Math. Japonica 40, No. 2(1994), 371–380

ON LOWER BOUND FOR GENERAL CONVEX POLYTOPES

ANTOINE DEZA

Received April 27, 1993

ABSTRACT. One of the fundamental questions in the combinatorial theory of convex polytopes is to determine the largest and the smallest number of vertices, edges etc. of a d-dimensional polytope with a given number m of facets. McMullen's upper bound theorem fully answers the first part of the question; and Barnette's lower bound theorem answers the second part for simple polytopes. In this paper we present a lower bound for the number of vertices of a general d-dimensional polytope with a given number m of facets. The tightness of this bound is proved using McMullen's conditions and bipyramids.

1. Main theorem. Convex polytopes are the d-dimensional analogues of 2-dimensional convex polygones and 3-dimensional convex polyhedra. A *polytope* is a bounded convex set in R^d that is the intersection of a finite number of closed halfspaces. The *faces* of a polytope are its intersections with supporting hyperplanes. The i-dimensional faces are called the *i-faces* and $f_i(P)$ denotes the number of *i-faces* of a polytope P; the d-tuple $(f_0(P), f_1(P), \ldots, f_{d-1}(P))$ is called the *f-vector* of P. In particular, 0-faces, 1-faces and $(d-1)$-faces are respectively called *vertices*, *edges* and *facets* of a d-dimensional polytope. One of the most important question in the combinatorial theory of convex polytopes is the determination of the largest and the smallest number of i-faces of a d-dimensional polytope with a given number of m of facets. Moreover, it is also interesting to find out which class of polytopes attains those bounds. General references to the topics discussed in this paper are [5, 6, 9]. In this section we first recall McMullen's upper bound theorem and Barnette's lower bound theorem for simple polytopes. Then we present a lower bound for general convex polytopes; the tightness of this bound is proved using McMullen's conditions and some particular types of polytopes.

The upper bound theorem was conjectured by Motzkin [10] in 1957 and proved by McMullen [7] in 1970. In order to state this theorem, we define for $i \geq 0$:

$$(1)\qquad u_i^d(m) = \sum_{j=0}^{d'} \binom{j}{i}\binom{m-d+j-1}{j} + \sum_{j=0}^{d''} \binom{d-j}{i}\binom{m-d+j-1}{j},$$

where $d' = \lfloor \frac{d}{2} \rfloor$ and $d'' = \lfloor \frac{d-1}{2} \rfloor$. Note that $d = d' + d'' + 1$.

We also recall that, with k a nonnegative integer, a polytope P is *k-neighbourly* if every k-subset of the set of the vertices of P is the vertex set of a proper face of P. A $\lfloor \frac{d}{2} \rfloor$-neighbourly polytope is simply called a *neighbourly* polytope. With those notations the upper bound theorem can be stated as follows:

Theorem 1.1. [7] *For any d-dimensional polytope P with m facets we have:*

[1] *Key words and phrases:* combinatorial geometry, convex polytopes, McMullen's conditions.

$$f_i(P) \le u_i^d(m) \qquad \textit{for } i = 0, \ldots, d-1.$$

Furthermore, if P is the dual of a neighbourly simplicial polytope, then

$$f_i(P) = u_i^d(m) \qquad \textit{for } i = 0, \ldots, d-1.$$

Remark 1.2. *Some calculation shows that:*

$$u_0^d(m) = \binom{m - d' - 1}{d''} + \binom{m - d'' - 1}{d'}.$$

The lower bound theorem was proved by Barnette [1, 2] in 1971-73. As for the upper bound theorem, we first need to define:

$$\varphi_i^d(m) = \begin{cases} (d-1)m - (d+1)(d-2), & \text{i=0;} \\ \binom{d}{i+1}m - \binom{d+1}{i+1}(d-1-i) & \text{i=1, \ldots , d-1.} \end{cases}$$

With this notation the lower bound theorem can be stated as follows:

Theorem 1.3. [1, 2] *For any simple d-dimensional polytope P with m facets we have:*

$$f_i(P) \ge \varphi_i^d(m) \qquad \textit{for } i = 0, \ldots, d-1.$$

Furthermore there are simple polytopes P with m facets such that

$$f_i(P) = \varphi_i^d(m) \qquad \textit{for } i = 0, \ldots, d-1.$$

While the upper bound theorem is valid for general convex polytopes, the lower bound theorem holds only for simple polytopes. In the next theorem we present a lower bound valid for general polytopes. First we define the following step function $l_0^d(m)$ by the relation:

$$l_0^d(m) = i \;\text{ if and only if }\; u_0^d(i-1) < m \le u_0^d(i).$$

This function is a sort of inverse function of $u_0^d(m)$. Moreover, we see in Section 4 that one can easily prove that $l_0^d(m)$ is a lower bound for the number of vertices of a polytope with m facets. The following theorem actually establishes the tightness of $l_0^d(m)$ and characterizes the class of polytopes which attain this lower bound.

