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McMullen's Conditions and Some Lower Bounds for General Convex Polytopes

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Abstract. We give a lower bound for the number of vertices of a general d-dimensional polytope with a given number m of i-faces for each $i = 0, \dots, \lfloor d/2 \rfloor - 1$. The tightness of those bounds is proved using McMullen's conditions. For m greater than a small constant, those lower bounds are attained by simplicial i-neighbourly polytopes.

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1. Main Theorem

Convex polytopes are the d-dimensional analogues of 2-dimensional convex polygones and 3-dimensional convex polyhedra. A *polytope* is a bounded convex set in R^d that is the intersection of a finite number of closed halfspaces. The *faces* of a polytope are its intersections with supporting hyperplanes. The i-dimensional faces are called the *i-faces* and $f_i(P)$ denotes the number of *i-faces* of a polytope P; the d-tuple $(f_0(P),\ f_1(P), \dots, f_{d-1}(P))$ is called the *f-vector* of P. In particular, 0-faces, 1-faces and $(d-1)$-faces are respectively called *vertices*, *edges* and *facets* of a d-dimensional polytope. One of the most important question in the combinatorial theory of convex polytopes is the determination of the largest and the smallest number of i-faces of a d-dimensional polytope with a given number of k-faces. Moreover, it is also interesting to find out which class of polytopes attains these bounds. General references to the topics discussed in our paper are [5], [6], [9]. In this section we first recall McMullen's upper bound theorem and Barnette's lower bound theorem for simplicial polytopes. Then we present our lower bounds for general convex polytopes.

The upper bound theorem was conjectured by Motzkin [10] in 1957 and proved by McMullen [7] in 1970. In order to state this theorem, we define for $i \geq 0$:

$$u_i^d(m) = \sum_{j=0}^{d'} \binom{j}{d-i-1} \binom{m-d+j-1}{j}$$

$$+\sum_{j=0}^{d''}\binom{d-j}{d-i-1}\binom{m-d+j-1}{j}, \tag{1}$$

where $d' = \lfloor d/2 \rfloor$ and $d'' = \lfloor (d-1)/2 \rfloor$. Note that $d = d' + d'' + 1$.

We also recall that, with k a nonnegative integer, a polytope P is *k-neighbourly* if every k-subset of the set of vertices of P is the vertex set of a proper face of P. A $\lfloor d/2 \rfloor$-neighbourly polytope is simply called a *neighbourly* polytope.

With those notations the upper bound theorem can be stated as follows:

THEOREM 1.1 [7]. *For any d-dimensional polyope P with m vertices we have:*

$$f_i(P) \le u_i^d(m) \quad \textit{for } i = 0, \dots, d-1.$$

Furthermore, if P is a neighbourly simplicial polytope, then

$$f_i(P) = u_i^d(m) \quad \textit{for } i = 0, \dots, d-1.$$

Remark 1.2. Some calculation shows that:

$$u_i^d(m) = \binom{m}{i+1} \quad \text{for } i = 0, \dots, d'-1.$$

The lower bound theorem was proved by Barnette [1], [2] in 1971–1973. As for the upper bound theorem, we first need to define:

$$\varphi_i^d(m) = \begin{cases} (d-1)m - (d+1)(d-2) & \text{if } i = d-1; \\ \binom{d}{i} m - \binom{d+1}{i+1} i & \text{if } i = 0, \dots, d-2. \end{cases}$$

With this notation the lower bound theorem can be stated as follows:

THEOREM 1.3 ([1], [2]). *For any simplicial d-dimensional polytope P with m vertices we have:*

$$f_i(P) \ge \varphi_i^d(m) \quad \textit{for } i = 0, \dots, d-1.$$

Furthermore, there are simplicial polytopes P with m vertices such that

$$f_i(P) = \varphi_i^d(m) \quad \textit{for } i = 0, \dots, d-1.$$

While the upper bound theorem is valid for general convex polytopes, the lower bound theorem holds only for simplicial polytopes. In the next theorem we present lower bounds valid for general polytopes.

First we define for $i = 0, \ldots, \lfloor d/2 \rfloor - 1$ the following step functions $l_i^d(m)$ by the relation:

$$l_i^d(m) = k \quad \text{if and only if} \quad \binom{k-1}{i+1} < m \leq \binom{k}{i+1}.$$

Those functions are a sort of inverse functions of $u_i^d(m)$. Moreover, we see in Section 4 that one can easily prove that $l_i^d(m)$ is a lower bound for the number of vertices of a polytope with m i-faces. The following theorem actually establishes the tightness of $l_i^d(m)$ and characterizes the class of polytopes which attain those lower bounds.

