Computational determination of the largest lattice polytope diameter

Nathan Chadder\textsuperscript{1}

\textit{Department of Computing and Software}
\textit{McMaster University}
\textit{Hamilton, Canada}

Antoine Deza\textsuperscript{2}

\textit{Department of Computing and Software}
\textit{McMaster University}
\textit{Hamilton, Canada}

Abstract

A lattice \((d, k)\)-polytope is the convex hull of a set of points in dimension \(d\) whose coordinates are integers between 0 and \(k\). Let \(\delta(d, k)\) be the largest diameter over all lattice \((d, k)\)-polytopes. We develop a computational framework to determine \(\delta(d, k)\) for small instances. We show that \(\delta(3, 4) = 7\) and \(\delta(3, 5) = 9\); that is, we verify for \((d, k) = (3, 4)\) and \((3, 5)\) the conjecture whereby \(\delta(d, k)\) is at most \(\lfloor (k + 1)d/2 \rfloor\) and is achieved, up to translation, by a Minkowski sum of lattice vectors.

Keywords: Lattice polytopes, edge-graph diameter, enumeration algorithm

\textsuperscript{1} Email: chaddens@mcmaster.ca
\textsuperscript{2} Email: deza@mcmaster.ca

https://doi.org/10.1016/j.endm.2017.10.019
1571-0653/© 2017 Elsevier B.V. All rights reserved.
1 Introduction

Finding a good bound on the maximal edge-diameter of a polytope in terms of its dimension and the number of its facets is not only a natural question of discrete geometry, but also historically closely connected with the theory of the simplex method, as the diameter is a lower bound for the number of pivots required in the worst case. Considering bounded polytopes whose vertices are rational-valued, we investigate a similar question where the number of facets is replaced by the grid embedding size.

The convex hull of integer-valued points is called a lattice polytope and if all the vertices are drawn from \( \{0, 1, \ldots, k\}^d \), it is referred to as a lattice \((d, k)\)-polytope. Naddef [7] showed in 1989 that \( \delta(d, 1) = d \), Kleinschmidt and Onn [6] generalized this result in 1992 showing that \( \delta(d, k) \leq kd \). In 2016, Del Pia and Michini [3] strengthened the upper bound to \( \delta(d, k) \leq kd - \lceil d/2 \rceil \) for \( k \geq 2 \), and showed that \( \delta(d, 2) = \lfloor 3d/2 \rfloor \). Pursuing Del Pia and Michini’s approach, Deza and Pournin [5] showed that \( \delta(d, k) \leq kd - \lceil 2d/3 \rceil - (k - 3) \) for \( k \geq 3 \), and that \( \delta(4, 3) = 8 \). The determination of \( \delta(2, k) \) was investigated independently in the early nineties by Thiele [8], Balog and Bárány [2], and Acketa and Žunić [1]. Deza, Manoussakis, and Onn [4] showed that \( \delta(d, k) \geq \lfloor (k + 1)d/2 \rfloor \) for all \( k \leq 2d - 1 \) and proposed Conjecture 1.1.

Conjecture 1.1 \( \delta(d, k) \leq \lfloor (k + 1)d/2 \rfloor \), and \( \delta(d, k) \) is achieved, up to translation, by a Minkowski sum of lattice vectors.

In Section 2, we propose a computational framework which drastically reduces the search space for lattice \((d, k)\)-polytopes achieving a large diameter. Applying this framework to \((d, k) = (3, 4)\) and \((3, 5)\), we determine in Section 3 that \( \delta(3, 4) = 7 \) and \( \delta(3, 5) = 9 \).

Theorem 1.2 Conjecture 1.1 holds for \((d, k) = (3, 4)\) and \((3, 5)\); that is, \( \delta(3, 4) = 7 \) and \( \delta(3, 5) = 9 \), and both diameters are achieved, up to translation, by a Minkowski sum of lattice vectors.