Theorem 1.4.

(*i*) *In even dimension, for any d-dimensional polytope P with m facets, we have:*

$$f_0(P) \ge l_0^d(m).$$

Furthermore there are simplicial $(\lfloor \frac{d}{2} \rfloor - 1)$-neighbourly polytopes P with m facets such that

$$f_0(P) = l_0^d(m) \qquad \textit{for } m \ge u_0^d(2d-1).$$

(*ii*) *In odd dimension, for any d-dimensional polytope P with m facets, we have:*

$$f_0(P) \ge l_0^d(m).$$

Furthermore there are simplicial $(\lfloor \frac{d}{2} \rfloor - 1)$-neighbourly polytopes P with m facets such that

$$f_0(P) = l_0^d(m) \qquad \textit{for } m \textit{ even and } m \ge u_0^d(d + \lfloor \tfrac{d}{2} \rfloor),$$

and there are polytopes P with m facets such that

$$f_0(P) \le l_0^d(m) + 1 \qquad \textit{for } m \textit{ odd and } m \ge u_0^d(2d-1).$$

Figures 3.1, 3.2 and 3.3 illustrate cases $d = 4, 5$ and 6. In particular we get a tight lower lower bound for any m in dimension 3, 4 and 6. In dimension 5, we have a tight lower bound except for $m = (r-3)(r-4) - 1$, $r \ge 8$; for those values, $l_0^5(m) + 1$ is attained.

Remark 1.5. *One can easily check that for d fixed, $l_0^d(m)$ is $O(\sqrt[\lfloor \frac{d}{2} \rfloor]{m})$.*

Before giving a complete proof of Theorem 1.4 in Section 3, we first recall in Section 2 the characterization of f-vector for simplicial polytopes. This characterization is used to prove the tightness of this lower bound for general polytopes.

2. Characterization of the f-vector of a simplicial polytope. In this section we present a characterization of the f-vector of a simplicial polytope. This characterization called McMullen's conditions was conjectured by McMullen [8] in 1971. The sufficiency of the conditions was proved by Billera and Lee [3, 4] in 1980-1981; the necessity was established by Stanley [11] in 1980.

For a d-tuple $f = (f_0, f_1, \dots, f_{d-1})$ of positive integers, we define the associated g-vector as:

$$g_i = \sum_{j=-1}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_j \qquad \text{for } i = -1, \dots, d,$$

with the conventions $f_{-1} = 1$ and $f_d = 0$.
Some calculation [9] shows that

$$(2) \qquad f_i = \sum_{j=-1}^{i} \binom{d-j}{d-i} g_j \qquad \text{for } i = 0, \dots, d-1.$$

For positive integers h and i, there exist uniquely determined positive integers $r_0, r_1, \dots, r_q$ with $q < i$ such that $h = \binom{r_0}{i} + \binom{r_1}{i-1} + \dots + \binom{r_q}{i-q}$. This representation is called the *i-canonical* representation of h. The i-canonical representation of 0 is 0. Then, for $j > i$, $h^{<j|i>}$ is defined by:

$$h^{<j|i>} = \binom{r_0 + j - i}{j} + \binom{r_1 + j - i}{j-1} + \dots + \binom{r_q + j - i}{j-q}.$$

We also recall that $d' = \lfloor \frac{d}{2} \rfloor$ and $d'' = \lfloor \frac{d-1}{2} \rfloor$.

Theorem 2.1. [3, 4, 11] *A d-tuple $f = (f_0, f_1, \dots, f_{d-1})$ of positive integers is the f-vector of a simplicial polytope if and only if the associated g-vector satisfies the following three conditions:*

(c_1)	$g_i = -g_{d-i-1}$	*for* $i = -1, 0, \dots, d''$,
(c_2)	$g_i \geq 0$	*for* $i = 0, \dots, d'-1$,
(c_3)	$g_i \leq g_{i-1}^{<i+1\|i>}$	*for* $i = 1, \dots, d'-1$.

