On the Skeleton of the Metric Polytope

Antoine Deza[1,5], Komei Fukuda[2], Dmitrii Pasechnik[3], and Masanori Sato[4]

[1] Institute of Statistical Mathematics, Prediction and Control, Tokyo, Japan
deza@ism.ac.jp
[2] ETH Zürich, Institute for Operations Research, Zürich, Switzerland
fukuda@ifor.math.ethz.ch
[3] Delft University of Technology, Operations Research, TWI, Netherlands
d.pasechnik@twi.tudelft.nl
[4] Tokyo Institute of Technology, Math. and Comput. Sci., Tokyo, Japan
[5] EHESS, Centre d'Analyse et de Mathématique Sociales, Paris, France.
deza@ehess.fr

Abstract. We consider convex polyhedra with applications to well-known combinatorial optimization problems: the metric polytope m_n and its relatives. For $n \leq 6$ the description of the metric polytope is easy as m_n has at most 544 vertices partitioned into 3 orbits; m_7 - the largest previously known instance - has 275 840 vertices but only 13 orbits. Using its large symmetry group, we enumerate orbitwise 1 550 825 600 vertices of the 28-dimensional metric polytope m_8. The description consists of 533 orbits and is conjectured to be complete. The orbitwise incidence and adjacency relations are also given. The skeleton of m_8 could be large enough to reveal some general features of the metric polytope on n nodes. While the extreme connectivity of the cuts appears to be one of the main features of the skeleton of m_n, we conjecture that the cut vertices do not form a cut-set. The combinatorial and computational applications of this conjecture are studied. In particular, a heuristic skipping the highest degeneracy is presented.

1 Introduction

Combinatorial polytopes, i.e. polytopes arising from combinatorial optimization problems, are often trivial for the very first cases and then suddenly the so-called combinatorial explosion occurs even for small instances. While these polytopes turn out to be quickly intractable for enumeration algorithm designed for general polytopes, tailor-made algorithms using their rich combinatorial features can exhibit surprisingly strong performances. For example, Christof and Reinelt [2] computed large instances of the *traveling salesman polytope*, the *linear ordering polytope* and the *cut polytope* exploiting their symmetry groups. In a similar vein, in addition to its symmetry group, we used its combinatorial structure to orbitwise enumerate the vertices of another combinatorial polytope: *the metric polytope*. Let first recall basic definitions and present some applications to well-known combinatorial optimization problems.

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The $\binom{n}{2}$-dimensional cut polytope c_n is usually introduced as the convex hull of the incidence vectors of all the cuts of K_n. More precisely, given a subset S of $V_n = \{1, 2, \ldots, n\}$, the *cut* determined by S consists of the pairs (i, j) of elements of V_n such that exactly one of i, j is in S. By $\delta(S)$ we denote both the cut and its incidence vector in $\mathbb{R}^{\binom{n}{2}}$; that is, $\delta(S)_{ij} = 1$ if exactly one of i, j is in S and 0 otherwise for $1 \leq i < j \leq n$. By abuse of notation, we use the term cut for both the cut itself and its incidence vector, so $\delta(S)_{ij}$ are considered as coordinates of a point in $\mathbb{R}^{\binom{n}{2}}$. The cut polytope c_n is the convex hull of all 2^{n-1} cuts, and the *cut cone* C_n is the conic hull of all $2^{n-1} - 1$ nonzero cuts. The cut polytope and one of its relaxation - the metric polytope - can also be defined in terms of a finite metric space in the following way. For all 3-sets $\{i, j, k\} \subset \{1, \ldots, n\}$, we consider the following inequalities:

$$x_{ij} - x_{ik} - x_{jk} \leq 0, \tag{1}$$

$$x_{ij} + x_{ik} + x_{jk} \leq 2. \tag{2}$$

(1) induce the $3\binom{n}{3}$ facets which define the *metric cone* M_n. Then, bounding the latter by (2) we obtain the metric polytope m_n. The $3\binom{n}{3}$ (resp. $\binom{n}{3}$) facets defined by (1) (resp. by (2)) can be seen as triangle (resp. perimeter) inequalities for distance x_{ij} on $\{1, 2, \ldots, n\}$. While the cut cone is the conic hull of all, up to a constant multiple, $\{0, 1\}$-valued extreme rays of the metric cone, the cut polytope c_n is the convex hull of all $\{0, 1\}$-valued vertices of the metric polytope. The link with finite metric spaces is the following: there is a natural $1 - 1$ correspondence between the elements of the metric cone and all the semi-metrics on n points, and the elements of the cut cone correspond precisely to the semi-metrics on n points that are isometrically embeddable into some l_1^m. It is easy to check that such minimal m is smaller or equal to $\binom{n}{2}$.

