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On the solitaire cone and its relationship to multi-commodity flows

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Abstract. The classical game of Peg Solitaire has uncertain origins, but was certainly popular by the time of LOUIS XIV, and was described by LEIBNIZ in 1710. The modern mathematical study of the game dates to the 1960s, when the solitaire cone was first described by BOARDMAN and CONWAY. Valid inequalities over this cone, known as pagoda functions, were used to show the infeasibility of various peg games. In this paper we study the extremal structure of solitaire cones for a variety of boards, and relate their structure to the well studied metric cone. In particular we give:

1. an equivalence between the multicommodity flow problem with associated dual metric cone and a generalized peg game with associated solitaire cone;
2. a related NP-completeness result;
3. a method of generating large classes of facets;
4. a complete characterization of 0-1 facets;
5. exponential upper and lower bounds (in the dimension) on the number of facets;
6. results on the number of facets, incidence and adjacency relationships and diameter for small rectangular, toric and triangular boards;
7. a complete characterization of the adjacency of extreme rays, diameter, number of 2-faces and edge connectivity for rectangular toric boards.

1. Introduction and basic properties

1.1. Introduction

Peg solitaire is a peg game for one player which is played on a board containing a number of holes. The most common modern version uses a cross-shaped board with 33 holes – see Fig. 1 – although a 37 hole board is common in France. Computer versions of the game now feature a wide variety of shapes, including rectangles and triangles. Initially the central hole is empty, the others contain pegs. If in some row (column resp.) two consecutive pegs are adjacent to an empty hole in the same row (column resp.), we may make a *move* by removing the two pegs and placing one peg in the empty hole. The objective of the game is to make moves until only one peg remains in the central hole.

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Variations of the original game, in addition to being played on different boards, also consider various alternate starting and finishing configurations.

The game itself has uncertain origins, and different legends attest to its invention by various cultures. An authoritative account with a long annotated bibliography can be found in the comprehensive book of BEASLEY [6]. The book mentions an engraving of BEREY, dated 1697, of a lady with a Solitaire board. The book also contains a quotation of LEIBNIZ [19] which was written for the Berlin Academy in 1710. Apparently the first theoretical study of the game that was published was done in 1841 by SUREMAIN DE MISSERY, and was reported in a paper by VALLOT [26]. The modern mathematical study of the game dates to the 1960s at Cambridge University. The group was led by CONWAY who has written a chapter in [8] on various mathematical aspects of the subject.

One of the problems studied by the Cambridge group is the following basic *feasibility* problem of peg solitaire:

For a given board B, starting configuration S and finishing configuration F, determine if there is a legal sequence of moves from S to F.

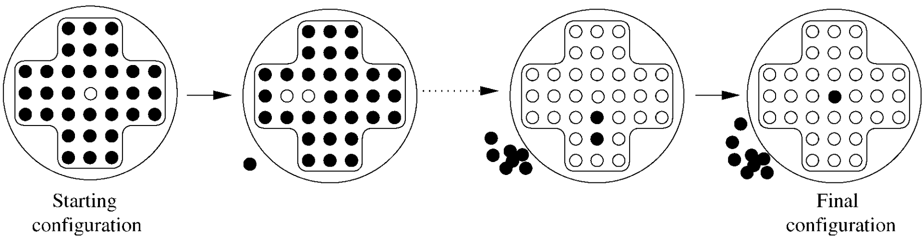

Fig. 1. A feasible English solitaire peg game with possible first and last moves

The complexity of the feasibility problem for the n by n game was shown by UEHARA AND IWATA [25] to be NP-complete, so easily checked necessary and sufficient conditions for feasibility are unlikely to exist. One of the earliest tools used to show the infeasibility of certain starting and finishing configurations is a polyhedral cone, which we will call the *solitaire cone*, S_B, corresponding to some given board B. This paper contains results on the extremal structure of this cone, which we describe in the next subsection.

