, it is possible to write h as a sum of a subset of generators $H_q^+(d, p)$. The last question is thus "*Is this problem coNP-complete?"*.

As for the linear optimization related questions, the hole related questions seem to be significantly harder for $q = \infty$. In particular, for $(q, d, p) = (\infty, d, 1)$, the questions investigate the holes of D_d.

5 Proofs for Sections 2.2 and 3

Let $G_q(d, p)$, respectively $G_q^+(d, p)$, denote the generators of $Z_q(d, p)$, respectively $Z_q^+(d, p)$. Recall that $\sigma_q(d, p)$, respectively $\sigma_q^+(d, p)$, denotes the sum of the generators of $Z_q(d, p)$, respectively $Z_q^+(d, p)$.

5.1 Proof for Section 2.2

5.1.1 Proof of Item (i) of Property 2.1

Proof Note that if the set G of generators of a zonotope Z is invariant under coordinate permutation or sign flip, then the same holds for Z. Let π denote a permutation or a sign flip, and consider a signed sum $\sum_{g \in G} \epsilon_g g$. Then, $\pi(\sum_{g \in G} \epsilon_g g) = \sum_{g \in G} \epsilon_g \pi(g)$ is also a signed sum of generators since G is permutation and sign flip invariant. In other words, the set of all signed sums is invariant under permutations and sign flips, and thus the same holds for the convex hull Z of all signed sums. Let $J_q(d, p)$ be the set of all $-g$ for $g \in G_q(d, p)$. The zonotope $\tilde{Z}_q(d, p)$ generated by $G_q(d, p) \cup J_q(d, p)$ is the image of $Z_q(d, p)$ by a homothety of factor 2, and thus shares the same symmetry group. One can check that the set of generators of $\tilde{Z}_q(d, p)$ is invariant under coordinate permutation or sign flip, thus the same holds for $\tilde{Z}_q(d, p)$, and consequently holds for $Z_q(d, p)$. □

5.1.2 Proof of Item (ii) of Property 2.1

Proof Consider the minimization problem $\{\min c^T x : x \in H_q(d, p)\}$ or, equivalently, $\min c^T x$ over all integer valued points of $H_q(d, p)$. Set $c = (d!\bar{x}^d, (d-1)!\bar{x}^{d-1}, \ldots, \bar{x})$ where $\bar{x} = (2p+1)^{d+1}$. Assuming that x is not the origin, let x_{i_0} denotes the first nonzero coordinate of x. Note that $x_{i_0} \geq 1$ by definition of $G_q(d, p)$, and $|x_i| \leq \bar{x}$. Thus, $c^T x \geq (d+1-i_0)!\bar{x}^{d+1-i_0} - \bar{x} \sum_{i_0 < i \leq d} (d+1-i)!\bar{x}^{d+1-i} > 0$.

In other words, the origin is the unique minimizer of a linear optimization instance over $H_q(d, p)$; that is, the origin is a vertex of $H_q(d, p)$. As $Z_q(d, p) = 2H_q(d, p) - \sigma_q(d, p)$, the point $-\sigma_q(d, p)$ is a vertex of $Z_q(d, p)$. By item (i) of Proposition 2.1, the point $\sigma_q(d, p)$ is a vertex of $Z_q(d, p)$, and thus $(\sigma_q(d, p) + \sigma_q(d, p))/2$ is a vertex of $H_q(d, p)$. □