Remark 2.2. *For a simplicial neighbourly polytope P, the dimension d and the number of vertices $f_0(P)$ are sufficient to determine the complete f-vector of P, moreover for $i = 0, \dots, d'-1$ we have: $f_i(P) = \binom{f_0}{i+1}$. Restating this in term of g_i, we can say that for a simplicial neighbourly polytope, the complete g-vector is fully determined by the dimension d and g_0. Moreover some calculation shows that $g_i(P) = \binom{g_0 + i}{i+1}$ for $i = 0, \dots, d'-1$.*

3. Lower bound for the number of vertices of a convex polytope with m facets. This section is devoted to the proof of Theorem 1.4. First we explain why we chose $l_0^d(m)$ as a candidate to be the lower bound for the number of vertices for a convex polytope with m facets. Then we introduce a family of $(\lfloor \frac{d}{2} \rfloor - 1)$-neighbourly simplicial polytopes. Finally, using this family and bipyramids, we prove the tightness of $l_0^d(m)$.

First using Theorem 1.1, for $i = 0$ we get the upper bound for $f_0(P)$, the number of vertices of a d-dimensional polytope P with m facets:

$$(3) \qquad f_0(P) \leq u_0^d(m), \text{ we recall that } u_0^d(m) = \binom{m - d' - 1}{d''} + \binom{m - d'' - 1}{d'}.$$

A dual version is, with m denoting the number of facets of a d-dimensional polytope P with f_0 vertices:

$$(4) \qquad m \leq u_0^d(f_0).$$

This last inequality led us to define $l_0^d(m)$ as the step function presented in Section 2. Indeed, the inequality (4) implies that a polytope, with m facets such as $m > u_0^d(f_0)$, has necessarily at least $f_0 + 1$ vertices. Therefore, with $f_0(P)$ denoting the number of vertices of a d-dimensional polytope P with m facets, we have:

$$f_0(P) \geq l_0^d(m),$$

which means that $l_0^d(m)$ is a lower bound for the number of vertices of a polytope with m facets. Now, we have to prove that this lower bound is attained. First, we recall that (3) is satisfied with equality if P is a dual neighbourly polytope. Therefore (4) is satisfied with equality if P is a neighbourly polytope with f_0 vertices, i.e. for $m = u_0^d(f_0)$, $f_0 \geq d+1$. Therefore the lower bound $l_0^d(m)$ is attained for $m = u_0^d(f_0)$ by neighbourly polytopes with f_0 vertices since obviously $l_0^d(u_0^d(f_0)) = f_0$. In other words $l_0^d(m)$ is a tight lower bound for $m = u_0^d(f_0)$, $f_0 \geq d+1$. Our objective is to prove the tightness of this lower bound for other values of m, i.e. for $u_0^d(f_0 - 1) < m < u_0^d(f_0)$ with $f_0 > d+2$. In order to do so we introduce a family of polytopes with suitable properties. We consider a $d-$tuple $f = (f_0, f_1, \ldots, f_{d-1})$ such that the associated g-vector $g = (g_{-1}, g_0, \ldots, g_d)$ satisfies the following conditions:

(a_1) $\quad g_i = -g_{d-i-1}$ $\quad$ for $i = -1, 0, \ldots, d''$,

(a_2) $\quad g_i = \binom{g_0 + i}{i+1}$ $\quad$ for $i = -1, \ldots, d' - 2$, with $g_0 \geq 0$,

(a_3) $\quad g_{d'-1} = \binom{g_0 + d' - 1}{d'} - \delta$ with $\delta \in \{0, 1, \ldots, \binom{g_0 + d' - 1}{d'}\}$.

We recall that the associated g-vector is given by:

$$g_i = \sum_{j=-1}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_j \qquad \text{for } i = -1, \ldots, d,$$

with the conventions $f_{-1} = 1$ and $f_d = 0$.

Lemma 3.1. *A d-tuple f such that the associated g-vector satisfies the conditions $(a_1), (a_2)$ and (a_3) is the f-vector of a simplicial polytope.*

Proof. To prove it we just need to check that McMullen's conditions presented in Section 3 are satisfied. (c_1), respectively (c_2), obviously holds using (a_1), respectively (a_2) and (a_3). To check (c_3), we first have to calculate the i-canonical representation of g_{i-1} for $i = 1, \ldots, d' - 1$. Using (a_2) we have:

$$g_{i-1} = \binom{g_0 + i - 1}{i} \qquad \text{for } i = 1, \ldots, d' - 1,$$

then the $i - canonical$ representation of g_{i-1} is obviously:

$$\binom{g_0 + i - 1}{i} = \binom{g_0 + i - 1}{i} \qquad \text{for } i = 1, \ldots, d' - 1,$$

i.e.