1.2. Basic properties

For ease of notation, we will mostly be concerned with rectangular boards which we represent by 0-1 matrices. A zero represents an empty hole and a one represents a peg. For example, let $S = [\ 1\ 0\ 1\ 1\ ]$ and $F = [\ 0\ 0\ 1\ 0\ ]$ be starting and finishing positions for the 1 by 4 board. This game is *feasible*, involving two moves and the intermediate position [1 1 0 0] – as shown in Fig. 2.

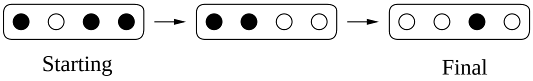

Fig. 2. A feasible game on the 1 by 4 board

For any move on an m by n board B we can define an m by n *move matrix* which has 3 non-zero entries: two entries of -1 in the positions from which pegs are removed and one entry of 1 for the hole receiving the new peg. The two moves involved in Fig. 2 are represented by $m_1 = [\ 0\ 1\ {-1}\ {-1}\ ]$ and $m_2 = [\ {-1}\ {-1}\ 1\ 0\ ]$. Clearly $F = S + m_1 + m_2$. By abuse of language, we use the term *move* for both the move itself and the move matrix. In general it is easily seen that if S, F define a feasible game of k moves there exist move matrices $m_1, \ldots, m_k$ such that

$$F - S = \sum_{i=1}^{k} m_i. \tag{1}$$

Lemma 1. *Equation 1 is necessary but not sufficient for the feasibility of a peg game.*

Proof. For example, for the following game – see Fig. 3 – $S = [\ 1\ 1\ 1\ 1\ ]$ and $F = [\ 0\ 0\ 0\ 1\ ]$. We have $F - S = [\ {-1}\ {-1}\ 1\ 0\ ] + [\ 0\ 1\ {-1}\ {-1}\ ] + [\ 0\ {-1}\ {-1}\ 1\ ]$, but S, F do not define a feasible game; in fact there are no legal moves!

□

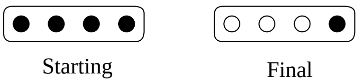

Fig. 3. An integer feasible but 0-1 infeasible game

Let us relax the conditions of the original peg game to allow any integer (positive or negative) number of pegs to occupy any hole. We call this game the *integer game*, and call the original game the *0-1 game*. Note that in a 0-1 game we require that in every position of the game a hole is either empty or contains a single peg. A *move* in the integer game is defined to correspond to the process of adding a move matrix to a given position. By the following lemma, we may identify an integer game S, F with the difference $F - S$.

Lemma 2. *Equation 1 is necessary and sufficient for the feasibility of the integer game.*

In other words, Lemma 2 states that all feasible integer games form the integer cone IS_B defined as the set of all non-negative integral combinations of moves.

Unfortunately deciding if $F - S$ can be expressed as the sum of move matrices seems to be a hard computational problem. In Sect. 2 we show that a variant of the game is NP-complete. A further relaxation of the game leads to a more tractable condition.

In the *fractional game* we allow any hole to contain a fractional (positive or negative) number of pegs. A *fractional move matrix* is obtained by multiplying a move matrix by any positive scalar. For example, let $S = [\,1\;1\;1\,]$, $F = [\,1\;0\;1\,]$. Then $F - S = [\,0\;-1\;0\,]$ $= \frac{1}{2}\,[\,-1\;-1\;1\,] + \frac{1}{2}\,[\,1\;-1\;-1\,]$ is a feasible fractional game and can be expressed as the sum of two fractional moves, but is not feasible as a 0-1 or integer game.