5.1.3 Proof of Item (iii) of Property 2.1

Proof We first show that the coordinates of the vertex $\sigma_q(d, p)$ are odd. As noted in the proof of item (iii) of Property 2.3, the i-th coordinate of $\sigma_q(d, p)$ is equal to the first coordinate of $\sigma_q(d - i + 1, p)$. Thus, it is enough to show that the first coordinate of $\sigma_q(d, p)$ is odd. Except for the first unit vector $(1, 0, \dots, 0)$, any generator g of $Z_q(d, p)$ with nonzero first coordinate can be paired with the generator $\bar{g}$ where $\bar{g}_1 = g_1$ and $\bar{g}_i = -g_i$ for $i \neq 1$. Thus, the sum of the first coordinates of the generators of $Z_q(d, p)$, excluding the first unit vector, is even. Hence, the first coordinate of $\sigma_q(d, p)$ is odd, and thus all the coordinates of $\sigma_q(d, p)$ are odd. Consider a vertex $v = \sum_{g \in G_q(d,p)} \epsilon(g) g$ of $Z_q(d, p)$. Since flipping the sign of an $\epsilon(g)$ does not change the parity of a coordinate of v, the coordinates of v have the same parity as the ones of $\sigma_q(d, p)$; i.e. are odd. In particular, the coordinates of a vertex of $Z_q(d, p)$ are nonzero and item (i) of Proposition 2.1 implies that the number of vertices of $Z_q(d, p)$ is a multiple of 2^d. □

5.1.4 Proof of Items (iv) and (v) of Property 2.1

Proof Let Z be a zonotope generated by integer-valued generators $m^j : j = 1, \dots, m(Z)$. Then, Z is, up to translation, a lattice (d, k)-polytope with $k \leq \max_{i=1,\dots,d} \sum_{1 \leq j \leq m(Z)} |m_i^j|$. Item (i) of Property 2.1 implies that the integer range of its coordinates is independent of the chosen coordinate. The same holds for $H_q(d, p)$, and, thus to determine the integer range of $H_q(d, p)$, it is enough to consider the first coordinates of its generators. Since the origin is a vertex of $H_q(d, p)$ and the first coordinate of its generator is nonnegative, the integer range of $H_q(d, p)$ is the sum of the first coordinates of its generators. For item (v), recall that the diameter of a zonotope is at most the number of its generators, and this inequality is satisfied with equality if no pair of generators are linearly dependent – which is the case for $Z_q(d, p)$ and $Z_q^+(d, p)$. □

5.1.5 Proof of Property 2.2

Proof Consider a generator $g \in G_q^+(d, p)$ and a coordinate permutation π. Since $\pi(g) \in G_q^+(d, p)$, $\pi(Z_q^+(d, p)) = \pi(\sum[-1, 1]G_q^+(d, p)) = \sum[-1, 1]$

$\pi(G_q^+(d,p)) = \sum[-1,1]G_q^+(d,p) = Z_q^+(d,p)$. As in the proof of item (ii) of Property 2.1, one can check that the origin is the unique minimizer of $\{\min c^T x : x \in H_q(d,p)\}$ with $c = (1,1,\ldots,1)$. Thus, the origin is a vertex of $H_q^+(d,p)$. As $Z_q^+(d,p) = 2H_q^+(d,p) - \sigma_q(d,p)$, the point $-\sigma_q(d,p)$ is a vertex of $Z_q^+(d,p)$. Since $Z_q^+(d,p)$ is invariant under the symmetries induced by coordinate permutations, $\sigma_q(d,p)$ is a vertex of $Z_q^+(d,p)$, and thus $(\sigma_q(d,p) + \sigma_q(d,p))/2$ is a vertex of $H_q^+(d,p)$. □