One of the motivations for the study of these polyhedra comes from their applications in combinatorial optimization, the most important being the max-cut and multicommodity flow problems. Given a graph $G = (V_n, E)$ and nonnegative weights w_e, $e \in E$, assigned to its edges, the *max-cut* problem consists in finding a cut $\delta(S)$ whose weight $\sum_{e \in \delta(S)} w_e$ is as large as possible. It is a well-known NP-complete problem. By setting $w_e = 0$ if e is not an edge of G, we can consider without loss of generality K_n the complete graph on V_n. Then the max-cut problem can be stated as a linear programming problem over the cut polytope c_n, as follows: $\max\ w^T x$ subject to $x \in c_n$. Since the metric polytope is a relaxation of the cut polytope, optimizing $w^T x$ over c_n instead of m_n provides an upper bound for the max-cut problem. Consider now the complete graph K_n; an instance of the *multicommodity flow* problem is given by two nonnegative vectors indexed by E: a capacity $c(e)$ and a requirement $r(e)$ for each $e \in E$. Let $U = \{e \in E : r(e) > 0\}$. If T denotes the subset of V_n spanned by the edges in U, then we say that the graph $G = (T, U)$ denotes the *support* of r. For each edge $e = (s, t)$ in the support of r, we seek a flow of $r(e)$ units between s and t in the complete graph. The sum of all flows along any edge $e' \in E$ must not exceed $c(e')$. If such a set of flows exists, we call c, r *feasible*. A necessary and sufficient condition for feasibility is: a pair c, r is feasible if and only if $(c-r)^T x \geq 0$ is valid over the metric cone, see [7]. For example, the triangle facet induced by (1) can be seen as an elementary solvable flow problem with $c(ij) = r(ik) = r(jk) = 1$ and $c(e) = r(e) = 0$ otherwise, so (1) corresponds to $(c - r)^T x \geq 0$ for x in the metric cone. Therefore, the metric cone is the dual cone to the cone of feasible multicommodity flow problems. For a detailed study of those polytopes and their applications in combinatorial optimization we refer to Deza and Laurent [4] and Poljak and Tuza [9].

2 Vertices of the Metric Polytope

2.1 Combinatorial and Geometric Properties

The polytope c_n is a $\binom{n}{2}$ dimensional $0-1$ polyhedron with 2^{n-1} vertices and m_n is a polytope of the same dimension with $4\binom{n}{3}$ facets inscribed in the cube $[0,1]^{\binom{n}{2}}$. We have $c_n \subseteq m_n$ with equality only for $n \leq 4$. It is easy to see that the point $\omega_n = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ is the center of gravity of both c_n and m_n and is also the center of the sphere of radius $r = \frac{1}{2}\sqrt{n(n-1)}$ where all the cuts lie. Any facet of the metric polytope contains a face of the cut polytope and the vertices of the cut polytope are vertices of the metric polytope. In fact, the cuts are precisely the integral vertices of the metric polytope. The metric polytope m_n wraps the cut polytope c_n very tightly. Indeed, in addition to the vertices, all edges and 2-faces of c_n are also faces of m_n, for 3-faces it is false for $n \geq 4$. Any two cuts are adjacent both on c_n and on m_n; in other words m_n is *quasi-integral*; that is, the skeleton of the convex hull of its integral vertices, i.e. the skeleton of c_n, is an induced subgraph of the skeleton of the metric polytope itself. We recall that the skeleton of a polytope is the graph formed by its vertices and edges. While the diameters of the cut polytope and the dual metric polytope satisfy $\delta(c_n) = 1$ and $\delta(m_n^*) = 2$, the diameters of their dual are conjectured to be $\delta(c_n^*) = 4$ and $\delta(m_n) = 3$.

One important feature of the metric and cut polytopes is their very large symmetry group. We recall that the *symmetry group* $Is(P)$ of a polytope P is the group of isometries preserving P. More precisely, for $n \geq 5$, $Is(m_n) = Is(c_n)$ and both are induced by permutations on $V_n = \{1, \ldots, n\}$ and *switching reflections by a cut* and, for $n \geq 5$, we have $|Is(m_n)| = 2^{n-1}n!$. Given a cut $\delta(S)$, the switching reflection $r_{\delta(S)}$ is defined by $y = r_{\delta(S)}(x)$ where $y_{ij} = 1 - x_{ij}$ if $(i,j) \in \delta(S)$ and $y_{ij} = x_{ij}$ otherwise. As these symmetries preserve the adjacency relations and the linear independency, all faces of m_n are partitioned into orbits of faces equivalent under permutations and switchings.

2.2 Vertices of the Metric Polytope

We recall some results on the vertices of the metric polytope and the Laurent-Poljak *dominant clique conjecture*. The cuts are the only integral vertices of m_n. All other vertices with are not fully fractional are so-called *trivial extensions* of a vertex of m_{n-1}. Consider the following two mappings

$$\begin{array}{lll} \mathbb{R}^{\binom{n-1}{2}} \longrightarrow \mathbb{R}^{\binom{n}{2}} & \mathbb{R}^{\binom{n-1}{2}} \longrightarrow \mathbb{R}^{\binom{n}{2}} & \\ v \longrightarrow \phi_0(v) & v \longrightarrow \phi_1(v) & \\ & & \\ \phi_0(v)_{ij} = v_{ij} & \phi_1(v)_{ij} = v_{ij} & \text{for } 1 \le i < j \le n-1 \\ \phi_0(v)_{i,n} = v_{1,i} & \phi_1(v)_{i,n} = 1 - v_{1,i} & \text{for } 2 \le i \le n-1 \\ \phi_0(v)_{1,n} = 0 & \phi_0(v)_{1,n} = 1. & \end{array}$$