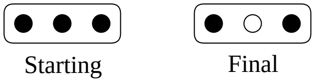

Fig. 4. A fractional feasible but integer (and therefore 0-1) infeasible game

Let B be a board and n_B the total number of possible moves on the board. The *solitaire cone* S_B is the set of all non-negative combinations of the n_B corresponding move matrices. Thus $F - S \in S_B$ if:

$$F - S = \sum_{i=1}^{n_B} y_i m_i, \quad y_i \geq 0, \quad i = 1, \ldots, n_B. \tag{2}$$

In the above definition it is assumed that the h_B holes in the board B are ordered in some way and that $F - S$ and m_i are h_B-vectors. When B is a rectangular m by n board $B_{m,n}$ it is convenient to display $F - S$ and m_i as m by n matrices, although of course all products should be interpreted as dot products of the corresponding mn-vectors.

We define the dual of the solitaire cone S_B as the the cone defined by the inequalities $m_i \cdot x \leq 0$, for each move matrix m_i defining S_B. The facets of the dual cone are therefore defined by triangle inequalities. These inequalities, combined in a very different way, also define the well studied metric cone, which we define in the next section. One of the motivations of our work was to explore the relationship between the solitaire and metric cones.

Lemma 3. *For $n \geq 4$ or $m \geq 4$, the solitaire cone $S_{m,n}$ (and its dual $S^*_{m,n}$) associated to the m by n board is a pointed full-dimensional cone.*

Proof. Consider $S_{1,n}$ with $n \geq 4$. Clearly the following $n \times n$ matrix lies in the span of the moves:

$$\begin{bmatrix} 1 & -1 & -1 & 0 & \ldots \\ 0 & -2 & 0 & \ldots & \\ & & \ddots & & \\ & \ldots & 0 & -2 & 0 \\ \ldots & 0 & -1 & -1 & 1 \end{bmatrix}$$

The first and last rows are simply moves and the other rows are sums of two different moves with the same triple. This matrix is obviously non-singular with determinant $(-2)^{n-2}$. Hence the associated solitaire cone $S_{1,n}$ is full-dimensional. For any m by n board with $m \geq 4$ or $n \geq 4$, we can extend the above idea: We get a block diagonal mn by mn matrix – one block for every row (or column), with each block being non singular. Hence $S_{m,n}$ is full-dimensional for $m \geq 4$ or $n \geq 4$. Since the dual cone clearly contains the $[1,2]^{mn}$ cube, $S^*_{m,n}$ is full-dimensional. Therefore both cones are pointed and full-dimensional.

□

Lemma 4. *The moves of the solitaire cone are extreme rays.*

Proof. All the generators – the moves – of the solitaire cone belong to the hyperplane $H : \sum_{i=1}^{h_B} x_i = -1$ and to the sphere $\Omega : \sum_{i=1}^{h_B} x_i^2 = 3$. In other words, the moves belong to a sphere of codimension 1 centered on the axis through the origin with normal $(1,\dots,1)$. This implies that the moves are extreme rays of the (pointed) solitaire cone.

□

The following result obtained in 1961 is credited to BOARDMAN (who apparently has not published anything on the subject) by BEASLEY [6], p. 87. We identify $F - S$ with the fractional game defined by S and F.

Proposition 1. *Equation 2 ($F - S \in S_B$) is necessary and sufficient for the feasibility of the fractional game; that is, the solitaire cone S_B is the cone of all feasible fractional games.*