Note that Conjecture 1.1 holds for all known values of \( \delta(d, k) \) given in Table 1, and hypothesizes, in particular, that \( \delta(d, 3) = 2d \). The new entries corresponding to \((d, k) = (3, 4)\) and \((3, 5)\) are entered in bold.
<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$d$</td>
<td>$d$</td>
<td>$\lfloor \frac{3d}{2} \rfloor$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1
The largest possible diameter $\delta(d, k)$ of a lattice $(d, k)$-polytope

2 Theoretical and Computational Framework

Since $\delta(2, k)$ and $\delta(d, 2)$ are known, we consider in the remainder of the paper that $d \geq 3$ and $k \geq 3$. While the number of lattice $(d, k)$-lattice polytopes is finite, a brute force search is typically intractable, even for small instances. Theorem 2.1, which recalls conditions established in [5], allows to drastically reduce the search space.

**Theorem 2.1** For $d \geq 3$, let $d(u, v)$ denote the distance between two vertices $u$ and $v$ in the edge-graph of a lattice $(d, k)$-polytope $P$ such that $d(u, v) = \delta(d, k)$. For $i = 1, \ldots, d$, let $F^0_i$, respectively $F^k_i$, denote the intersection of $P$ with the facet of the cube $[0, k]^d$ corresponding to $x_i = 0$, respectively $x_i = k$. Then, $d(u, v) \leq \delta(d - 1, k) + k$, and the following conditions are necessary for the inequality to hold with equality:

1. $u + v = (k, k, \ldots, k)$,
2. any edge of $P$ with $u$ or $v$ as a vertex is $\{-1, 0, 1\}$-valued,
3. for $i = 1, \ldots, d$, $F^0_i$, respectively $F^k_i$, is a $(d - 1)$-dimensional face of $P$ with diameter $\delta(F^0_i) = \delta(d - 1, k)$, respectively $\delta(F^k_i) = \delta(d - 1, k)$.

Thus, to show that $\delta(d, k) < \delta(d - 1, k) + k$, it is enough to show that there is no lattice $(d, k)$-polytope admitting a pair of vertices $(u, v)$ such that $d(u, v) = \delta(d, k)$ and the conditions (1), (2), and (3) are satisfied. The computational framework to determine, given $(d, k)$, whether $\delta(d, k) = \delta(d - 1, k) + k$ is outlined below and illustrated for $(d, k) = (3, 4)$ or $(3, 5)$. 

Algorithm to determine whether $\delta(d, k) < \delta(d - 1, k) + k$

**Step 1: Initialization**
Determine the set $F_{d-1,k}^*$ of all the lattice $(d - 1, k)$-polytopes $P$ such that $\delta(P) = \delta(d - 1, k)$. For example, for $(d, k) = (3, 4)$, the determination of all the 335 lattice $(2, 4)$-polygons $P$ such that $\delta(P) = 4$ is straightforward.

**Step 2: Symmetries**
Consider, up to the symmetries of the cube $[0, k]^d$, the possible entries for a pair of vertices $(u, v)$ such that $u + v = \{k, k, \ldots, k\}$. For example, for $(d, k) = (3, 4)$, the following 9 vertices cover all possibilities for $u$ up to symmetry: (0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 1, 1), (0, 1, 2), (0, 2, 2), (1, 1, 1), (1, 1, 2), and (1, 2, 2), where $v = (4, 4, 4) - u$. The set $F_{d-1,k}^*$ can be used to further reduce the search space. For example, for $(d, k) = (3, 4)$, one can check that the points (1, 2) and (2, 2) belong to any element of $F_{2,4}^*$. Thus, the points (1, 1, 2) and (1, 2, 2) can be ruled out from the possible entries for $u$.

**Step 3: Shelling**
For each of the possible pairs $(u, v)$ determined during Step 2, consider all possible ways for 2d elements of the set $F_{d-1,k}^*$ determined during Step 1 to form the 2d facets of $P$ lying on a facet of the cube $[0, k]^d$. For example, for $(d, k) = (3, 4)$ and $u = (0, 0, 0)$, we must find 6 elements of $F_{2,4}^*$, 3 with (0, 0) as a vertex, and 3 with (4, 4) as a vertex. In addition, if an edge of an element of $F_{d-1,k}^*$ with $u$ or $v$ as vertex is not $\{-1, 0, 1\}$-valued, this element is disregarded.