The vertices $\phi_0(v)$ and $\phi_1(v)$ are called trivial extensions of v. Note that $\phi_1(v) = r_{\delta(\{n\})}(\phi_0(v))$. In other words, the new vertices are the fully fractional ones. The $(\frac{1}{3}, \frac{2}{3})$-valued fully fractional vertices are well studied and include the anticut orbit formed by the 2^{n-1} *anticuts* $\bar{\delta}(S) = \frac{2}{3}(1, \ldots, 1)$-$\frac{1}{3}\delta(S)$. If $G = (V_n, E)$ is a connected graph, we denote by d_G its path metric, where $d_G(i,j)$ is the length of a shortest path from i to j in G for $i \neq j \in V_n$. Then $\tau(d_G) = max(d_G(i,j) + d_G(i,k) + d_G(j,k) : i,j,k \in G)$ is called the *triameter of* G and we set $x_G = \frac{2}{\tau(d_G)} d_G$. Any vertex of m_n of the form x_G for some graph is called a *graphic* vertex, see Fig. 1 for the graphs of 2 graphic $(\frac{1}{3}, \frac{2}{3})$-valued vertices of m_8. Note that for any connected graph $G = (V_n, E)$, we have $\tau(d_G) \le 2(n-1)$ and that any $(\frac{1}{3}, \frac{2}{3})$-valued vertex of m_n is (up to switching) graphic. Let the *incidence* Icd_v denotes the number of facets containing the vertex v and the *adjacency* Icd_v denotes the number of vertices adjacent to v (i.e. forming an edge with v). The following is straigtforward to prove.

Proposition 1. *The vertices of the metric polytope m_n are partitioned into orbits of its symmetry group. Let v be a vertex of m_n,* Icd_v *its incidence,* Adj_v *its adjacency, O_v the orbit generated by the action of $Is(m_n)$ on v, and $\tilde{v}$ the canonical representative of O_v. Then* $\mathrm{Icd}_v = \mathrm{Icd}_{\tilde{v}}$, $\mathrm{Adj}_v = \mathrm{Adj}_{\tilde{v}}$ *and $O_v = O_{\tilde{v}}$.*

Since $m_3 = c_3$ and $m_4 = c_4$, the vertices of m_3 and m_4 are made of 4 and 8 cuts forming 1 orbit. The 32 vertices of m_5 are 16 cuts and 16 anticuts, i.e., form 2 orbits. The metric polytope m_6 has 544 vertices, see [8], partitioned into 3 orbits: cuts, anticuts and 1 orbit of trivial extensions; and m_7 has 275 840 vertices, see [3], partitioned into 13 orbits: cuts, anticuts, 3 orbits of trivial extensions, 3 $(\frac{1}{3}, \frac{2}{3})$-valued orbits and 5 other fully fractional orbits. See Table 1, where the 13 canonical representative vertices of the metric polytope on 7 nodes are given with their incidence and adjacency.

Property 1. Let v be a vertex of m_n and $\delta(S)$ any cut. Then one has: $Icd_v \le Icd_{\delta(S)} = 3\binom{n}{3}$ with equality only for $v = \delta(S)$. Moreover, if v is a trivial extension, $Icd_v \le 2\binom{n}{3}$ and, if v is fully fractional, $Icd_v \le Icd_{\bar{\delta}(S)} = \binom{n}{3}$ with equality only for $v = \bar{\delta}(S)$.

Property 1 is illustrated in Table 1 where the orbits $O_{\tilde{v}_i}$ are ordered by decreasing values of the incidence $Icd_{\tilde{v}_i}$. The first orbit $O_{\tilde{v}_1}$ is the cut orbit and all fully fractional orbits are after the anticut orbit $O_{\tilde{v}_5}$. The trivial extension orbits are $O_{\tilde{v}_i}$ for $i = 2, 3$ and 4.

Table 1. The 13 orbits of vertices of m_7

Orbit $O_{\tilde{v}_i}$	Canonical representative vertex $\tilde{v}_i$	$Icd_{\tilde{v}_i}$	$Adj_{\tilde{v}_i}$	$\lvert O_{\tilde{v}_i}\rvert$
$O_{\tilde{v}_1}$	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$	105	55 226	64
$O_{\tilde{v}_2}$	$\frac{2}{3}(1,1,1,1,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0)$	49	496	2 240
$O_{\tilde{v}_3}$	$\frac{2}{3}(1,1,1,1,0,1,1,1,1,1,0,1,1,1,1,1,1,1,1,1,1)$	45	594	6 720
$O_{\tilde{v}_4}$	$\frac{2}{3}(1,1,1,1,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$	40	763	1 344
$O_{\tilde{v}_5}$	$\frac{2}{3}(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$	35	896	64
$O_{\tilde{v}_6}$	$\frac{1}{4}(1,2,3,1,2,1,1,2,2,1,2,1,1,2,3,2,3,2,1,2,1)$	30	96	20 160
$O_{\tilde{v}_7}$	$\frac{2}{5}(2,1,1,1,1,2,2,1,1,1,1,2,1,1,1,2,1,1,2,1,2)$	28	57	23 040
$O_{\tilde{v}_8}$	$\frac{1}{3}(1,1,1,1,1,1,2,2,1,1,1,2,1,1,1,1,1,1,2,2,2)$	26	76	4 480
$O_{\tilde{v}_9}$	$\frac{2}{7}(1,2,3,2,1,2,1,2,1,2,1,1,2,1,1,1,2,2,1,1,1)$	25	30	40 320
$O_{\tilde{v}_{10}}$	$\frac{1}{5}(3,2,3,3,1,1,1,2,2,2,2,3,3,3,3,4,4,2,2,4,2)$	25	27	16 128
$O_{\tilde{v}_{11}}$	$\frac{1}{3}(1,1,1,1,1,1,2,2,1,1,1,2,1,1,1,2,1,1,2,2,2)$	23	39	40 320
$O_{\tilde{v}_{12}}$	$\frac{1}{6}(1,2,4,2,2,2,1,3,3,3,3,2,2,2,4,2,2,2,4,4,4)$	23	24	80 640
$O_{\tilde{v}_{13}}$	$\frac{1}{3}(2,2,1,1,1,2,2,1,1,1,1,2,1,1,1,2,1,1,2,1,2)$	22	46	40 320
Total				275 840

Conjecture 1. [8] Any vertex of the metric polytope is adjacent to a cut.