Lemma 1 and 2 and Proposition 1 are illustrated in Fig. 5

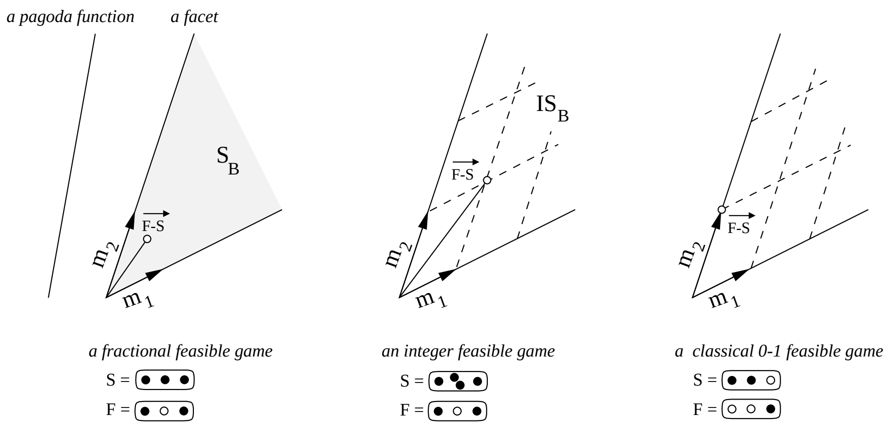

Fig. 5. Respectively fractional, integer and 0-1 feasible solitaire peg games

The condition $F - S \in S_B$ is therefore a necessary condition for the feasibility of the original peg game and, more usefully, provides a certificate for the infeasibility of certain games. The certificate of infeasibility is any inequality valid for S_B which is violated by $F - S$. According to [6], p. 71, these inequalities "were developed by J.H. Conway and J.M. Boardman in 1961, and were called *pagoda functions* by Conway...". They are also known as *resource counts*, and are discussed in some detail in CONWAY [8]. The strongest such inequalities are those that support the facets of S_B. For example, the facet given in Fig. 14 induces an inequality $a \cdot x \le 0$ that is violated by $F - S$ with S, F given in Fig. 6 ($(F - S) \cdot a = 2 > 0$). This implies that this game is not feasible even as a fractional game and, therefore, not feasible as an integer game or classical 0-1 game either.

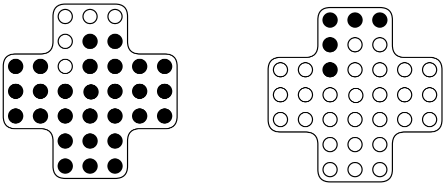

Fig. 6. An infeasible classical solitaire peg game

Other tools to show the infeasibility of various peg games include the so-called *rule-of-three* which simply amounts to color the board by diagonals of α, β and γ (in either direction). Then, with $\#\alpha$ ($\#\beta, \#\gamma$ resp.) denoting the number of pegs in an α-colored (β, γ resp.) holes, one can check that the parity of $\#\alpha - \#\beta$, $\#\beta - \#\gamma$ and $\#\gamma - \#\alpha$ is an invariant for the moves. The rule-of-three was apparently first exposed in 1841 by SUREMAIN DE MISSERY; see BEASLEY's book [6] for a detailed historical background. The rule-of-three can be used, for example, to show that on the classical cross-shaped English 33-board, starting with the initial configuration given in Fig. 1, the only reachable final configurations with *exactly one* peg are – besides the configuration of Fig. 1 – the four configurations of Fig. 7.

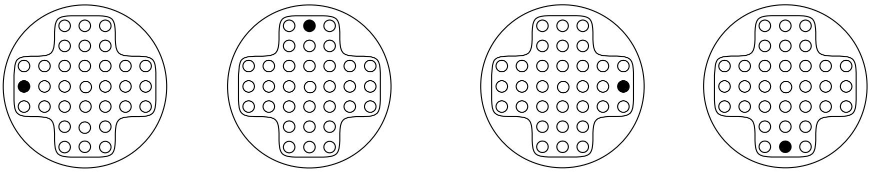

Fig. 7. Four other feasible final configurations

Another necessary condition generalizing the rule-of-three – the *solitaire lattice criterion* – is to check if $F - S$ belongs to the *solitaire lattice* generated by all integer linear combinations of moves, that is:

$$F - S = \sum_{i=1}^{n_B} y_i m_i, \quad y_i \in \mathbb{Z}, \; i = 1, \ldots, n_B \; .$$

While the lattice criterion is shown to be equivalent to the rule-of-three for the classical English 33-board and French 37-board as well as for any $m \times n$ board, the lattice criterion is stronger than the rule-of-three for games played on more complex boards. In fact, for a wide family of boards presented in this paper, the lattice criterion exponentially outperforms the rule-of-three; see DEZA AND ONN [12].

