Solutions to the Oberwolfach problem for orders 18 to 40

A. Deza\textsuperscript{1}, F. Franek\textsuperscript{1}, W. Hua\textsuperscript{1}, M. Meszka\textsuperscript{2}, and A. Rosa\textsuperscript{3}

\textsuperscript{1} Department of Computing and Software
McMaster University, Hamilton, Ontario, Canada L8S 4K1
email: deza/franek/hua@mcmaster.ca
\textsuperscript{2} Faculty of Applied Mathematics
AGH University of Science and Technology
al. Mickiewicza 30 30-059 Kraków, Poland
email: meszka@agh.edu.pl
\textsuperscript{3} Department of Mathematics and Statistics
McMaster University, Hamilton, Ontario, Canada L8S 4K1
email: rosa@mcmaster.ca

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Abstract. The Oberwolfach problem (OP) asks whether $K_n$ (for $n$ odd) or $K_n$ minus a 1-factor (for $n$ even) admits a 2-factorization where each 2-factor is isomorphic to a given 2-factor $F$. The order $n$ and the type of the 2-factor $F$ are the parameters of the problem. For $n \leq 17$, the existence of a solution has been resolved for all possible parameters. There are also many special types of 2-factors for which solutions to OP are known. We provide solutions to OP for all orders $n$, $18 \leq n \leq 40$.

The computational results for higher orders were obtained using the SHARCNET high-performance computing cluster.

1 Introduction

A 2-factor of a graph $G$ is a 2-regular spanning subgraph of $G$. A 2-factorization of $G$ is an edge-disjoint partition of the edge set of $G$ into 2-factors. Determining if the complete graph $K_{2k+1}$ has a 2-factorization where the 2-factors are isomorphic to each other is known as the Oberwolfach problem. The problem, when generalized to graphs of even order, asks if $K_{2k} \setminus I$, where $I$ is a 1-factor, has a 2-factorization where the 2-factors are isomorphic to each other.

More specifically, an instance OP($n; a_1, \ldots, a_m$) of the Oberwolfach problem asks if there is a 2-factorization of $K_n$ ($K_n \setminus I$ for even $n$) such that each 2-factor is isomorphic to $C_{a_1} \cup \ldots \cup C_{a_m}$. Decomposing a graph of order $n$ into 2-factors necessitates that we have $\sum a_i = n$, $a_i \geq 3$.

The problem was first introduced by Gerhard Ringel and named the Oberwolfach problem as it was inspired by a question whether participants at a mathematical meeting at the Oberwolfach Institute could be seated during various dinners at the conference so that everybody would sit next to any other participant exactly once.
Since the problem was introduced, many papers on the topic have appeared. With the exception of four cases (OP(6; 3^2), OP(12; 3^4), OP(9; 4, 5), and OP(11; 3^2, 5)) for which solutions are known not to exist, solutions were produced for all orders \( n \leq 17 \) (see \[AB06\]) and for many special cases (for instance OP(\(n; r^k, n-r k\)) for all \( n \geq 6kr-1 \), see \[HJ01\]). A comprehensive survey by B. Alspach can be found in \[CD96\], with more up-to-date results in \[CD06\].

2 Methods

If we consider looking for solutions computationally, the naive brute-force approach runs in \( O((n!)^{\frac{n-1}{2}}) \) time, and the problem is already intractable for \( n \geq 18 \). We present five methods of constructing possible solutions for different instances, all but one depending only on \( n \). These methods facilitate significant reduction of the search and were successfully used to construct solutions for all orders between 18 and 40. More precisely, we were able to construct solutions for OP(\(n; \cdot\)), 18 \( \leq n \leq 40 \), with the exception of OP(18; 3^6) and OP(33; 3^11); however, solutions for both of these instances are well known since these correspond to an NKTS(18), a nearly Kirkman triple system of order 18 (cf., e.g., \[VS93,MR01\]), and to a KTS(33), a Kirkman triple system of order 33 (cf., e.g., \[RW71\]).