THEOREM 1.4. *For any d-dimensional polytope P with m i-faces we have*:

$$f_0(P) \geq l_i^d(m) \quad \textit{for } i = 0, \ldots; \lfloor d/2 \rfloor - 1.$$

Furthermore, for m greater than a small constant c_i^d, there are simplicial i-neighbourly polytopes P with m i-faces such that:

$$f_0(P) = l_i^d(m) \quad \textit{for } i = 0, \ldots, \lfloor d/2 \rfloor - 1.$$

Remark 1.5. One can easily check that:

(i) for each $i = 0, \ldots, \lfloor d/2 \rfloor - 1$, for d fixed, $l_i^d(m)$ is $O(\sqrt[i+1]{m})$,

(ii) $c_1^d = 3/2(d-1)(2d-1)$.

Before giving a complete proof of Theorem 1.4 in Section 3, we first recall in Section 2 the characterization of f-vector for simplicial polytopes. This characterization is used to prove the tightness of those lower bounds for general polytopes.

2. Characterization of the f-Vector of a Simplicial Polytope

In this section we present a characterization of the f-vector of a simplicial polytope. This characterization, called McMullen's conditions, was conjectured by McMullen [8] in 1971. The sufficiency of the conditions was proved by Billera and Lee [3], [4] in 1980–1981; the necessity was established by Stanley [11] in 1980.

For a d-tuple $f = (f_0, f_1, \ldots, f_{d-1})$ of positive integers, we define the associated g-vector as:

$$g_j = \sum_{k=-1}^{j-1} (-1)^{j-k-1} \binom{d-k}{d-j+1} f_k \quad \text{for } j = 0, \ldots, d+1,$$

with the conventions $f_{-1} = 1$ and $f_d = 0$.

Some calculation [9] shows that

$$f_j = \sum_{k=0}^{j+1} \binom{d-k+1}{d-j} g_k \quad \text{for } j = 0, \ldots, d-1. \tag{2}$$

For positive integers h and i, there exist uniquely determined positive integers $r_0, r_1, \ldots, r_q$ with $q < i$ such that

$$h = \binom{r_0}{i} + \binom{r_1}{i-1} + \cdots + \binom{r_q}{i-q}.$$

This representation is called the *i-canonical* representation of h. The i-canonical representation of 0 is 0.

Then, for $j > i$, $h^{\langle j|i \rangle}$ is defined by:

$$h^{\langle j|i \rangle} = \binom{r_0+j-i}{j} + \binom{r_1+j-i}{j-1} + \cdots + \binom{r_q+j-i}{j-q}.$$

We also recall that $d' = \lfloor d/2 \rfloor$ and $d'' = \lfloor (d-1)/2 \rfloor$.

THEOREM 2.1 ([3], [4], [11]). *A d-tuple $f = (f_0, f_1, \ldots, f_{d-1})$ of positive integers is the f-vector of a simplicial polytope if and only if the associated g-vector satisfies the following three conditions*:

$$\begin{aligned}
&(c_1)\ g_j = -g_{d-j+1} && \textit{for } j = 0, \ldots, d''+1,\\
&(c_2)\ g_j \geq 0 && \textit{for } j = 1, \ldots, d' \textit{and } g_0 = 1,\\
&(c_3)\ g_j \leq g_{j-1}^{\langle j|j-1 \rangle} && \textit{for } j = 2, \ldots, d'.
\end{aligned}$$

Remark 2.2. For a simplicial i-neighbourly polytope P, the dimension d and the number of vertices $f_0(P)$ are sufficient to determine the first i terms of the f-vector of P; moreover, for $k = 0, \ldots, i-1$ we have:

$$f_k(P) = \binom{f_0}{k+1}.$$

Restating this in term of g_k, we can say that for a simplicial i-neighbourly polytope, the first i terms of the g-vector: $g_1, \ldots, g_i$ are fully determined by the dimension d and g_1. Moreover, some calculation shows that

$$g_k(P) = \binom{g_1+k-1}{k} \quad \text{for } k = 1, \ldots, i.$$

3. Lower Bound for the Number of Vertices of a Convex Polytope with m i-faces

This section is devoted to the proof of Theorem 1.4. First we explain why we chose $l_i^d(m)$ as a candidate to be the lower bound for the number of vertices for a convex polytope with m i-faces. Then we introduce a family of i-neighbourly simplicial polytopes. Finally, using this family, we prove the tightness of $l_i^d(m)$.