Remark 1. In later sections we will consider the toric closure of a board B which is more symmetric. These boards allow moves that traverse the boundary of the board. For example, the game given in Fig. 8 is feasible on a toric closure of the 1 by 4 board.

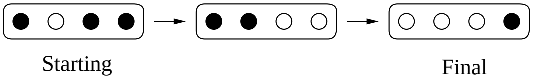

Fig. 8. A feasible classical game on the toric closure of the 1 by 4 board

Finding all facets of the cone S_B is an example of a *convex hull* or *vertex enumeration* problem, for which various computer programs are available. The computational results in this paper were obtained using the double description method *cdd* implemented by FUKUDA [14], and the reverse search method *lrs* implemented by AVIS [3]. The diameters of cones were computed using *graphy* implemented by FUKUDA [14]. We made use of these codes to completely generate all facets for some small boards as reported in later sections (such as the 95 444 facets for the toric closure of the 4 by 4 board). For realistically sized boards the corresponding solitaire cones are too large for these programs. For example the original peg solitaire game gives rise to a cone with 76 extreme rays in 33 dimensions. The reverse search vertex enumeration code *lrs* applied to this cone generated over 300,000 facets from about 6 million bases of the cone that were generated before the program was terminated. An unbiased estimate obtained from *lrs* suggests that the solitaire cone has about 9.2 million facets and 12 billion bases. If correct, this estimate puts the size of the cone just beyond problems currently solvable by a parallel version of *lrs*.

The rest of the paper is organized as follows. We give some new results on facets of S_B for various boards in Sect. 3. We present an algorithm for generating a large class of facets, and which generates all 0-1 facets for some boards. These facets are considerably more complex than the 0-1 facets of the dual of the metric cone, which are generated by cuts in the complete graph. In Sect. 4 we give some results and conjectures on the combinatorial and geometric properties of the solitaire cone. We investigate the diameter, edge connectivity, adjacency and incidence relationships of the solitaire cone and its dual. In Sect. 5 we recall some properties of the cone generated by the $\{0, 1\}$-valued facets of the solitaire cone, and some other related polyhedra. Finally in Sect. 6 we conclude by comparing and contrasting the solitaire and metric cones. The Appendix contains the proofs of Lemma 6 and Theorem 8.

2. Relationship with multicommodity flows and the metric cone

The solitaire cone is generated by a set of extreme rays, each of which is all zero except for three non-zero components which are $1, -1, -1$. In this section we relate the solitaire cone to another cone with the same property, the *flow cone* which is dual to the *metric cone*. This cone arises in the study of multicommodity flows (which we abbreviate to multiflows), and we will show similarities between this problem and the peg game. For more details on the relationship between the metric cone and multicommodity flows, see the survey paper of AVIS AND DEZA [5]. We give a brief sketch here.

2.1. *Metric cone and multicommodity flows*

Let K_n denote the complete graph on n vertices. We write the edge with endpoints i and j as either ij or ji. For each edge ij we define a non-negative integral *capacity* c_{ij} and *demand* d_{ij}. Since the same edge is labeled in two ways, we identify c_{ij} with c_{ji}, and perform similar identifications for other vectors indexed by edges. Let c and d denote the respective capacity and demand vectors of length $\binom{n}{2}$. Let $1 \le r \ne s \ne t \le n$ be three distinct indices. A *flow* on the triple (r, s, t) is a vector

$$f_{ij}^{rst} = \begin{cases} 1 & i = r,\ j = s \\ -1 & i = r,\ j = t \\ -1 & i = s,\ j = t \\ 0 & \text{otherwise.} \end{cases}$$

