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Computational determination of the largest lattice polytope diameter

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Abstract

A lattice (d, k)-polytope is the convex hull of a set of points in dimension d whose coordinates are integers between 0 and k. Let $\delta(d, k)$ be the largest diameter over all lattice (d, k)-polytopes. We develop a computational framework to determine $\delta(d, k)$ for small instances. We show that $\delta(3, 4) = 7$ and $\delta(3, 5) = 9$; that is, we verify for $(d, k) = (3, 4)$ and $(3, 5)$ the conjecture whereby $\delta(d, k)$ is at most $\lfloor (k + 1)d/2 \rfloor$ and is achieved, up to translation, by a Minkowski sum of lattice vectors.

Keywords: Lattice polytopes, edge-graph diameter, enumeration algorithm

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1 Introduction

Finding a good bound on the maximal edge-diameter of a polytope in terms of its dimension and the number of its facets is not only a natural question of discrete geometry, but also historically closely connected with the theory of the simplex method, as the diameter is a lower bound for the number of pivots required in the worst case. Considering bounded polytopes whose vertices are rational-valued, we investigate a similar question where the number of facets is replaced by the grid embedding size.

The convex hull of integer-valued points is called a lattice polytope and if all the vertices are drawn from $\{0, 1, \dots, k\}^d$, it is referred to as a lattice (d, k)-polytope. Let $\delta(d, k)$ be the largest edge-diameter over all lattice (d, k)-polytopes. Naddef [7] showed in 1989 that $\delta(d, 1) = d$, Kleinschmidt and Onn [6] generalized this result in 1992 showing that $\delta(d, k) \le kd$. In 2016, Del Pia and Michini [3] strengthened the upper bound to $\delta(d, k) \le kd - \lceil d/2 \rceil$ for $k \ge 2$, and showed that $\delta(d, 2) = \lfloor 3d/2 \rfloor$. Pursuing Del Pia and Michini's approach, Deza and Pournin [5] showed that $\delta(d, k) \le kd - \lceil 2d/3 \rceil - (k - 3)$ for $k \ge 3$, and that $\delta(4, 3) = 8$. The determination of $\delta(2, k)$ was investigated independently in the early nineties by Thiele [8], Balog and Bárány [2], and Acketa and Žunić [1]. Deza, Manoussakis, and Onn [4] showed that $\delta(d, k) \ge \lfloor (k + 1)d/2 \rfloor$ for all $k \le 2d - 1$ and proposed Conjecture 1.1.

Conjecture 1.1 *$\delta(d, k) \le \lfloor (k + 1)d/2 \rfloor$, and $\delta(d, k)$ is achieved, up to translation, by a Minkowski sum of lattice vectors.*

In Section 2, we propose a computational framework which drastically reduces the search space for lattice (d, k)-polytopes achieving a large diameter. Applying this framework to $(d, k) = (3, 4)$ and $(3, 5)$, we determine in Section 3 that $\delta(3, 4) = 7$ and $\delta(3, 5) = 9$.

Theorem 1.2 *Conjecture 1.1 holds for $(d, k) = (3, 4)$ and $(3, 5)$; that is, $\delta(3, 4) = 7$ and $\delta(3, 5) = 9$, and both diameters are achieved, up to translation, by a Minkowski sum of lattice vectors.*

Note that Conjecture 1.1 holds for all known values of $\delta(d, k)$ given in Table 1, and hypothesizes, in particular, that $\delta(d, 3) = 2d$. The new entries corresponding to $(d, k) = (3, 4)$ and $(3, 5)$ are entered in bold.

$d \backslash k$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	2	3	4	4	5	6	6	7	8	8
3	3	4	6	**7**	**9**					
4	4	6	8							
$\vdots$	$\vdots$	$\vdots$								
d	d	$\lfloor \frac{3d}{2} \rfloor$								

Table 1
The largest possible diameter $\delta(d, k)$ of a lattice (d, k)-polytope

2 Theoretical and Computational Framework

Since $\delta(2, k)$ and $\delta(d, 2)$ are known, we consider in the remainder of the paper that $d \geq 3$ and $k \geq 3$. While the number of lattice (d, k)-lattice polytopes is finite, a brute force search is typically intractable, even for small instances. Theorem 2.1, which recalls conditions established in [5], allows to drastically reduce the search space.