Conjecture 1 underlines the extreme connectivity of the cuts. Recall that the cuts form a clique in both the cut and metric polytopes. Therefore, if Conjecture 1 holds, the cuts would be a dominant clique in the skeleton of m_n implying that its diameter would satisfy $\delta(m_n) \leq 3$.

3 Orbitwise Enumeration Algorithm

As stated in Proposition 1, the *neighborhood*, that is, the set of vertices adjacent to a given vertex, is equivalent up to permutations and switchings for all vertices belonging to the same orbit. This property leads to the following *orbitwise enumeration algorithm*. The main two subroutines are the computation of the canonical representative $\tilde{v}$ of the orbit generated by a vertex v and the enumeration of the neighborhood $N_{\tilde{v}}$ of the vertex $\tilde{v}$. Starting from an initial vertex v_{start} the algorithm computes the canonical representative $\tilde{v}_{start}$, enumerates its neighborhood $N_{\tilde{v}_{start}}$, identifies new orbits contained in $N_{\tilde{v}_{start}}$, updates the list L of canonical representatives and then picks up the next canonical representative in L whose neighborhood is not yet computed. The algorithm terminates when there is no more such canonical representative in L and outputs L. Since the skeleton of a polytope is connected, this algorithm finds all orbits.

Orbitwise Enumeration Algorithm

begin

find an initial vertex v_{start};

compute the canonical representative $\tilde{v}_{start}$ of the orbit $O_{\tilde{v}_{start}}$;

mark $\tilde{v}_{start}$ with 0; /* neighborhood not yet computed */

initialized the list of canonical representatives $L := \{\tilde{v}_{start}\}$;

while L contains a 0-marked vertex $\tilde{v}_i$ **do**

begin

compute the neighborhood $N_{\tilde{v}_i}$ of $\tilde{v}_i$;

for each vertex v adjacent to $\tilde{v}_i$

compute the canonical representative $\tilde{v}$ of the orbit $O_{\tilde{v}}$;

if $\tilde{v} \notin L$ then mark $\tilde{v}$ with 0 and $L := L \cup \{\tilde{v}\}$; **endif**;

endfor;

mark $\tilde{v}_i$ with 1; /* neighborhood computed */ ;

endwhile;

sort L by decreasing values of $Icd_{\tilde{v}_i}$, decreasing $Adj_{\tilde{v}_i}$ and increasing $|O_{\tilde{v}_i}|$;

output L;

end.

Lemma 1. *Let I be the number of orbits,* $\mathrm{Icd}_{\tilde{v}_i}$ *and* $\mathrm{Adj}_{\tilde{v}_i}$ *the incidence and the adjacency of the orbit $O_{\tilde{v}_i}$ for $i = 1, \ldots, I$. The neighborhood enumeration subroutine is called exactly I times and each neighborhood is generated by* $\mathrm{Icd}_{\tilde{v}_i}$ *facets. The canonical representative computation subroutine is called exactly* $\sum_{\tilde{v}_i} \mathrm{Adj}_{\tilde{v}_i}$ *times.*

Remark 1.

1. The orbitwise enumeration algorithm performs I classic vertex enumerations for smaller sub-polytopes (one for each orbit of neighborhoods) instead of performing one large classic vertex enumeration (the whole polytope).
2. The computation is independent of the choice of the initial vertex v_{start}. Among the known vertices of m_n; an easy choice for v_{start} is the anticut $\bar{\delta}(\emptyset) = \frac{2}{3}(1, \ldots, 1)$.
3. In case of very high degeneracy, the subroutine computing the canonical representative has to be called a large number of times and some of the neighborhoods might represent a large fraction of the whole polytope. It is the case for the neighborhood of a cut $N_{\delta(S)}$ as $Icd_{\delta(S)} = 3\binom{n}{3}$.

For $i = 1, \ldots, I$, the algorithm gets the orbitwise incidence $Icd_{\tilde{v}_i}$ (by simply checking which inequality is satisfied with equality) as input for the neighborhood enumeration subroutine and produces the adjacency $Adj_{\tilde{v}_i}$ as output of this subroutine. By counting the number of times a vertex equivalent to the canonical representative $\tilde{v}_j$ is found in $N_{\tilde{v}_i}$, we get the *orbitwise adjacency table*, that is, the $I \times I$ matrix Adj with $Adj_{i,j} = Adj_{\tilde{v}_i, \tilde{v}_j}$ the number of vertices of the orbit $O_{\tilde{v}_j}$ adjacent to $\tilde{v}_i$. The orbits $O_{\tilde{v}_i}$ are ordered first by decreasing values of the incidence $Icd_{\tilde{v}_i}$, then by decreasing adjacency $Adj_{\tilde{v}_i}$ and then by increasing orbitsize $|O_{\tilde{v}_i}|$. Let us assume we know the size of one orbit; for example, we have