$$g_{i-1}^{<i+1|i>} = \binom{g_0 + i}{i+1} \qquad \text{for } i = 1, \ldots, d' - 1,$$

thus

$$(5) \qquad g_{i-1}^{<i+1|i>} = g_i \qquad \text{for } i = 1, \ldots, d' - 2.$$

(5) implies that (c_3) holds for $i = 1, \ldots, d' - 2$. To complete the proof we have to check that (c_3) holds for $i = d' - 1$. Using (5) we notice that (a_3) can be read as:

$$g_{d'-1} \leq g_{d'-2}^{<d'|d'-1>},$$

which is the desired inequality and completes the proof. ∎

The next lemma give us more details about the f-vector of a simplicial polytope such that the associated g-vector satisfies the conditions $(a_1), (a_2)$ and (a_3).

Lemma 3.2. *Let P_δ be a polytope of the class of simplicial polytopes such that the associated g-vector satisfies the conditions $(a_1), (a_2)$ and (a_3); we have:*

(i) $\quad$ *P_δ is a $(\lfloor \frac{d}{2} \rfloor - 1)$-neighbourly polytope with $g_0 + d + 1$ vertices.*

(ii) P_δ *has* $u_0^d(g_0 + d + 1) - \delta$ *facets in even dimension and,*
(iii) P_δ *has* $u_0^d(g_0 + d + 1) - 2\delta$ *facets in odd dimension.*

Proof. Since g_i for $i = -1, 0 \dots, d$ are given by (a_1), (a_2) and (a_3); we are able to calculate the f-vector of P_δ using (2). If we set $j = -1$ and $j = 0$ in (2), we have

$$f_0(P_\delta) = d + 1 + g_0,$$

thus P_δ has $g_0 + d + 1$ vertices.

Then, to determine the degree of neighbourliness of P_δ, using Remark 2.2, we notice that (a_2) means that P_δ has the same g_j as a simplicial neighbourly polytope for $j = -1, \dots, d' - 2$. Now, using (2), we remark that for $i = 0, \dots, d' - 2$, f_i depends only on g_j for $j = -1, \dots, i$. This implies that the $f_i(P_\delta)$ are the same as for a neighbourly polytope with $g_0 + d + 1$ vertices for $i = 0, \dots, d' - 2$, which means that P_δ is a $(d' - 1)$-neighbourly polytope and completes the proof of part (i) of Lemma 3.2. Moreover, using the same argument, we obtain that P_0 is an neighbourly polytope with $g_0 + d + 1$ vertices. To complete the proof of Lemma 3.2, we have to evaluate $f_{d-1}(P_\delta)$, the number of facets of P_δ. Using (2), we have:

$$\begin{aligned} f_{d-1}(P_\delta) &= \sum_{j=-1}^{d-1} (d-j) g_j \\ &= \sum_{j=-1}^{d'} (d-j) g_j + \sum_{j=d'+1}^{d-1} (d-j) g_j \\ &= \sum_{j=-1}^{d'} (d-j) g_j - \sum_{j=0}^{d''-1} (j+1) g_j. \qquad (using(a_1)) \end{aligned}$$

Since that (a_1) implies $g_{d'} = 0$ in odd dimension, we have:

$$f_{d-1}(P_\delta) = \begin{cases} \sum_{j=-1}^{d'-1} (d-2j-1) g_j + g_{d'-1} & \text{in even dimension} \\ \sum_{j=-1}^{d'-1} (d-2j-1) g_j + 2 g_{d'-1} & \text{in odd dimension} \end{cases}$$

$$(6) \qquad = \begin{cases} \sum_{j=-1}^{d'-1} (d-2j-1) g_j + \binom{g_0 + d' - 1}{d'} - \delta & \text{in even dimension} \\ \sum_{j=-1}^{d'-1} (d-2j-1) g_j + 2\binom{g_0 + d' - 1}{d'} - 2\delta & \text{in odd dimension} \end{cases}$$

Since P_0 is a neighbourly polytope with $g_0 + d + 1$ vertices, we have:

$$f_{d-1}(P_\delta) = u_0^d(g_0 + d + 1) \qquad \text{for } \delta = 0, \text{ in even and odd dimension.}$$

This together with (6) implies:

$$f_{d-1}(P_\delta) = \begin{cases} u_0^d(g_0 + d + 1) - \delta & \text{in even dimension} \\ u_0^d(g_0 + d + 1) - 2\delta & \text{in odd dimension} \end{cases}$$

which completes the proof of Lemma 3.2. ∎

Remark 3.3. *It is not surprising to find a different result for* f_{d-1} *in even and in odd dimension since simplicial polytopes satisfy the relation:* $d\, f_{d-1} = 2\, f_{d-2}$ *which implies that a simplicial polytope has a even number of facets in odd dimension.*