5.1.6 Proof of Items (i) and (ii) of Property 2.3

Proof Given a canonical vertex v of $Z_q(d,p)$, let c be a vector such that v is the unique maximizer of $\{\max c^T x : x \in Z_q(d,p)\}$. Up to infinitesimal perturbations, we can assume that the coordinates of c are pairwise distinct and nonzero. Note that each coordinate c_i of c is positive as otherwise flipping the sign of $v_i > 0$ would yield a point in $Z_q(d,p)$ with higher objective value than v. Assume that $c_i < c_j$ for some $i < j$. Then, $v_i = v_j$ as otherwise permuting v_i and v_j would yield a point in $Z_q(d,p)$ with higher objective value than v. Let $\pi_{ij}(c)$ be obtained by permuting c_i and c_j. Then, one can check that v is the unique maximizer of $\{\max \pi_{ij}(c)^T x : x \in Z_q(d,p)\}$. Assume, by contradiction, that $v' \in Z_q(d,p)$ satisfies $\pi_{ij}(c)^T v' \geq \pi_{ij}(c)^T v$. Then, $c^T \pi_{ij}(v') = \pi_{ij}(c)^T v' \geq \pi_{ij}(c)^T v = c^T v$ which implies $\pi_{ij}(v') = v$, and hence $v' = v$, since v is the unique maximizer of $\{\max c^T x : x \in Z_q(d,p)\}$. Thus, successive appropriate permutations yield a vector $\pi(c)$ with $\pi(c)_1 > \cdots > \pi(c)_d > 0$ such that v is the unique maximizer of $\{\max c^T x : x \in Z_q(d,p)\}$. For item (ii), one can check that $\sigma_1(d,2) = (2d-1, 2d-3, \ldots, 1)$ is the unique maximizer of $\{\max c^T x : x \in Z_1(2,p)\}$ for any c satisfying $c_1 > \cdots > c_d > 0$. Thus, by item (i) of Property 2.3, $\sigma_1(d,2)$ is the unique canonical vertex of $Z_1(d,2)$ and the vertices of $Z_1(d,2)$ are the $2^d d!$ coordinate permutations and sign flips of $\sigma_1(d,2)$. □

5.1.7 Proof of Item (iii) of Property 2.3

Proof We first note that the i-th coordinate of $\sigma_q(d,p)$ is equal to the first coordinate of $\sigma_q(d-i+1,p)$. The statement trivially holds for $i = 1$. For $i > 1$, consider a generator g of $Z_q(d,p)$ with $g_i \neq 0$ and $g_{i_0} > 0$ for some $i_0 < i$, then g can be paired with the generator $\bar{g}$ where $g_i = -\bar{g}_i$ and $g_{i_0} = \bar{g}_{i_0}$. Thus, the sum of all the i-th coordinates of the generators of $Z_q(d,p)$ is equal to the sum of the generators of $Z_q(d,p)$ such that the first $i-1$ coordinates are zero. In other words, the i-th coordinate of $\sigma_q(d,p)$ is equal to the first coordinate of $\sigma_q(d-i+1,p)$. For example, for finite q, $\sigma_q(d,1) = (1,\ldots,1)$ and $Z_q(d,1)$ is the $\{-1,1\}^d$-cube. Then, note that for $q = \infty$ or $p \neq 1$ the first coordinate of $\sigma_q(d-i+1,p)$, which is the grid embedding size of $H_q(d-i+1,p)$, is strictly decreasing with i increasing. Thus, the action of the symmetry group of $Z_q(d,p)$ on $\sigma_q(d,p)$ generates $2^d d!$ distinct vertices of $Z_q(d,p)$. For instance, one can check the i-th coordinate of $\sigma_\infty(d,1)$ is 3^{d-i}. □