2.1 The method used for \( n \equiv 3 \pmod{4} \)

This case is the simplest, and most of the remaining cases are variations of this one. Rather than search for \( k = \frac{n-1}{2} \) edge-disjoint 2-factors of \( K_n \), we seek a single base 2-factor satisfying certain properties. Developing then this base 2-factor according to a prescribed group (i.e. letting this group act on the base 2-factor) produces the remaining 2-factors of the 2-factorization.

We identify the vertex set of \( K_n \) with the set \( V = Z_k \times \{1, 2\} \cup \{\infty\} \), and let \( \alpha : V \to V \) be such that \( \alpha(\infty) = \infty \), \( \alpha(j_i) = j_i + 1 \pmod{k} \), \( j_i \in Z_k, i = 1, 2 \).

We then apply Bose’s well-known “method of pure and mixed differences” (cf., e.g., \[MH86\]).

An edge of \( F \) between \( s, t \in Z_k \times \{i\} \) \( (i = 1, 2) \), is of pure difference \( j \), \( j \leq \frac{k}{2} \), of type \( i \) if and only if \( t - s \equiv j \pmod{k} \). An edge of \( F \) between \( s \in Z_k \times \{1\}, t \in Z_k \times \{2\} \) is of mixed difference \( j \) if and only if \( t - s \equiv j \pmod{k} \). Let us call an edge joining \( \infty \) to an element of \( Z_k \times \{i\} \) an \( i \)-infinity edge.

Necessary and sufficient conditions for the base 2-factor \( F \) to produce a 2-factorization are:

1. \( F \) contains exactly one \( i \)-infinity edge for \( i = 1, 2 \)
2. \( F \) contains exactly one edge of pure difference \( j \) of type \( i \) for \( i = 1, 2 \), \( 1 \leq j \leq \frac{k-1}{2} \)
3. \( F \) contains exactly one edge of mixed difference \( j \) for \( 0 \leq j < k \)
With the given $\alpha$ which is an automorphism of the resulting 2-factorization, and the conditions on the base 2-factor, exhibiting solutions for these instances is much faster because of the reduced search space. Finding a base 2-factor at this point can be done using brute-force backtracking with reasonable pruning.

This method was successfully used for orders 19, 23, 27, 31, 35, and 39.

2.2 The method for $n \equiv 0 \pmod{4}$

Similarly to the first case, we seek a base 2-factor which yields, upon an action of a group of order $k = \frac{n-2}{2}$ on it, a 2-factorization of $K_n \setminus I$. Here we identify the vertex set of $K_n$ with $Z_k \times \{1, 2\} \cup \{\infty_1, \infty_2\}$, and let $\alpha : V \to V$ be given by $\alpha(\infty_i) = \infty_1$, $\alpha(j_i) = j_i + 1 \pmod{k}$, $j \in Z_k, i = 1, 2$.

The conditions on the base 2-factor $F$ for a 2-factorization remain the same, except that exactly one mixed difference is forbidden from $F$; the edges with this mixed difference, together with the edge $\{\infty_1, \infty_2\}$ form the 1-factor $I$ deleted from $K_n$. In our computations for this case, we forbade the mixed difference 1.

This method was successfully used for orders 20, 24, 28, 32, 36, and 40.

2.3 The method for $n \equiv 2 \pmod{4}$

This method is the same as in the previous case, except that instead of forbidding a mixed difference, we forbid the pure differences $\frac{k}{2}$ of type $i$, $i = 1, 2$.

This method was successfully used for orders 18, 22, 26, 30, 34 and 38 with the exception of OP(18; 3\(6\)) which, as stated before, is already known.

2.4 The method for $n \equiv 1 \pmod{4}$

The method from the first case, where $n \equiv 3 \pmod{4}$, can be generalized to obtain a method for $n \equiv 1 \pmod{4}$. After choosing an infinity element $\infty$, the remaining vertices are partitioned into $r$ sets of size $w = \frac{n-1}{r}$, where $r$ is the largest power of two that divides $n-1$.