Proof of Theorem 1.4 At the beginning of this section we noticed that $l_0^d(m)$ was a lower bound for the number of vertices of a polytope with m facets. Then we added that $l_0^d(m)$ was attained for $m = u_0^d(f_0)$ by neighbourly polytopes with f_0 vertices, $f_0 \geq d+1$. Therefore, to complete the proof of Theorem 1.4 we need to fill the gap between $u_0^d(f_0)$ and $u_0^d(f_0-1)$ with polytopes having f_0 vertices, $f_0 > d+1$. The candidates are of course the P_δ with $g_0 = f_0 - d - 1$. We separately consider the even and odd dimensional case.

In even dimension, Lemma 3.2 implies that, for a given $g_0 = f_0 - d - 1$, as δ increases by 1 from 0 to $\binom{f_0 - d' - 2}{d'}$, $f_{d-1}(P_\delta)$, the number of facets of P_δ, decreases by 1 from $u_0^d(f_0)$ to $u_0^d(f_0) - \binom{f_0 - d' - 2}{d'}$. As P_δ has f_0 vertices, these numbers completely fill the gap between two neighbourly polytopes with $f_0 - 1$ and f_0 vertices if the following inequality holds:

$$\begin{aligned}
\binom{f_0 - d' - 2}{d'} &\geq u_0^d(f_0) - u_0^d(f_0 - 1)\\
&= \binom{f_0 - d'}{d'} + \binom{f_0 - d' - 1}{d' - 1} - \binom{f_0 - d' - 1}{d'} - \binom{f_0 - d' - 2}{d' - 1}\\
&= \binom{f_0 - d' - 1}{d' - 1} + \binom{f_0 - d' - 2}{d' - 2}\\
&= \tfrac{f_0 - 2}{d' - 1} \binom{f_0 - d' - 2}{d' - 2},
\end{aligned}$$

$$\begin{aligned}
\text{hence} \quad & \tfrac{(f_0 - 2d')\,(f_0 - 2d' - 1)}{d'\,(f_0 - 2)} \geq 1,\\
\text{thus} \quad & f_0 \geq 4d',\\
\text{thus} \quad & m \geq u_0^d(2d - 1),
\end{aligned}$$

which completes the proof of part (i) of Theorem 1.4.

In odd dimension, Lemma 3.2 implies that, for a given $g_0 = f_0 - d - 1$, as δ increases by 1 from 0 to $\binom{f_0 - d' - 2}{d'}$, $f_{d-1}(P_\delta)$, the number of facets of P_δ, decreases by 2 from $u_0^d(f_0)$ to $u_0^d(f_0) - 2\binom{f_0 - d' - 2}{d'}$. As P_δ has f_0 vertices, these numbers fill the gap between two neighbourly polytopes with $f_0 - 1$ and f_0 vertices for even m if the following inequality holds:

$$\begin{aligned}
2\binom{f_0 - d' - 2}{d'} &\geq u_0^d(f_0) - u_0^d(f_0 - 1)\\
&= 2\binom{f_0 - d' - 1}{d'} - 2\binom{f_0 - d' - 2}{d'},
\end{aligned}$$

$$\begin{aligned}
\text{hence} \quad & \binom{f_0 - d' - 2}{d'} \geq \binom{f_0 - d' - 2}{d' - 1},\\
\text{thus} \quad & f_0 \geq 3\,d' + 1,\\
\text{thus} \quad & m \geq u_0^d(d + \lfloor \tfrac{d}{2} \rfloor),
\end{aligned}$$

which completes the proof of part (ii) of Theorem 1.4 for m even. To complete the proof, we have to consider the case of an odd number m of facets in odd dimension. We have to find polytopes with f_0 or $f_0 + 1$ vertices for m odd and $u_0^d(f_0 - 1) < m < u_0^d(f_0)$. In order to do so, we first recall the definition of a *bipyramid.* Let Q be a $(d-1)$-dimensional polytope, and let I be a closed line segment, such that the intersection of the relative interior of Q and the relative interior of I is a single point. Then the d-dimensional polytope $P = conv(Q \cup I)$ is called a d-dimensional *bipyramid* with *basis* Q. Moreover, one can easily check that we have:

$$\begin{cases} f_0(P) = f_0(Q) + 2,\\ f_{d-1}(P) = 2\,f_{d-1}(Q). \end{cases}$$

Then we define a *degenerate bipyramid.* Let Q be a $(d-1)$-dimensional polytope, let F be a facet of Q, and let I be a closed line segment, such that the intersection of Q and the relative interior of I is a single point; and such that the intersection of the relative interior of F and the relative interior of I is a single point. Then the d-dimensional polytope $P = conv(Q \cup I)$ is called a d-dimensional *degenerate bipyramid* with *basis* Q. Moreover, one can easily check that we have:

$$(7) \qquad \begin{cases} f_0(P) = f_0(Q) + 2,\\ f_{d-1}(P) = 2\,f_{d-1}(Q) - 1. \end{cases}$$