5.1.8 Proof of Item (iv) of Property 2.3

Proof The statement trivially holds for $d = 1$. For $d \geq 2$, we show by induction that the vertices of $Z^+_\infty(d, 1)$ include $\sigma(d)$ satisfying $0 = \sigma_1(d) < \cdots < \sigma_d(d) = 2^{d-1}$. The base case holds for $d = 2$ as $\sigma(2) = (0, 2)$ is a vertex of $Z^+_\infty(2, 1)$. Assume such a vertex $\sigma(d)$ exists, and thus $\sigma(d) = \sum_{g \in G^+_\infty(d,1)} \epsilon(g)g$ for some $\epsilon(g)$ and $\sigma(d)$ is the unique maximizer of $\{\max c(d)^T x : x \in Z^+_\infty(d, 1)\}$ for some $c(d)$. The generators of $Z^+_\infty(d + 1, 1)$ consist of the $2^d - 1$ vectors $(g, 0)$ obtained by appending 0 to a generator of $Z^+_\infty(d, 1)$, the $2^d - 1$ vectors $(g, 1)$ obtained by appending 1, and the unit vector e_{d+1}. Consider the point $s(d + 1) = e_{d+1} + \sum_{g \in G^+_\infty(d,1)} (g, 1) - \sum_{g \in G^+_\infty(d,1)} \epsilon(g)(g, 0) = (2^{d-1}, \ldots, 2^{d-1}, 2^d) - (\sigma(d), 0)$; that is, $s(d + 1) = (2^{d-1} - \sigma_1(d), \ldots, 2^{d-1} - \sigma_{d-1}(d), 0, 2^d)$. Thus, the coordinates of $s(d + 1)$ are pairwise distinct and a suitable permutation of $s(d + 1)$ yields a point $\sigma(d + 1)$ satisfying $0 = \sigma_1(d + 1) < \cdots < \sigma_{d+1}(d + 1) = 2^d$. To show that $\sigma(d + 1)$ is a vertex of $Z^+_\infty(d + 1, 1)$, one can check that $\sigma(d + 1)$ is the unique maximizer of $\{\max c(d + 1)^T x : x \in Z^+_\infty(d + 1, 1)\}$ where $c(d + 1) = (-c(d), c_{d+1})$ for sufficiently large c_{d+1}. Thus, for $d \geq 2$, a point $\sigma(d)$ satisfying $0 = \sigma_1(d) < \cdots < \sigma_d(d) = 2^{d-1}$ is a vertex of $Z^+_q(d, p)$. Zonotopes being centrally symmetric, $-\sigma(d)$ is a vertex of $Z^+_q(d, p)$ and the same holds for the distinct $2d!$ permutations of $\pm\sigma(d)$. □

5.2 *Proof for Section 3*

5.2.1 Proof of Property 3.1

Proof One can check that the generators of $H_1(d, 2)$ consist of $\binom{d}{1}$ unity vectors and $2\binom{d}{2}$ vectors $\{\ldots, 1, \ldots, \pm1, \ldots\}$. Thus, the diameter of $H_1(d, 2)$ is $\binom{d}{1} + 2\binom{d}{2} = d^2$. Similarly, one can check that the sum of the first coordinates of the generators of $H_1(d, 2)$ is $2d - 1$. Note that $H_1(d, 2)$ is the permutahedron of type B_d. Then, one can check that, in addition to the previously determined generators of $H_1(d, 2)$, the generators of $H_1(d, 3)$ consist of $2\binom{d}{2}$ vectors $\{\ldots, 1, \ldots, \pm2, \ldots\}$, $2\binom{d}{2}$ vectors $\{\ldots, 2, \ldots, \pm1, \ldots\}$, and $4\binom{d}{3}$ vectors $\{\ldots, 1, \ldots, \pm1, \ldots, \pm1, \ldots\}$. Thus, the diameter of $H_1(d, 3)$ is $\binom{d}{1} + 6\binom{d}{2} + 4\binom{d}{3} = d(d + 2)(2d - 1)/3$. Similarly, one can check that the sum of the first coordinates of the generators of $H_1(d, 3)$ is $\binom{d-1}{0} + 8\binom{d-1}{1} + 4\binom{d-1}{2} = 2d^2 + 2d - 3$. Furthermore, one can check that, in addition to the previously determined generators of $H_1(d, 3)$, the generators of $H_1(d, 4)$ consist of $2\binom{d}{2}$ vectors $\{\ldots, 1, \ldots, \pm3, \ldots\}$, $2\binom{d}{2}$ vectors $\{\ldots, 3, \ldots, \pm1, \ldots\}$, $4\binom{d}{3}$ vectors $\{\ldots, 1, \ldots, \pm1, \ldots, \pm2, \ldots\}$, $4\binom{d}{3}$ vectors $\{\ldots, 1, \ldots, \pm2, \ldots, \pm1, \ldots\}$, $4\binom{d}{3}$ vectors $\{\ldots, 2, \ldots, \pm1, \ldots, \pm1, \ldots\}$, and $8\binom{d}{4}$ vectors $\{\ldots, 1, \ldots, \pm1, \ldots,$