$|O_{\tilde{\delta}(S)}| = 2^{n-1}$. Then, from the matrix Adj, we can usually get the size of the other orbits using the following easy relation: $Adj_{\tilde{v}_i,\tilde{v}_j} \times |O_{\tilde{v}_i}| = Adj_{\tilde{v}_j,\tilde{v}_i} \times |O_{\tilde{v}_j}|$. See, for example, Table 2 where the orbitwise adjacency table is given for the metric polytope on 7 nodes. The first row of Table 2 lists orbitwise $N_{\tilde{v}_1}$; that is, the 55 226 neighboors of a vertex belonging to $O_{\tilde{v}_1}$, that is a cut. For example, $Adj_{\tilde{v}_1,\tilde{v}_4} = 945$ in the fourth column means that a cut is adjacent to 945 vertices belonging to the orbit $O_{\tilde{v}_4}$. Since all the facets incident to the origin $\delta(\emptyset)$ are precisely the $3\binom{n}{3}$ triangle facets, an extreme ray of the metric cone M_n corresponds to each vertex adjacent to $\tilde{v}_1 = \delta(\emptyset)$. In other words, the adjacency $Adj_{\delta(S)} = Adj_{\tilde{v}_1}$ of a cut equals the number of extreme rays of the metric cone M_n. We recall that the 41 orbits under permutations of the extreme rays of M_7 were found by Grishukhin [6].

Table 2. Orbitwise adjacency table of the skeleton of m_7

	$O_{\tilde{v}_1}$	$O_{\tilde{v}_2}$	$O_{\tilde{v}_3}$	$O_{\tilde{v}_4}$	$O_{\tilde{v}_5}$	$O_{\tilde{v}_6}$	$O_{\tilde{v}_7}$	$O_{\tilde{v}_8}$	$O_{\tilde{v}_9}$	$O_{\tilde{v}_{10}}$	$O_{\tilde{v}_{11}}$	$O_{\tilde{v}_{12}}$	$O_{\tilde{v}_{13}}$	$Adj_{\tilde{v}_i}$
$O_{\tilde{v}_1}$	63	980	3 570	945	56	7 560	5 400	1 120	6 930	2 772	6 930	10 080	8 820	55 226
$O_{\tilde{v}_2}$	28	24	132	45	3	18	0	12	0	36	72	108	18	496
$O_{\tilde{v}_3}$	34	44	126	42	4	60	48	8	48	12	48	48	72	594
$O_{\tilde{v}_4}$	45	75	210	20	3	60	0	20	0	60	60	180	30	763
$O_{\tilde{v}_5}$	56	105	420	63	0	0	0	0	0	252	0	0	0	896
$O_{\tilde{v}_6}$	24	2	20	4	0	8	8	2	4	0	8	8	8	96
$O_{\tilde{v}_7}$	15	0	14	0	0	7	7	0	7	0	0	0	7	57
$O_{\tilde{v}_8}$	16	6	12	6	0	9	0	0	0	0	9	18	0	76
$O_{\tilde{v}_9}$	11	0	8	0	0	2	4	0	0	0	1	0	4	30
$O_{\tilde{v}_{10}}$	11	5	5	5	1	0	0	0	0	0	0	0	0	27
$O_{\tilde{v}_{11}}$	11	4	8	2	0	4	0	1	1	0	0	4	4	39
$O_{\tilde{v}_{12}}$	8	3	4	3	0	2	0	1	0	0	2	0	1	24
$O_{\tilde{v}_{13}}$	14	1	12	1	0	4	4	0	4	0	4	2	0	46

Remark 2. The output, that is, the list L of canonical representatives $\tilde{v}_i$ for $i = 1, \ldots, I$, is extremely compact. Apart from vertex enumeration, the algorithm computes the orbits invariants $Adj_{\tilde{v}_i}, Icd_{\tilde{v}_i}$ and $|O_{\tilde{v}_i}|$. The orbitwise adjacency table Adj reveals the skeleton. The total number of vertices is simply $\sum_{\tilde{v}_i} |O_{\tilde{v}_i}|$ and the full list of vertices can be generated by the action of the symmetry group on each representative $\tilde{v}_i$.

4 Generating Vertices of the Metric Polytope

The heuristics presented in this section are valid for other combinatorial polytopes, but for convenience we restrict ourselves to the metric polytope. Insertion algorithms usually handle high degeneracy better than pivoting algorithms, see [1] for a detailed presentation of the main vertex enumeration methods. The metric polytope m_n is quite degenerate (the cut incidence $Icd_{\delta(S)} = 3\binom{n}{3}$ is much larger than the dimension $d = \binom{n}{2}$). Thus we choose an insertion algorithm for the neighborhood enumeration subroutine: the *cddlib* implementation of the double description method [5]. In the remainder, we always assume that the neighborhood enumeration subroutine is performed by an insertion algorithm. Item 3 of Remark 1 indicates that even the neighborhoods of highly degenerate polytopes might lie beyond the range of problems currently solvable by insertion algorithms. In Sect. 4.2, we present heuristics addressing this issue.

4.1 A Conjecture on the Skeleton of the Metric Polytope

If true, the Laurent-Poljak Conjecture 1 would give the following computational implication: the enumeration of the extreme rays of M_n gives all the orbits of the m_n. Since the number of extreme rays of the metric cone $|M_n| = Adj_{\delta(S)}$ might be a large fraction of the number of vertices of the metric polytope, the computational gain would be limited. Therefore, we propose a *no cut-set conjecture* which can be seen as complementary to the Laurent-Poljak conjecture both graphically and computationally.