Theorem 2.1 *For $d \geq 3$, let $d(u, v)$ denote the distance between two vertices u and v in the edge-graph of a lattice (d, k)-polytope P such that $d(u, v) = \delta(d, k)$. For $i = 1, \ldots, d$, let F_i^0, respectively F_i^k, denote the intersection of P with the facet of the cube $[0, k]^d$ corresponding to $x_i = 0$, respectively $x_i = k$. Then, $d(u, v) \leq \delta(d-1, k) + k$, and the following conditions are necessary for the inequality to hold with equality:*

(1) $u + v = (k, k, \ldots, k)$,

(2) *any edge of P with u or v as a vertex is $\{-1, 0, 1\}$-valued,*

(3) *for $i = 1, \ldots, d$, F_i^0, respectively F_i^k, is a $(d-1)$-dimensional face of P with diameter $\delta(F_i^0) = \delta(d-1, k)$, respectively $\delta(F_i^k) = \delta(d-1, k)$.*

Thus, to show that $\delta(d, k) < \delta(d-1, k) + k$, it is enough to show that there is no lattice (d, k)-polytope admitting a pair of vertices (u, v) such that $d(u, v) = \delta(d, k)$ and the conditions (1), (2), and (3) are satisfied. The computational framework to determine, given (d, k), whether $\delta(d, k) = \delta(d-1, k) + k$ is outlined below and illustrated for $(d, k) = (3, 4)$ or $(3, 5)$.

Algorithm to determine whether $\delta(d,k) < \delta(d-1,k)+k$

Step 1*:* Initialization
Determine the set $\mathcal{F}^*_{d-1,k}$ of all the lattice $(d-1,k)$-polytopes P such that $\delta(P) = \delta(d-1,k)$. For example, for $(d,k) = (3,4)$, the determination of all the 335 *lattice* $(2,4)$*-polygons P such that $\delta(P) = 4$ is straightforward.*

Step 2*:* Symmetries
Consider, up to the symmetries of the cube $[0,k]^d$, the possible entries for a pair of vertices (u,v) such that $u + v = \{k,k,\ldots,k\}$. For example, for $(d,k) = (3,4)$, the following 9 vertices cover all possibilities for u up to symmetry: $(0,0,0)$, $(0,0,1)$, $(0,0,2)$, $(0,1,1)$, $(0,1,2)$, $(0,2,2)$, $(1,1,1)$, $(1,1,2)$, *and* $(1,2,2)$, *where $v = (4,4,4) - u$. The set $\mathcal{F}^*_{d-1,k}$ can be used to further reduce the search space. For example, for $(d,k) = (3,4)$, one can check that the points* $(1,2)$ *and* $(2,2)$ *belong to any element of $\mathcal{F}^*_{2,4}$. Thus, the points* $(1,1,2)$ *and* $(1,2,2)$ *can be ruled out from the possible entries for u.*

Step 3*:* Shelling
For each of the possible pairs (u,v) determined during Step 2*, consider all possible ways for* $2d$ *elements of the set $\mathcal{F}^*_{d-1,k}$ determined during Step 1 to form the* $2d$ *facets of P lying on a facet of the cube $[0,k]^d$. For example, for $(d,k) = (3,4)$ and $u = (0,0,0)$, we must find* 6 *elements of $\mathcal{F}^*_{2,4}$,* 3 *with* $(0,0)$ *as a vertex, and* 3 *with* $(4,4)$ *as a vertex. In addition, if an edge of an element of $\mathcal{F}^*_{d-1,k}$ with u or v as vertex is not $\{-1,0,1\}$-valued, this element is disregarded.*

Note that since the choice of an element of $\mathcal{F}^*_{d-1,k}$ defines the vertices of P belonging to a facet of the cube $[0,k]^d$, the choice for the next element of $\mathcal{F}^*_{d-1,k}$ to form a shelling is significantly restricted. In addition, if the set of vertices and edges belonging to the current elements of $\mathcal{F}^*_{d-1,k}$ considered for a shelling includes a path from u to v of length at most $\delta(d-1,k)+k-1$, a shortcut between u and v exists and the last added elements of $\mathcal{F}^*_{d-1,k}$ can be disregarded. A shortcut between u and v may be found using variants of Theorem 2.1 even if u and v are not connected by the set of current edges.