Let the $r$ sets be $Z_w \times \{i\}, 1 \leq i \leq r$. We now seek $\frac{r}{2}$ base 2-factors in $K_n$ satisfying certain conditions outlined below such that under the automorphism $\alpha$ given by $\alpha(\infty) = \infty_1$, $\alpha(j_i) = j_i + 1 \pmod{w}$, $j \in Z_w, i = 1, \ldots, r$, a 2-factorization of $K_n$ (with all 2-factors isomorphic) results. Such a 2-factorization is said to be $r$-rotational. In this sense, a solution in the case where $n \equiv 3 \pmod{4}$ is a 2-rotational 2-factorization.

Generalizing from section 2.1, we say that an edge between $s \in Z_w \times \{i\}, t \in Z_w \times \{j\}$ has mixed difference $k$ of type $(i, j)$ provided $t - s \equiv k \pmod{w}$.

Necessary and sufficient conditions on the 2-factors for an $r$-rotational solution are:

1. Exactly one edge in the union of the 2-factors is an $i$-infinity edge, $1 \leq i \leq r$
2. Exactly one edge in the union of the 2-factors has a pure difference $j$ of type $i$, $1 \leq i \leq r$, $1 \leq j \leq \frac{w-1}{2}$
3. Exactly one edge in the union of the 2-factors has a mixed difference $k$ of type $(i, j)$, $1 \leq i < j \leq r$, $0 \leq k < w$

This method was successfully used for orders 21, 25, 29, and 37. One will note that for $n = 33$, we have $r = 32$, and so the method here is reduced to an exhaustive search. In this case, we show, by a different method, how to exhibit solutions for all instances except for OP(33;311).

**Different approach for $n \equiv 1 \pmod{4}$** Another method can be used to deal with the case $n \equiv 1 \pmod{4}$. Suppose that $w = \frac{w-1}{2}$; hence $w$ is even. After choosing the infinity element $\infty$, partition the remaining vertices into two sets $Z_w \times \{i\}, i = 1, 2$. To construct a 2-factor $F$, consider two cases.

Suppose first that $F$ contains a cycle of length at least 5. We need $F$ to satisfy the following conditions:

1. $F$ has exactly one $i$-infinity edge for $i = 1, 2$
2. One component of $F$ of size at least 5 contains the sequence of vertices $(0, 1, (\frac{w}{2}), 2, 0)$$
3. $F$ contains exactly one edge of each pure difference of type 1 and exactly one edge of each pure difference of type 2
4. Every mixed difference except $\frac{w}{2}$ appears exactly once in $F$

The action of the permutation $\alpha$ on the vertex set of $F$ produces $w$ 2-factors $F_0, F_1, \ldots, F_{w-1}$, each isomorphic to $F$. To get a required 2-factorization, in the 2-factor $F_k$, for $k = \frac{w}{2}, \frac{w}{2} + 1, \ldots, w - 1$, we have to replace the path $k_1, (\frac{w}{2} + k_1)\times (\frac{w}{2} + k_2)$, with $k_1, (\frac{w}{2} + k_2), (\frac{w}{2} + k_1), k_2$. This operation replaces two edges of pure difference $\frac{w}{2}$ with two edges of mixed difference $\frac{w}{2}$ in each 2-factor $F_k$, and yields a proper 2-factorization.

Consider now the case when $F$ contains one cycle of length 4 and one cycle of length 3. We want to find $F'$ satisfying the following conditions.

1. $F'$ contains the cycle $0, 1, \infty, (\frac{w}{2}), 2, 0_1$, where $j \neq 0, \frac{w}{2}$
2. $F'$ contains the cycle $0_2, (\frac{w}{2}), 2, i_1, 0_2$, where $i \neq 0, \frac{w}{2}$
3. $F'$ contains exactly one edge of each pure difference in $1, 2, \ldots, \frac{w}{2} - 1$ of type 1, and exactly one edge of each pure difference in $1, 2, \ldots, \frac{w}{2}$ of type 2
4. Every mixed difference appears exactly once in $F'$

Similarly to the above, $\alpha$ produces $w$ 2-factors $F'_0, F'_1, \ldots, F'_{w-1}$. To get a proper 2-factorization, in the 2-factor $F'_k$, for $k = \frac{w}{2}, \frac{w}{2} + 1, \ldots, w - 1$, we have to replace the path $k_1, \infty, (\frac{w}{2} + k_1)$ with the edge $k_1, (\frac{w}{2} + k_1)$ and moreover the edge $k_2, (\frac{w}{2} + k_2)$ with the path $k_2, \infty, (\frac{w}{2} + k_2)$. Notice that such replacement does not change the structure of $F'_k$.