Conjecture 2. For $n \geq 6$, the restriction of the skeleton of the metric polytope m_n to the non-cut vertices is connected.

For any pair of vertices, while Conjecture 1 implies that there is a path made of cuts joining them, Conjecture 2 means that there is a path made of non-cuts vertices joining them. In other words, the cut vertices would form a *dominating set* but not a *cut-set* in the skeleton of m_n. On the other hand, while Conjecture 1 means that the enumeration of the metric cone M_n is enough to obtain the metric polytope m_n; Conjecture 2 means that we can obtain m_n without enumerating $M_n = N_{\delta(\emptyset)}$, see Sect. 4.2. Note that for arbitrary graphs these are clearly independent. Both are strongly believed to be true and hold for $n \leq 7$.

4.2 Heuristic: Skipping High Degeneracy

If Conjecture 2 holds, all orbits can be found by the following metric cone skipping heuristic: disregard $\tilde{v}$ if $\tilde{v} = \delta(S)$. In other words, disregard the neighborhood of the cuts, that is, essentially the metric cone M_n. This neighborhood is believed to be by a large margin the largest, as we expect that $Adj_{\delta(S)} \gg Adj_{v \neq \delta(S)}$, see Property 1 and Item 1 of Remark 3. In other words, the heuristic removes the hardest neighborhood enumeration. Note that cuts are easy to recognize as $\delta(S)$ is uniquely characterized by its incidence: $Icd_{\delta(S)} = 3\binom{n}{3}$. Therefore, disregarding the metric cone M_n, consists simply in choosing a non-cut as initial vertex v_{start} and modifying the main loop of the orbitwise enumeration algorithm in the following way:

Metric Cone Skipping Heuristic

> **if** $\tilde{v} \notin L$ then $L := L \cup \{\tilde{v}\}$;
>   **if** $Icd_v = 3\binom{n}{3}$ then mark $\tilde{v}$ with 1
>   else mark $\tilde{v}$ with 0; **endif**;
> **endif**;

For the metric on 8 nodes, while $Adj_{\delta(S)} \geq 119\,269\,588$ - i.e. 7.7% of the total number of vertices of m_8 - the enumeration of the other 532 neighborhoods generates (with multiplicity) $\sum_{\tilde{v}_i \neq \delta(S)} Adj_{\tilde{v}_i} = 780\,711$ vertices - i.e. less than 0.05% of the total number of vertices. One can easily get the neighborhood of the cut from the orbitwise adjacency table Adj. Taking the column and the row corresponding to the cuts as we have: $Adj_{\delta(S),\tilde{v}_i} \times 2^{n-1} = Adj_{\tilde{v}_i,\delta(S)} \times |O_{\tilde{v}_i}|$ where $Adj_{\tilde{v}_i,\delta(S)}$ is the number of cuts adjacent to $\tilde{v}_i$.

Proposition 2. *If true, Conjecture 2 would be a certificate that the "Metric Cone Skipping Heuristic" gives a complete description of the metric polytope by generating only a very small fraction of the vertices.*

One can further decrease the computation time by skipping not only the orbit with the highest incidence (the cuts) but all orbits with arbitrarily set in advance upper bound Icd_{max} on the incidence. Skipping high degeneracy consists simply in the following modification of the main loop of the orbitwise enumeration algorithm:

Skipping High Degeneracy Heuristic

> **if** $\tilde{v} \notin L$ then $L := L \cup \{\tilde{v}\}$;
>   **if** $Icd_v > Icd_{max}$ then mark $\tilde{v}$ with 1
>   else mark $\tilde{v}$ with 0; **endif**;
> **endif**;

In this case, a certificate for a complete description is that the restriction of the skeleton of m_n to $O_{\tilde{v}_{start}}$ and the low incidence orbits $O_{\tilde{v}_i\,:\,Icd_{\tilde{v}_i} \leq Icd_{max}}$ is connected. This heuristic is particularly suitable for partial enumeration purpose and the choice of the initial vertex v_{start} could become a critical factor, see Item 2 of Proposition 5. For the metric polytope on 8 nodes, we can take $v_{start} = \tilde{v}_4$; that is, the trivial extension with the fourth highest incidence $Icd_{\tilde{v}_4} = 74$. We expected this type of vertex to be connected to many orbits and, indeed, the neighborhood $N_{\tilde{v}_4}$ contains representatives of 450 different orbits out of 533. Another choice is $v_{start} = \tilde{v}_{41}$ with $Icd_{\tilde{v}_{41}} = 42$ and $Adj_{\tilde{v}_{41}} = 533$. The neighborhood computation subroutine was restricted to vertices satisfying $Icd_{\tilde{v}_i} \leq 40 < \frac{1}{2}\binom{n+1}{3}$ - i.e. halfway from the dimension $\binom{n}{2}$ to the anticut incidence $Icd_{\bar{\delta}(S)} = \binom{n}{3}$. Besides $N_{\tilde{v}_{41}}$, the algorithm computed 485 neighborhoods with low incidences generating $\sum_{\tilde{v}_i : Icd_{\tilde{v}_i} \leq 40} Adj_{\tilde{v}_i} = 63\,095$ vertices - i.e. less than 0.005% of the total number of vertices. Still, this heuristic approach proved to be enough as this tiny number of vertices contains representatives of all 533 orbits. Another remarkable feature is that since all the 485 neighborhoods are generated by few facets - 389 have even less than 34 facets - the neighborhood enumeration subroutine is performing extremely well. Similarly to the previous *metric cone skipping heuristic* case, missing entries of the table *Adj* (i.e. the rows corresponding to the orbits with high incidence $Icd_{\tilde{v}_i} > Icd_{max}$) can be computed using the relations: $Adj_{\tilde{v}_i,\tilde{v}_j} \times |O_{\tilde{v}_i}| = Adj_{\tilde{v}_j,\tilde{v}_i} \times |O_{\tilde{v}_j}|$. In particular, we can first compute $N_{\bar{\delta}(S)}$; that is, all nonzero values of $Adj_{\bar{\delta}(S),\tilde{v}_j}$ and, using the fact that $|O_{\bar{\delta}(S)}| = 2^{n-1}$, get all corresponding $|O_{\tilde{v}_j}|$ and then use them iteratively to obtain some of the remaining unknown $|O_{\tilde{v}_i}|$.