Then we remark that, for d odd

$$\frac{2\,u_0^{d-2}(f_0-1)}{u_0^d(f_0)} = \frac{(f_0-1)}{(f_0-d'-1)} \geq 1,$$

$$\Rightarrow \quad (8) \qquad 2\,u_0^{d-2}(f_0-1) \geq u_0^d(f_0).$$

and also that, for d odd

$$\frac{2\,u_0^{d-2}(f_0-2)}{u_0^d(f_0)} = \frac{(f_0-2d'-1)\,(f_0-2)}{(f_0-d'-1)\,(f_0-d'-2)} \leq 1,$$

$$\Rightarrow \quad (9) \qquad 2\,u_0^{d-2}(f_0-2) \leq u_0^d(f_0).$$

Now, we are able to construct polytopes in odd dimension d with f_0 or f_0+1 vertices and an odd number m of facets such as $u_0^d(f_0-1) < m < u_0^d(f_0)$. First, we consider a d-dimensional degenerate bipyramid P with basis a $(d-1)$-dimensional polytope Q with f_0-2 vertices such as $u_0^{d-2}(f_0-3) < f_{d-2}(Q) \leq u_0^{d-2}(f_0-2)$. As d is odd, $d-1$ is even and we can use part (i) of Theorem 1.4, which means there are such polytopes Q if $f_0 \geq 2d$. Therefore, using (7), with $f_0 \geq 2\,d$, there are degenerate bipyramids P such as:

$$\begin{cases} P \text{ has } f_0 \text{ vertices}, \\ f_{d-1}(P) = 2\,f_{d-1}(Q) - 1. \end{cases}$$

$$\Rightarrow \quad \begin{cases} P \text{ has } f_0 \text{ vertices}, \\ 2\,u_0^{d-2}(f_0-3) + 1 \leq f_{d-1}(P) \leq 2\,u_0^{d-2}(f_0-2) - 1. \end{cases}$$

Thus, using (8) and (9), there are degenerate bipyramids P such as:

$$\begin{cases} P \text{ has } f_0 \text{ vertices}, \\ u_0^d(f_0-1) + 1 \leq f_{d-1}(P) \leq 2\,u_0^{d-2}(f_0-2) - 1, \end{cases}$$

which means that for $f_0 \geq 2d$, there are degenerate bipyramids P with m facets such that

$$(10) \qquad f_{d-1}(P) = l_0^d(m) \text{ for } m \text{ odd and } u_0^d(f_0-1) < m < 2\,u_0^{d-2}(f_0-2).$$

Then, we consider a d-dimensional degenerate bipyramid P with basis a $(d-1)$-dimensional polytope Q with f_0-1 vertices such as $u_0^{d-2}(f_0-2) < f_{d-2}(Q) \leq u_0^{d-2}(f_0-1)$. As d is odd, $d-1$ is even and we can use part (i) of Theorem 1.4, which means there are such polytopes Q if $f_0 \geq 2\,d-1$. Therefore, using (7), with $f_0 \geq 2\,d-1$, there are degenerate bipyramids P such as:

$$\begin{cases} P \text{ has } f_0+1 \text{ vertices}, \\ f_{d-1}(P) = 2\,f_{d-1}(Q) - 1. \end{cases}$$

$$\Rightarrow \quad \begin{cases} P \text{ has } f_0+1 \text{ vertices}, \\ 2\,u_0^{d-2}(f_0-2) + 1 \leq f_{d-1}(P) \leq 2\,u_0^{d-2}(f_0-1) - 1. \end{cases}$$

Thus, using (8) and (9), there are degenerate bipyramids P such as:

$$\begin{cases} P \text{ has } f_0+1 \text{ vertices}, \\ 2\,u_0^{d-2}(f_0-2) + 1 \leq f_{d-1}(P) \leq u_0^d(f_0) - 1, \end{cases}$$

which means that for $f_0 \geq 2d-1$, there are degenerate bipyramids P with m facets such that

$$(11) \qquad f_{d-1}(P) = l_0^d(m) + 1 \text{ for } m \text{ odd and } 2\,u_0^{d-2}(f_0-2) < m < u_0^d(f_0).$$

Finally, (10) and (11) complete the proof of part (ii), and therefore the proof of Theorem 1.4.