First, using Theorem 1.1 and Remark 1.2, we get the upper bound for the number m of i-faces of a d-dimensional polytope P with f_0 vertices:

$$m \leq \binom{f_0}{i+1} \quad \text{for } i = 0, \ldots, d' - 1. \tag{3}$$

This last inequality led us to define $l_i^d(m)$ as the step function presented in Section 2. Indeed, inequality (3) implies that a polytope, with m i-faces, such as

$$m > \binom{f_0}{i+1},$$

has necessarily at least $f_0 + 1$ vertices. Therefore, with $f_0(P)$ denoting the number of vertices of a d-dimensional polytope P with m i-faces, we have:

$$f_0(P) \geq l_i^d(m) \quad \text{for } i = 0, \ldots, d' - 1,$$

which means that $l_i^d(m)$ is a lower bound for the number of vertices of a polytope with m i-faces. Now, we have to prove that this lower bound is attained. First, we recall that (3) is satisfied with equality if P is a neighbourly polytope with f_0 vertices, i.e. for

$$m = \binom{f_0}{i+1}, \; f_0 \geq d + 1.$$

Therefore the lower bound $l_i^d(m)$ is attained for

$$m = \binom{f_0}{i+1}$$

by neighbourly polytopes with f_0 vertices since, obviously,

$$l_i^d\left(\binom{f_0}{i+1}\right) = f_0.$$

In other words, $l_i^d(m)$ is a tight lower bound for

$$m = \left(\binom{f_0}{i+1}\right), \; f_0 \geq d + 1.$$

Our objective is to prove the tightness of this lower bound for other values of m, i.e. for

$$\binom{f_0 - 1}{i+1} < m < \binom{f_0}{i+1}$$

with $f_0 > d+2$. In order to do so we introduce a family of polytopes with suitable properties.

For each $i = 0,\dots,d'-1$, we consider a d-tuple $f = (f_0, f_1,\dots,f_{d-1})$ such that the associated g-vector $g = (g_0,\dots,g_{d+1})$ satisfies the following conditions:

$$(\mathrm{a}_1^i)\ g_j = -g_{d-j+1} \qquad \text{for } j = 0,\dots,d''+1,$$

$$(\mathrm{a}_2^i)\ g_j = \binom{g_1+j-1}{j} \qquad \text{for } j = 0,\dots,i, \text{ with } g_1 \geq 0,$$

$$(\mathrm{a}_3^i)\ g_{i+1} = \binom{g_1+i}{i+1} - \delta \qquad \text{with } \delta \in \left\{0, 1,\dots, \binom{g_1+i}{i+1}\right\},$$

$$(\mathrm{a}_4^i)\ g_j = 0 \qquad \text{for } i+1 < j < d'+1.$$

We recall that the associated g-vector is given by:

$$g_j = \sum_{k=-1}^{j-1} (-1)^{j-k-1} \binom{d-k}{d-j+1} f_k \quad \text{for } j = 0,\dots,d+1,$$

with the conventions $f_{-1} = 1$ and $f_d = 0$.

LEMMA 3.1. *For each $i = 0,\dots,d'-1$, a d-tuple f such that the associated g-vector satisfies conditions* (a_1^i), (a_2^i), (a_3^i) *and* (a_4^i) *is the f-vector of a simplicial polytope.*

Proof. To prove it we just need to check the McMullen's conditions presented in Section 3 are satisfied. (c_1), respectively (c_2), obviously holds using (a_1^i), respectively (a_2^i), (a_3^i) and (a_4^i). To check (c_3), we first have to calculate the $(j-1)$-canonical representation of g_{j-1} for $j = 2\dots,d'$. Using (a_2^i) we have:

$$g_{j-1} = \binom{g_1+j-2}{j-1} \quad \text{for } j = 2,\dots,i+1,$$

then the $(j-1)$-canonical representation of g_{j-1} is obviously

$$\binom{g_1+j-2}{j-1} = \binom{g_1+j-2}{j-1} \quad \text{for } j = 2,\dots,i+1,$$

that is,

$$g_{j-1}^{\langle j|j-1\rangle} = \binom{g_1+j-1}{j} \quad \text{for } j = 2,\dots,i+1,$$

thus

$$g_{j-1}^{\langle j|j-1\rangle} = g_j \quad \text{for } j = 2,\dots,i. \tag{4}$$

(4) implies that (c$_3$) holds for $j = 2, \ldots, i$. As (a$_4^i$) implies that (c$_3$) holds for $j = i+2, \ldots, d'$, to complete the proof we have to check that (c$_3$) holds for $j = i+1$. Using (4) we notice that (a$_3^i$) can be read as:

$$g_{i+1} \leq g_i^{\langle i+1|i\rangle},$$

which is the desired inequality and completes the proof. □

The next lemma give us more details about the f-vector of a simplicial polytope such that the associated g-vector satisfies conditions (a$_1^i$), (a$_2^i$), (a$_3^i$) and (a$_4^i$).