5 Vertices of the Metric Polytope on 8 Nodes

Using the metric cone skipping heuristic presented in Sect. 4.2, we enumerate 533 orbits of m_8. The list of canonical representative with their adjacency and incidence and, especially, the adjacency table *Adj* being too large to be included in this paper, we refer to `http://www.is.titech.ac.jp/~ deza/deza.html` where a detailed presentation is available. For example, the anticut row of *Adj* has only 15 nonzero entries $Adj_{\bar{\delta}(S),\tilde{v}_j}$. A summary description is given in Proposition 4.

Proposition 3.

1. *The metric polytope m_8 has at least 1 550 825 600 vertices and the metric cone M_8 has at least 119 269 588 extreme rays; we conjecture that both descriptions are complete.*
2. *For $i = 1, \ldots, 533$ each orbit representative $\tilde{v}_i$ is adjacent to at least 2 cuts implying that the* Laurent-Poljak *dominant clique conjecture holds for these 533 orbits of m_8.*

Proposition 4. *The 1 550 825 600 vertices of the metric polytope on* 8 *nodes are partitioned into* 533 *orbits:*

(i) *1 cut orbit $O_{\delta(S)}$ with* $\mathrm{Icd}_{\delta(S)} = 168$, $\mathrm{Adj}_{\delta(S)} \geq 119\,269\,588$ *and* $|O_{\delta(S)}| = 128$
(ii) *28 trivial extensions orbits $O_{\tilde{v}_i}$ with* $\mathrm{Icd}_{\tilde{v}_i} = 88, 79, 74, \ldots, 42$, $\mathrm{Adj}_{\tilde{v}_i} = 137\,758, \ldots, 127$ *and* $|O_{\tilde{v}_i}| = 1\,290\,240, \ldots, 3\,584$
(iii) *504 fully fractional orbits $O_{\tilde{v}_i}$:*
1 *anticut orbit $O_{\bar{\delta}(S)}$ with* $\mathrm{Icd}_{\bar{\delta}(S)} = 56$, $\mathrm{Adj}_{\bar{\delta}(S)} = 52\,367$ *and* $|O_{\bar{\delta}(S)}| = 128$
37 $(\frac{1}{3}, \frac{2}{3})$*-valued orbits $O_{\tilde{v}_i}$ with* $\mathrm{Icd}_{\tilde{v}_i} = 44, 40, \ldots, 28$, $\mathrm{Adj}_{\tilde{v}_i} = 6\,285, 5\,247, \ldots, 28$ *and* $|O_{\tilde{v}_i}| = 5\,160\,960, \ldots, 35\,840$
466 *non* $(\frac{1}{3}, \frac{2}{3})$*-valued fully fractional orbits $O_{\tilde{v}_i}$ with* $\mathrm{Icd}_{\tilde{v}_i} = 48, 45, \ldots, 29$, $\mathrm{Adj}_{\tilde{v}_i} = 22\,300, 4\,906, \ldots, 29$ *and* $|O_{\tilde{v}_i}| = 5\,160\,960, \ldots, 40\,320$.

Proposition 5.

1. *Exactly two of the 533 orbits of m_8 described in Proposition 4 are orbits of simple vertices; that is, satisfying* $\mathrm{Icd}_{\tilde{v}_i} = \mathrm{Adj}_{\tilde{v}_i} = \binom{n}{2}$. *Both representative vertices $\tilde{v}_{532}$ and $\tilde{v}_{533}$ are graphic* $(\frac{1}{3}, \frac{2}{3})$*-valued vertices, see Fig. 1.*