Acknowledgements The author would like to express his gratitude to Professor Komei Fukuda of the University of Tsukuba, Tokyo, for many stimulating discussions and his constant support.

References

1. Barnette D., *The minimum number of vertices of a simple polytope*, Israel J.Math. **10**, (1971), 121-125.
2. Barnette D., *A proof for the lower bound conjecture for convex polytopes*, Pacific J.Math. **46**, (1973), 349-354.

3. Billera L.J. and Lee C.W., *Sufficiency of McMullen's conditions for f-vectors of simplicial polytopes*, Bull. Amer. Math. Soc. (N.S.) **2**, (1980), 181-185.
4. Billera L.J. and Lee C.W., *A proof of sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes*, J. Combin. Theory, Ser. A **31**, (1981), 237-255.
5. Brøndsted A., *An Introduction to Convex Polytopes*, Graduate Texts in Mathematics, **90**, New York-Heidelberg-Berlin, Springer-Verlag, 1983.
6. Grünbaum B., **Convex Polytopes**, London-New York-Sydney Wiley, 1967.
7. McMullen P., *The maximum numbers of faces of a convex polytope*, Mathematika **17**, (1970), 179-184.
8. McMullen P., *The numbers of faces of simplicial polytopes*, Israel J. Math. **9**, (1971), 559-570.
9. McMullen P. and Shephard G.C., *Convex Polytopes and the Upper Bound Conjecture*, London Mathematical Society Lecture Note Series, Vol.3. Cambridge University Press, Cambridge, (1971).
10. Motzkin T.S., *Comonotone curves and polyhedra. Abstract 111*, Bull. Am. Math. Soc. **63**, (1957), 35.
11. Stanley R.P., *The number of faces of a simplicial convex polytope*, Adv. Math. **35**, (1980), 236-238.

Université de Paris-sud, centre d'Orsay, 91405 France and Department of Information Sciences, Tokyo Institute of Technology, 2-12-1 Ohokayama, Meguro-ku, Tokyo 152, Japan.
e-mail:deza@is.titech.ac.jp