The presented method was successfully used for $n = 33$ in all cases except in one case where the 2-factor has to have all its cycles of length 3.
3 Conclusion

Using variations on the 2-rotational approach, we presented five methods for restricting a search for possible solutions for the Oberwolfach problem. The methods were successful and we were able to obtain by computer search solutions for all orders $18 \leq n \leq 40$, with two exceptions, both of which have known solutions. For the higher orders we used SHARCNET computing facilities, due to the size of the search. Our results further substantiate the conjecture that $\text{OP}(6; 3^2)$, $\text{OP}(9; 4, 5)$, $\text{OP}(11; 3^2, 5)$, $\text{OP}(12; 3^4)$ are the only instances which have no solutions.

Since each order has many types of 2-factors to be considered (and the bigger the order, the more types to deal with, see Table 1), a complete listing of results is too large for inclusion in this paper. In the appendix, we list some selected results. A comprehensive set of results can be found available at http://optlab.mcmaster.ca/~oberwolfach/.

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References


A Appendix

A.1 Selected Results

Only selected results are listed here; a comprehensive listing can be found at http://optlab.mcmaster.ca/~oberwolfach/.

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Table 1. Number of OP instances by order

\[(0_1 \infty_1 0_2 1_1 2_1 4_1 7_1 1_2 3_1 4_2 5_2 7_2 \infty_2 5_1 2_2 6_1 6_2 3_2)\]
\[(0_1 \infty_1 0_2 1_1 2_1 4_1 7_1 1_2 3_1 4_2 6_2 7_2 2_2 6_1 3_2) (5_1 5_2 \infty_2)\]
\[(0_1 \infty_1 0_2 1_1 2_1 4_1 7_1 1_2 3_1 6_2 5_1 5_2 7_2 4_2) (6_1 3_2 2_2 \infty_2)\]
\[(0_1 \infty_1 0_2 1_1 2_1 4_1 7_1 1_2 3_1 6_2 6_1 7_2 4_2) (5_1 2_2 3_2 5_2 \infty_2)\]
\[(0_1 \infty_1 0_2 1_1 2_1 4_1 7_1 1_2 3_1 6_2 3_2 4_2) (5_1 2_2 \infty_2 6_1 7_2 5_2)\]

\[\vdots\]

\[(0_1 \infty_1 0_2 1_1 3_1 1_2) (2_1 5_1 7_2) (4_1 4_2 \infty_2) (6_1 7_1 2_2) (3_2 5_2 6_2)\]
\[(0_1 \infty_1 0_2 1_1 2_1) (3_1 6_1 1_2 7_1 3_2) (4_1 2_2 5_1 6_2 \infty_2) (4_2 5_2 7_2)\]
\[(0_1 \infty_1 0_2 1_1 2_1) (3_1 6_1 2_2 3_2 6_2) (4_1 4_2 5_1 5_2) (5_1 7_2 1_2 \infty_2)\]
\[(0_1 \infty_1 0_2 1_1 2_1) (3_1 3_2 5_1 6_2) (4_1 7_1 1_2) (6_1 2_2 \infty_2) (4_2 5_2 7_2)\]
\[(0_1 \infty_1 0_2 1_1) (2_1 4_1 7_1 2_2) (3_1 5_2 7_2 \infty_2) (5_1 1_2 6_2) (6_1 3_2 4_2)\]