$\pm 1, \dots, \pm 1, \dots\}$. Thus, the diameter of $H_1(d,4)$ is $\binom{d}{1} + 10\binom{d}{2} + 16\binom{d}{3} + 8\binom{d}{4} = d(d^3 + 2d^2 + 2d - 2)/3$. Similarly, one can check that the sum of the first coordinates of the generators of $H_1(d,4)$ is $\binom{d-1}{0} + 16\binom{d-1}{1} + 20\binom{d-1}{2} + 8\binom{d-1}{3}$. Finally, item (v) corresponds to Proposition 2.4. □

5.2.2 Proof of Property 3.2

Proof One can check that the generators of $H_2(d,2)$ consist of $\binom{d}{1}$ unity vectors, $2\binom{d}{2}$ vectors $\{\dots, 1, \dots, \pm 1, \dots\}$, $4\binom{d}{3}$ vectors $\{\dots, 1, \dots, \pm 1, \dots, \pm 1, \dots\}$, and $8\binom{d}{4}$ vectors $\{\dots, 1, \dots, \pm 1, \dots, \pm 1, \dots, \pm 1, \dots\}$. Thus, the diameter of $H_2(d,2)$ is $\sum_{0 \le j \le 3} 2^j \binom{d}{j+1}$. Similarly, one can check that the sum of the first coordinates of the generators of $H_2(d,2)$ is $\sum_{0 \le j \le 3} 2^j \binom{d-1}{j}$. □

5.2.3 Proof of Property 3.3

Proof One can check that $H_\infty(d,1)$ has $(3^d - 1)/2$ generators consisting of all $\{-1,0,1\}$-valued vectors which first nonzero coordinate is positive. Out of the 5^d $\{-2,-1,0,1,2\}$-valued vectors, 3^d are $\{-2,0,2\}$-valued. Thus, keeping the ones which first nonzero coordinate is positive, $H_\infty(d,2)$ has $(5^d - 3^d)/2$ generators. Similarly, one can check that the sum of the first coordinates of the generators of $H_\infty(d,2)$ is $3 \times 5^d - 5 \times 3^d$. The generators (i,j) of $H_\infty(2,p)$ such that $||(i,j)||_\infty \le 1$ are $(1,0)$, $(0,1)$, $(1,1)$ and $(1,-1)$. For a given $i > 1$, there are $2\phi(i)$ generators (i,j) such that $||(i,j)||_\infty > 1$ and $j < i$. Thus, there are $4 \sum_{2 \le j \le p} \phi(j)$ generators (i,j) such that $||(i,j)||_\infty > 1$. Thus, the diameter of $H_\infty(2,p)$ is $4 \sum_{1 \le j \le p} \phi(j)$. □

5.2.4 Proof of Property 3.4

Proof One can check that the generators of $H_1^+(d,2)$ consist of $\binom{d}{1}$ unity vectors and $\binom{d}{2}$ vectors $\{\dots, 1, \dots, 1, \dots\}$. Thus, the diameter of $H_1^+(d,2)$ is $\binom{d}{1} + \binom{d}{2} = \binom{d+1}{2}$. Similarly, one can check that the sum of the first coordinates of the generators of $H_1^+(d,2)$ is d. Note that $H_1^+(2,p)$ is the Minkowski sum of the permutahedron with the $\{0,1\}^d$-cube. One can check that, in addition to the previously determined generators of $H_1^+(2,p)$, the generators of $H_1^+(d,3)$ consist of $\binom{d}{3}$ vectors $\{\dots, 1, \dots, 1, \dots, 1, \dots,\}$, $\binom{d}{2}$ vectors $\{\dots, 1, \dots, 2, \dots\}$, and $\binom{d}{2}$ vectors $\{\dots, 2, \dots, 1, \dots\}$. Thus $H_1^+(d,3)$ has $\binom{d}{3} + 3\binom{d}{2} + \binom{d}{1}$ generators. Similarly, one can check that the sum of the first coordinates of the generators of $H_1^+(d,3)$ is