Note that since the choice of an element of $F_{d-1,k}^*$ defines the vertices of $P$ belonging to a facet of the cube $[0, k]^d$, the choice for the next element of $F_{d-1,k}^*$ to form a shelling is significantly restricted. In addition, if the set of vertices and edges belonging to the current elements of $F_{d-1,k}^*$ considered for a shelling includes a path from $u$ to $v$ of length at most $\delta(d - 1, k) + k - 1$, a shortcut between $u$ and $v$ exists and the last added elements of $F_{d-1,k}^*$ can be disregarded. A shortcut between $u$ and $v$ may be found using variants of Theorem 2.1 even if $u$ and $v$ are not connected by the set of current edges.

**Step 4: Inner points**
For each choice of 2d elements of $F_{d-1,k}^*$ forming a shelling obtained during Step 3, consider the $\{1, 2, \ldots, k - 1\}$-valued points not in the convex hull of the vertices of the 2d elements of $F_{d-1,k}^*$ forming a shelling. Each such
\{1, 2, \ldots, k-1\}\text{-valued point is considered as a potential vertex of }P\text{ in a binary tree. If the current set of edges includes a path from }u\text{ to }v\text{ of length at most }\delta(d-1,k)+k-1\text{, a shortcut between }u\text{ and }v\text{ exists and the corresponding node of the binary tree can be disregarded, and the the binary tree is pruned at this node.}

A convex hull and diameter computation are performed for each node of the obtained binary tree. If there is a node yielding a diameter of }\delta(d-1,k)+k\text{ we can conclude that }\delta(d,k)=\delta(d-1,k)+k.\text{ Otherwise, we can conclude that }\delta(d,k)<\delta(d-1,k)+k.\text{ For example, for }(d,k)=(3,5),\text{ no choice of }6\text{ elements of }\mathcal{F}_{2,5}^*\text{ forming a shelling such that }d(u,v)\geq10\text{ exist, and thus Step 4 is not executed.}

### 3 Computational Results

For \((d,k)=(3,4)\), shellings exist for which path lengths are not decidable by the algorithm without convex hull computations. However, none of these shellings achieves a diameter of at least 8. For \((d,k)=(3,5)\) the algorithm stops at Step 3, as there is no combination of 6 elements of }\mathcal{F}_{2,5}^*\text{ which form a shelling such that }d(u,v)\geq\delta(2,5)+5.\text{ A shortcut from }u\text{ to }v\text{ is typically found early on in the shelling, which leads to the algorithm terminating quickly. Consequently, }\delta(3,4)<8\text{ and }\delta(3,5)<10.\text{ Since the Minkowski sum of }\((1,0,0),(0,1,0),(0,0,1),(0,1,1),(1,0,1),(1,1,0)\text{, and }\((1,1,1)\text{ forms a lattice }\text{(3,4)-polytope with diameter 7, we conclude that }\delta(3,4)=7.\text{ Similarly, since the Minkowski sum of }\((1,0,0),(0,1,0),(0,0,1),(0,1,1),(1,0,1),(1,1,0),(0,1,-1),(1,0,-1),(1,-1,0)\text{ forms, up to translation, a lattice }\text{(3,5)-polytope with diameter 9, we conclude that }\delta(3,5)=9.\text{ Computations for additional values of }\delta(d,k)\text{ are currently underway. In particular, the same algorithm may determine whether }\delta(d,k)=\delta(d-1,k)+k\text{ or }\delta(d-1,k)+k-1\text{ for }\(d,k)=(5,3)\text{ and }\text{(4,4) provided the set of all lattice }\text{(d-1,k)-polytopes achieving }\delta(d-1,k)\text{ is determined for }\(d,k)=(5,3)\text{ and }\text{(4,4). Similarly, the algorithm could be adapted to determine whether }\delta(d,k)<\delta(d-1,k)+k-1\text{ provided the set of all lattice }\text{(d-1,k)-polytopes achieving }\delta(d-1,k)\text{ or }\delta(d-1,k)-1\text{ is determined. For example, the adapted algorithm may determine whether }\delta(3,6)=10.\)
Acknowledgement

This work was partially supported by the Natural Sciences and Engineering Research Council of Canada Discovery Grant program (RGPIN-2015-06163).

References