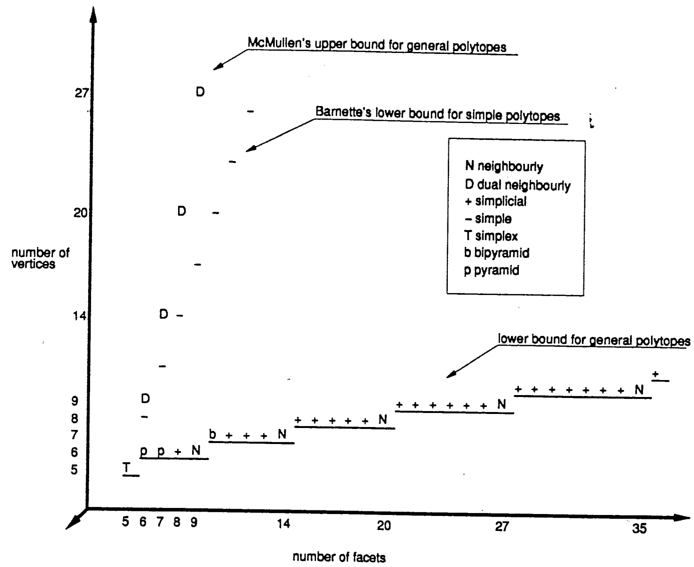

FIGURE 3.1. lower bound for the number of vertices of a 4-dimensional polytope

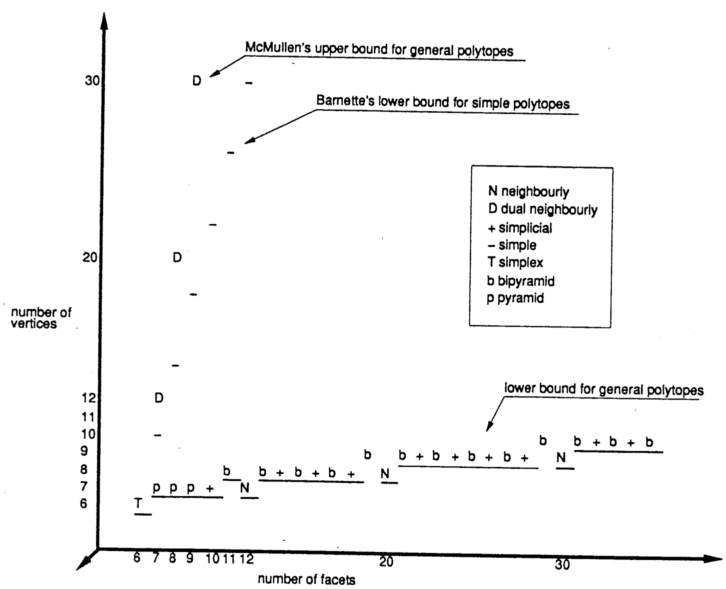

FIGURE 3.2. lower bound for the number of vertices of a 5-dimensional polytope

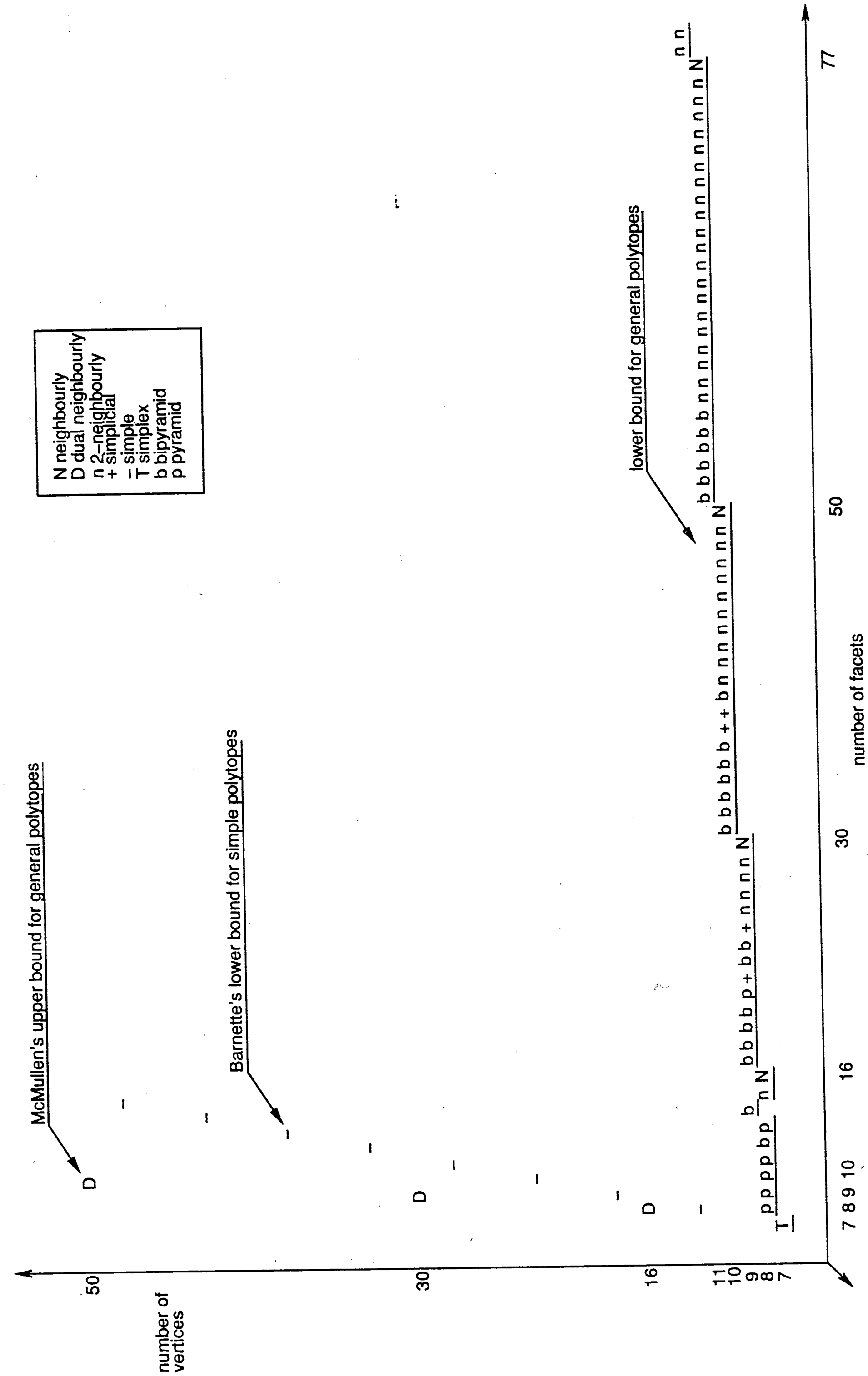

FIGURE 3.3. lower bound for the number of vertices of a 6-dimensional polytope