$\binom{d-1}{2} + 4\binom{d-1}{1} + \binom{d}{0}$. Out of the generators of $H_1(2, p)$, $\sum_{2\leq j\leq p} \phi(j)$ have a negative coordinate. Thus, the diameter of $H_1^+(2, p)$ is $1 + \sum_{1\leq j\leq p} \phi(j)$. Similarly, one can check that the sum of the first coordinates of the generators of $H_1^+(2, p)$ is $1 + \sum_{2\leq j\leq p} j\phi(j)/2$. □

5.2.5 Proof of Property 3.5

Proof One can check that the generators of $H_2^+(d, 2)$ consist of $\binom{d}{i}$ vectors with exactly i ones for $i = 1, 2, 3$, and 4. Thus, the diameter of $H_2^+(d, 2)$ is $\binom{d+1}{2} + \binom{d+1}{4}$. Similarly, one can check that the sum of the first coordinates of the generators of $H_2^+(d, 2)$ is $\binom{d}{1} + \binom{d}{3}$. □

5.2.6 Proof of Property 3.6

Proof One can check that $H_\infty^+(d, 1)$ has $2^d - 1$ generators consisting of all $\{0, 1\}$-valued vectors except the origin. Thus, the diameter of $H_\infty^+(d, 1)$ is $2^d - 1$. Similarly, one can check that the sum of the first coordinates of the generators of $H_\infty^+(d, 1)$ is 2^{d-1}. Out of the 3^d $\{0, 1, 2\}$-valued vectors, 2^d are $\{0, 2\}$-valued. Thus, the diameter of $H_\infty^+(d, 2)$ is $3^d - 2^d$. Similarly, one can check that the sum of the first coordinates of the generators of $H_\infty^+(d, 2)$ is $3^d - 2^d$. The generators (i, j) of $H_\infty^+(2, p)$ such that $||(i, j)||_\infty \leq 1$ are $(1, 0)$, $(0, 1)$, and $(1, 1)$. For a given $i > 1$, there are $\phi(i)$ generators (i, j) such that $||(i, j)||_\infty > 1$ and $j < i$. Thus, there are $2 \sum_{2\leq j\leq p} \phi(j)$ generators (i, j) such that $||(i, j)||_\infty > 1$. Thus, the diameter of $H_\infty^+(2, p)$ is $1 + 2 \sum_{1\leq j\leq p} \phi(j)$. □

Acknowledgements The authors thank the anonymous referees, Johanne Cohen, Nathann Cohen, Komei Fukuda, and Aladin Virmaux for valuable comments and for informing us of reference [31], Emo Welzl and Günter Ziegler for helping us access Thorsten Thiele's Diplomarbeit, Dmitrii Pasechnik for pointing out reference [29], and Vincent Pilaud for pointing out that $Z_1(d, 2)$ is the permutahedron of type B_d. This work was partially supported by the Natural Sciences and Engineering Research Council of Canada Discovery Grant program (RGPIN-2015-06163), by the Digiteo Chair C&O program at Université de Paris Sud, and by the Dresner Chair at the Technion.

References

1. D. Acketa, J. Žunić, On the maximal number of edges of convex digital polygons included into an $m \times m$-grid. J. Comb. Theory A **69**, 358–368 (1995)
2. X. Allamigeon, P. Benchimol, S. Gaubert, M. Joswig, Log-barrier interior point methods are not strongly polynomial. SIAM J. Appl. Algebra Geom. **2**, 140–178 (2018)