LEMMA 3.2. *For each $i = 0, \ldots, d'-1$, let P_i^δ be a polytope of the class of simplicial polytopes such that the associated g-vector satisfies the conditions* (a$_1^i$), (a$_2^i$), (a$_3^i$) *and* (a$_4^i$); *we have*:

(i) *P_i^δ is an i-neighbourly polytope with $g_1 + d + 1$ vertices.*

(ii) *P_i^δ has $\binom{g_1+d+1}{i+1} - \delta$ i-faces.*

Proof. Since g_j for $j = 0, \ldots, d+1$ are given by (a$_1^i$), (a$_2^i$), (a$_3^i$) and (a$_4^i$); we are able to calculate the f-vector of P_i^δ using (2). If we set $j = 0$ in (2), we have

$$f_0(P_i^\delta) = d + 1 + g_1,$$

thus P_i^δ has $g_1 + d + 1$ vertices.

Then, to determine the degree of neighbourliness of P_i^δ, using Remark 2.2, we notice that (a$_3^i$) means that P_i^δ has the same g_k as a simplicial neighbourly polytope for $k = 0, \ldots, i$. Now, using (2), we remark that for $j = 0, \ldots, i-1$, f_j depends only on g_k for $k = 0, \ldots, j+1$. This implies that the $f_j(P_i^\delta)$ are the same as for a neighbourly polytope with $g_1 + d + 1$ vertices for $j = 0, \ldots, i-1$, which means that P_i^δ is an i-neighbourly polytope and completes the proof of part (i) of Lemma 3.2. Moreover, using the same argument, we obtain that for each $i = 0, \ldots, d'-1$, P_i^0 is an $(i+1)$-neighbourly polytope with $g_1 + d + 1$ vertices.

To complete the proof of Lemma 3.2, we have to evaluate $f_i(P_i^\delta)$, the number of i-faces of P_i^δ. Using (2), we have:

$$\begin{aligned} f_i(P_i^\delta) &= \sum_{k=0}^{i+1} \binom{d-k+1}{d-i} g_k \\ &= \sum_{k=0}^{i} \binom{d-k+1}{d-i} g_k + g_{i+1} \\ &= \sum_{k=0}^{i} \binom{d-k+1}{d-i} g_k + \binom{g_1+i}{i+1} - \delta \quad (\text{using}(a_3^i)). \end{aligned} \tag{5}$$

Since P_i^0 is an $(i+1)$-neighbourly polytope with g_1+d+1 vertices, we have:

$$f_i(P_i^\delta) = \binom{g_1+d+1}{i+1} \quad \text{for } \delta = 0,$$

and this, together with (5), implies:

$$f_i(P_i^\delta) = \binom{g_1+d+1}{i+1} - \delta,$$

which completes the proof of Lemma 3.2. □

PROOF OF THEOREM 1.4. At the beginning of this section we noticed that $l_i^d(m)$ was a lower bound for the number of vertices of a polytope with m i-faces. Then we added that $l_i^d(m)$ was attained for

$$m = \binom{f_0}{i+1}$$

by neighbourly polytopes with f_0 vertices, $f_0 \geq d+1$. Therefore, to complete the proof of Theorem 1.4 we need to fill the gap between

$$\binom{f_0}{i+1} \quad \text{and} \quad \binom{f_0-1}{i+1}$$

with polytopes having f_0 vertices, $f_0 > d+1$. The candidates are, of course, P_i^δ with $g_1 = f_0 - d - 1$.

Lemma 3.2 implies that, for a given $g_1 = f_0 - d - 1$, as δ increases by 1 from

$$0 \quad \text{to} \quad \binom{f_0-d-1+i}{i+1},$$

the number of i-faces of P_i^δ, decreases by 1 from

$$\binom{f_0}{i+1} \quad \text{to} \quad \binom{f_0}{i+1} - \binom{f_0-d-1+i}{i+1}.$$

As P_i^δ has f_0 vertices, these numbers completely fill the gap between two neighbourly polytopes with f_0-1 and f_0 vertices if the following inequality holds:

$$\binom{f_0-d-1+i}{i+1} + 1 \geq \binom{f_0}{i+1} - \binom{f_0-1}{i+1}.$$

Hence

$$\binom{f_0 - d - 1 + i}{i + 1} + 1 \geq \binom{f_0 - 1}{i}.$$

For a given dimension d, the left-hand side is $O(f_0^{i+1})$ while the right-hand side is $O(f_0^i)$. Therefore, for f_0 greater than some constant, i.e. for m greater than a constant c_i^d, the above inequality holds. In other words, $l_i^d(m)$ is attained by i-neighbourly polytopes for $m \geq c_i^d$, which completes the proof of Theorem 1.4. □

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