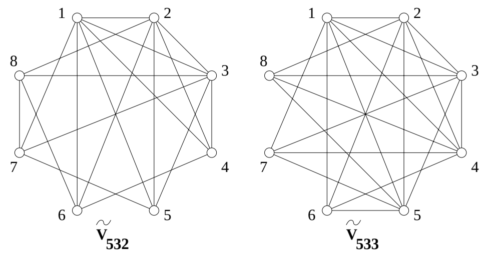

Fig. 1. Two simple graphic vertices of m_8

2. *The* $O_{\tilde{v}_{532}}$ *row of the adjacency table has only 3 nonzero entries:* $\mathrm{Adj}_{\tilde{v}_{532},\delta(S)} = 14$, $\mathrm{Adj}_{\tilde{v}_{532},\tilde{v}_4} = 7$ *and* $\mathrm{Adj}_{\tilde{v}_{532},\tilde{v}_{41}} = 7$ *with* $\mathrm{Icd}_{\tilde{v}_{41}} = 42$. *It implies that, among the 532 non-cut orbits, the vertices of* $O_{\tilde{v}_4}$ *and* $O_{\tilde{v}_{41}}$ *are the only initial vertices such that the restriction of the skeleton of* m_n *to* $O_{\tilde{v}_{start}}$ *and* $O_{\tilde{v}_i : \mathrm{Icd}_{\tilde{v}_i} \leq 41}$ *could be connected. One can easily check that it holds for both* $v_{start} = \tilde{v}_4$ *and* $v_{start} = \tilde{v}_{41}$.
3. *One can easily check that the set* $N_{\tilde{v}_{532}}$ *of the neighbors of* $\tilde{v}_{532}$ *and the set* $N_{r_{\delta(\{1\})}(\tilde{v}_{533})}$ *of the neighbors of the switching of* $\tilde{v}_{533}$ *by the cut* $\delta(\{1\})$ *are disjoint. It implies that the diameter* $\delta(m_8^{533})$ *of the restriction of the skeleton of* m_8 *to the 533 orbits described in Proposition 4 satisfies* $\delta(m_8^{533}) \geq 3$. *Since the* LAURENT-POLJAK *conjecture holds for these 533 orbits, see Item 2 of Proposition 3, we have* $\delta(m_8^{533}) = 3$.

Remark 3.

1. For $n \leq 7$, we have $Adj_{\tilde{v}} < Adj_{\delta(S)}$ for $\tilde{v} \neq \delta(S)$ and it is conjectured in [3] to be true for any n. For $n = 8$, it holds for the 533 orbits described in Proposition 4 as we have $865 \times Adj_{\tilde{v}_i} < Adj_{\delta(S)}$ for $i = 2 \ldots 533$.
2. For any n, we have $|O_{\tilde{v}}| \leq |Is(m_n)| = 2^{n-1}n!$. While for $n \leq 7$, this inequality is strict for all orbits, it is satisfied with equality for the largest (fully fractional) orbits of m_8.
3. It is conjectured, see [3], that for n large enough, at least one vertex of m_n is simple. While it is false $n = 6$ and 7, Item 1 of Proposition 5 implies that it holds for $n = 8$.

6 Conclusions

We presented an orbitwise enumeration algorithm for combinatorial polytopes with large symmetry group. In particular, we computed 1 550 825 600 vertices of a highly degenerate 28-dimensional polytope defined by its 224 facets: the metric polytope on 8 nodes. The description consists of only 533 canonical representatives and we conjecture it is complete. The orbitwise incidence, adjacency and skeleton are also given. While the extreme connectivity of the cuts (LAURENT-POLJAK conjectured they form a dominating set) appears to be one of the main features of the skeleton of m_n, we conjecture that the cut vertices do not form a cut-set in the skeleton of m_n. The combinatorial and computational applications of this conjecture are studied. In particular, a heuristic skipping the metric cone is presented. The algorithm can be parallelized very easily and, combined with the heuristic, higher-dimensional instances of the metric polytope and other combinatorial polyhedra vertex enumeration problems could be solvable. While the largest previously computed metric polytope m_7 has only 13 orbits of vertices, m_8 has at least 533 orbits and therefore could be large enough to reveal some general features of the metric polytope on n nodes. In particular, the skeleton of m_8 suggests the orbitwise adjacency relations between the cuts, anticuts, the trivial extensions and the fully fractional orbits: The row $Adj_{\delta(S),j}$ and the column $Adj_{i,\delta(S)}$ should have only nonzero entries (Laurent-Poljak dominating set conjecture). The anticuts are mainly orbitwise adjacent to (few) trivial extensions and the fully fractional orbits are badly orbitwise connected among themselves. The trivial extensions are well connected among themselves and not so well to the fully fractional orbits but still the restriction to the non-cut orbits is connected (no cut-set conjecture).

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References

1. Avis D., Bremner D., Seidel R.: How good are convex hull algorithms? Computational Geometry: Theory and Applications **7** (1997) 265–301
2. Christof T., Reinelt G.: Decomposition and Parallelization Techniques for Enumerating the Facets of 0/1-Polytopes. Preprint, Heidelberg University (1998)
3. Deza A, Deza M., Fukuda F.: On skeletons, diameters and volumes of metric polyhedra. Lecture Notes in Computer Science, Vol. 1120. Springer-Verlag, Berlin Heidelberg New York (1996) 112–128
4. Deza M., Laurent M.: Geometry of cuts and metrics. Algorithms and Combinatorics Vol. 15 Springer-Verlag, Berlin Heidelberg New York (1997)
5. Fukuda K.: cddlib reference manual, version 0.86, IFOR, ETHZ, Zürich, Switzerland (1999)
6. Grishukhin V. P.: Computing extreme rays of the metric cone for seven points. European Journal of Combinatorics **13** (1992) 153–165
7. Iri M.: On an extension of maximum-flow minimum-cut theorem to multicommodity flows. Journal of the Operational Society of Japan **13** (1970-1971) 129-135
8. Laurent M, Poljak S.: The metric polytope. In: Balas E., Cornuejils G., Kannan R. (eds.): Integer Programming and Combinatorial Optimization (1992) 247–286
9. Poljak S., Tuza Z.: Maximum Cuts and Large Bipartite Subgraphs. In: Cook W., Lovasz L., Seymour P. D. (eds.): DIMACS **20** (1995) 181–244