3. A. Balog, I. Bárány, On the convex hull of the integer points in a disc, in *Proceedings of the Seventh Annual Symposium on Computational Geometry* (1991), pp. 162–165
4. S. Borgwardt, J. De Loera, E. Finhold. The diameters of transportation polytopes satisfy the Hirsch conjecture. Mathematical Programming, (to appear)
5. N. Chadder, A. Deza, Computational determination of the largest lattice polytope diameter. Electron. Notes Discrete Math. **62**, 105–110 (2017)
6. C. Colbourn, W. Kocay, D. Stinson, Some NP-complete problems for hypergraph degree sequences. Discrete Appl. Math. **14**, 239–254 (1986)
7. A. Del Pia, C. Michini, On the diameter of lattice polytopes. Discrete Comput. Geom. **55**, 681–687 (2016)
8. A. Deza, A. Deza, Z. Guan, L. Pournin, Distance between vertices of lattice polytopes. AdvOL Report 2018/1, McMaster University, 2018
9. A. Deza, A. Levin, S.M. Meesum, S. Onn, Optimization over degree sequence. arXiv:1706.03951, 2017
10. A. Deza, G. Manoussakis, S. Onn, Primitive zonotopes. Discrete Comput. Geom. **60**, 27–39 (2018)
11. A. Deza, L. Pournin, Improved bounds on the diameter of lattice polytopes. Acta Mathematica Hungarica, **154**, 457–469 (2018)
12. D. Eppstein, Zonohedra and zonotopes. Math. Educ. Res. **5**, 15–21 (1996)
13. P. Erdős, T. Gallai, Graphs with prescribed degrees of vertices (in Hungarian). Matematikai Lopak **11**, 264–274 (1960)
14. K. Fukuda, Lecture notes: Polyhedral computation. http://www-oldurls.inf.ethz.ch/personal/fukudak/lect/pclect/notes2015/
15. M. Grötschel, L. Lovász, A. Schrijver, *Geometric Algorithms and Combinatorial Optimization* (Springer, Berlin, 1993)
16. B. Grünbaum, *Convex Polytopes* (Springer, Graduate Texts in Mathematics, Berlin, 2003)
17. J. Humphreys, *Reflection Groups and Coxeter Groups* (Cambridge University Press, Cambridge Studies in Advanced Mathematics, Cambridge, 1990)
18. G. Kalai, D. Kleitman, A quasi-polynomial bound for the diameter of graphs of polyhedra. Bull. Am. Math. Soc. **26**, 315–316 (1992)
19. P. Kleinschmidt, S. Onn, On the diameter of convex polytopes. Discrete Math. **102**, 75–77 (1992)
20. C. Klivans, V. Reiner, Shifted set families, degree sequences, and plethysm. Electron. J. Comb. **15** (2008)
21. R. Liu, Nonconvexity of the set of hypergraph degree sequences. Electron. J. Comb. **20**(1) (2013)
22. M. Melamed, S. Onn, Convex integer optimization by constantly many linear counterparts. Linear Algebra Its Appl. **447**, 88–109 (2014)
23. N.L.B. Murthy, M.K. Srinivasan, The polytope of degree sequences of hypergraphs. Linear Algebra Its Appl. **350**, 147–170 (2002)
24. D. Naddef, The Hirsch conjecture is true for $(0, 1)$-polytopes. Math. Program. **45**, 109–110 (1989)
25. S. Onn, *Nonlinear Discrete Optimization*. (European Mathematical Society, Zurich Lectures in Advanced Mathematics, 2010)
26. S. Onn, U.G. Rothblum, Convex combinatorial optimization. Discrete Comput. Geom. **32**, 549–566 (2004)
27. F. Santos, A counterexample to the Hirsch conjecture. Ann. Math. **176**, 383–412 (2012)
28. N. Sloane (ed.), The on-line encyclopedia of integer sequences. https://oeis.org
29. I. Soprunov, J. Soprunova, Eventual quasi-linearity of the Minkowski length. Eur. J. Comb. **58**, 110–117 (2016)
30. N. Sukegawa, Improving bounds on the diameter of a polyhedron in high dimensions. Discrete Math. **340**, 2134–2142 (2017)
31. T. Thiele, *Extremalprobleme für Punktmengen* (Diplomarbeit, Freie Universität Berlin, 1991)
32. M. Todd, An improved Kalai-Kleitman bound for the diameter of a polyhedron. SIAM J. Discrete Math. **28**, 1944–1947 (2014)
33. G. Ziegler, *Lectures on Polytopes* (Springer, Graduate Texts in Mathematics, Berlin, 1995)