Step 4. Inner points
For each choice of $2d$ *elements of $\mathcal{F}^*_{d-1,k}$ forming a shelling obtained during Step* 3*, consider the $\{1,2,\ldots,k-1\}$-valued points not in the convex hull of the vertices of the* $2d$ *elements of $\mathcal{F}^*_{d-1,k}$ forming a shelling. Each such*

$\{1, 2, \ldots, k-1\}$*-valued point is considered as a potential vertex of* P *in a binary tree. If the current set of edges includes a path from* u *to* v *of length at most* $\delta(d-1, k)+k-1$*, a shortcut between* u *and* v *exists and the corresponding node of the binary tree can be disregarded, and the the binary tree is pruned at this node.*

A convex hull and diameter computation are performed for each node of the obtained binary tree. If there is a node yielding a diameter of $\delta(d-1, k) + k$ *we can conclude that* $\delta(d, k) = \delta(d-1, k) + k$*. Otherwise, we can conclude that* $\delta(d, k) < \delta(d-1, k) + k$*. For example, for* $(d, k) = (3, 5)$*, no choice of* 6 *elements of* $\mathcal{F}^*_{2,5}$ *forming a shelling such that* $d(u, v) \geq 10$ *exist, and thus Step 4 is not executed.*

3 Computational Results

For $(d, k) = (3, 4)$, shellings exist for which path lengths are not decidable by the algorithm without convex hull computations. However, none of these shellings achieves a diameter of at least 8. For $(d, k) = (3, 5)$ the algorithm stops at Step 3, as there is no combination of 6 elements of $\mathcal{F}^*_{2,5}$ which form a shelling such that $d(u, v) \geq \delta(2, 5)+5$. A shortcut from u to v is typically found early on in the shelling, which leads to the algorithm terminating quickly. Consequently, $\delta(3, 4) < 8$ and $\delta(3, 5) < 10$. Since the Minkowski sum of $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, and $(1, 1, 1)$ forms a lattice $(3, 4)$-polytope with diameter 7, we conclude that $\delta(3, 4) = 7$. Similarly, since the Minkowski sum of $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, $(0, 1, -1)$, $(1, 0, -1)$, and $(1, -1, 0)$ forms, up to translation, a lattice $(3, 5)$-polytope with diameter 9, we conclude that $\delta(3, 5) = 9$. Computations for additional values of $\delta(d, k)$ are currently underway. In particular, the same algorithm may determine whether $\delta(d, k) = \delta(d-1, k)+k$ or $\delta(d-1, k)+k-1$ for $(d, k) = (5, 3)$ and $(4, 4)$ provided the set of all lattice $(d-1, k)$-polytopes achieving $\delta(d-1, k)$ is determined for $(d, k) = (5, 3)$ and $(4, 4)$. Similarly, the algorithm could be adapted to determine whether $\delta(d, k) < \delta(d-1, k) + k - 1$ provided the set of all lattice $(d-1, k)$-polytopes achieving $\delta(d-1, k)$ or $\delta(d-1, k) - 1$ is determined. For example, the adapted algorithm may determine whether $\delta(3, 6) = 10$.

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References

[1] Dragan Acketa and Joviša Žunić, *On the maximal number of edges of convex digital polygons included into an* $m \times m$*-grid*, Journal of Combinatorial Theory A **69** (1995), 358–368.

[2] Antal Balog and Imre Bárány, *On the convex hull of the integer points in a disc*, Proceedings of the Seventh Annual Symposium on Computational Geometry (1991), 162–165.

[3] Alberto Del Pia and Carla Michini, *On the diameter of lattice polytopes*, Discrete and Computational Geometry **55** (2016), 681–687.

[4] Antoine Deza, George Manoussakis, and Shmuel Onn, *Primitive zonotopes*, Discrete and Computational Geometry (to appear).

[5] Antoine Deza and Lionel Pournin, *Improved bounds on the diameter of lattice polytopes*, arXiv:1610.00341 (2016).

[6] Peter Kleinschmidt and Shmuel Onn, *On the diameter of convex polytopes*, Discrete Mathematics **102** (1992), 75–77.

[7] Dennis Naddef, *The Hirsch conjecture is true for* $(0,1)$*-polytopes*, Mathematical Programming **45** (1989), 109–110.

[8] Torsten Thiele, *Extremalprobleme für Punktmengen*, Master thesis, Freie Universität, Berlin, 1991.