Fig. 1. Base 2-factors for \(n = 18\)
Fig. 2. Base 2-factors for $n = 19$

\[
\begin{align*}
& (0_1 \infty 0_2 1_2 2_4 3_1 4_5 7_1 2_3 5_9 8_3 2_7 6_1 8_2 6_2 3_3 2_4) \\
& (0_1 \infty 0_2 1_2 2_4 3_1 4_5 7_1 2_3 5_9 8_3 2_7 6_1 8_2 6_2 3_3 2_4) \\
& (0_1 \infty 0_2 1_2 2_4 3_1 4_5 7_1 2_3 5_9 8_3 2_7 6_1 8_2 6_2 3_3 2_4) \\
& (0_1 \infty 0_2 1_2 2_4 3_1 4_5 7_1 2_3 5_9 8_3 2_7 6_1 8_2 6_2 3_3 2_4) \\
& (0_1 \infty 0_2 1_2 2_4 3_1 4_5 7_1 2_3 5_9 8_3 2_7 6_1 8_2 6_2 3_3 2_4) \\
& \vdots
\end{align*}
\]

Fig. 3. Base 2-factors for $n = 20$

\[
\begin{align*}
& \{ (0_1 \infty 0_3 1_1 2_4 3_0 1_2 2_4 3_2 2_3 0_4 3_3 4_1 4_3 2_3 1_4), \}
& \{ (0_2 \infty 0_4 1_2 4_3 0_0 1_1 2_3 1_4 3_3 1_1 2_4 3_2 1_4 2_4 3_3 1_4), \}
& \{ (0_1 \infty 0_3 1_1 2_4 3_0 1_2 2_4 3_2 2_3 0_4 3_3 4_1 4_3 2_3 1_4), \}
& \{ (0_2 \infty 0_4 1_2 4_3 0_0 1_1 2_3 1_4 3_3 1_1 2_4 3_2 1_4 2_4 3_3 1_4), \}
& \vdots
\end{align*}
\]

Fig. 4. Base 2-factors for $n = 21$
(01 81 82 02 11 21 41 71 31 91 141 51 12 61 32 101 42 131 22 151 52 62 102 121 132 111 152 92 112 142 72 122 ∞)
(01 81 82 02 11 21 41 71 31 91 141 51 12 61 32 101 42 131 22 151 52 62 112 72 142 121 132 102 122 ∞) (111 92 152)
(01 81 82 02 11 21 41 71 31 91 141 51 12 61 32 101 42 131 22 151 52 62 112 132 92 152 111 122 ∞) (121 102 72 142)
(01 81 82 02 11 21 41 71 31 91 141 51 12 61 32 101 42 131 22 151 52 62 122 92 111 152 102 ∞) (121 132 112 72 142)
(01 81 82 02 11 21 41 71 31 91 141 51 12 61 32 101 42 131 22 151 52 62 122 92 112 72 142) (111 132 121 ∞ 102 152)

...}

(01 81 82 02 11) (21 41 71 31 91) (51 101 12 22) (61 72 141 102) (111 62 121 142) (131 32 132 112) (151 42 152 ∞) (52 92 122)
(01 81 82 02 11) (21 41 71 31 91) (51 101 12 72) (61 22 121 152) (111 52 ∞) (131 102 142) (141 92 122) (151 32 42) (62 112 132)
(01 81 82 02 11) (21 41 71 111) (31 91 141 12) (51 22 61 72) (101 52 102 142) (121 32 131 62) (151 42 112 ∞) (92 122 132 152)
(01 81 82 02 11) (21 41 71 111) (31 91 141 12) (51 22 61 122) (101 112 102 142) (121 52 72) (131 152 ∞) (151 42 92) (32 62 132)
(01 81 82 02 11) (21 41 71 111) (31 91 42) (51 101 12) (61 92 152) (121 142 ∞) (131 22 72) (141 112 122) (151 32 52) (62 102 132)

**Fig. 5.** Base 2-factors for \( n = 33 \), with some \( a_i \geq 5 \)

\[
(\sigma_1 \infty \times \sigma_1 \infty) \ (21 \ 31 \ 51 \ 91) \ (41 \ 71 \ 131 \ 152) \ (61 \ 22 \ 122 \ 112) \ (101 \ 151 \ 32 \ 102) \ (111 \ 42 \ 72 \ 53) \ (131 \ 02 \ 82) \ (121 \ 92 \ 132) \ (141 \ 12 \ 62)
\]

**Fig. 6.** Base 2-factors for \( OP(33; 4^6, 3^3) \) and \( OP(33; 4^3, 3^7) \)