The *integral multiflow problem* is to try to find an integral combination of flows, z, such that $z \ge d - c$. If this is possible we say that c, d is *integer feasible*. Such a vector z defines a set of d_{ij} paths between each pair of vertices i and j in K_n. No edge ij in K_n can appear in more than c_{ij} paths. If $z = d - c$, we say the multiflow is *saturated*. This corresponds to the case where each edge ij is contained in exactly c_{ij} paths defined by z. In the example shown in Fig. 9, we have $n = 4$, $d = [\ 3\ 0\ 0\ 0\ 0\ 3\ ]$ and $c = [\ 0\ 5\ 3\ 4\ 1\ 0\ ]$. Then $d - c \le 3f^{123} + 2f^{341} + f^{342} = z$. The problem is integer feasible and the multiflow is unsaturated as there is a residual capacity of 1 on arc 1, 3.

EVEN, ITAI AND SHAMIR – see GAREY AND JOHNSON [15] p. 217 – proved the following result showing that integer feasibility is an intractable problem in general.

Theorem 1. *It is NP-complete to decide if c, d is integer feasible for the multiflow problem, even if c is a 0-1 vector and d has only two non-zero components.*

The *fractional multiflow problem* is to try to express $d - c$ as a non-negative combination of flows. If this is possible we say that c, d is *fractionally feasible*. For a fractionally feasible problem it is always possible to find a saturated multiflow, since any excess capacity on edge rs can be used by combining flows f^{rst} and f^{srt} (with possibly fractional multipliers). For example, set $n = 5$, $d = [\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ ]$ and $c = [\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ ]$. We identify the flow problem c, d with the vector $d - c$. This problem has the fractional solution

$$d - c = \frac{1}{2}(f^{124} + f^{125} + f^{134} + f^{135} + f^{234} + f^{235})$$

but no integral solution. Fractional multiflows lead to the study of the *flow cone* which is the set of all non-negative combinations of the $3\binom{n}{3}$ possible flow vectors:

$$F_n = \left\{z : z = \sum_{r,s,t} y_{rst} f^{rst} \ \text{ with } \ y_{rst} \ge 0 \text{ and } 1 \le r < s < t \le n\right\}. \qquad (3)$$

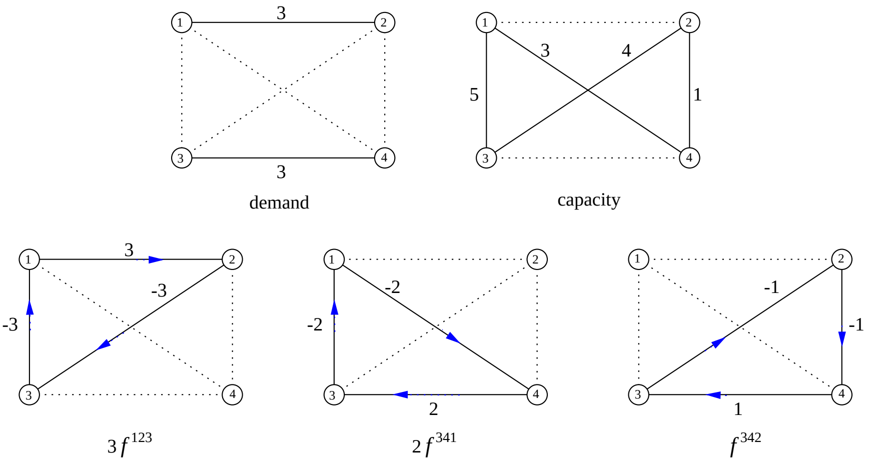

Fig. 9. An integer feasible multicommodity flow problem

F_3 is illustrated in Fig. 10. The dual of the flow cone is the much studied *metric cone*; see for example, AVIS [1], DEZA, DEZA AND FUKUDA [11] and LOMONOSOV [20]):

$$M_n = \{z : z_{rs} \le z_{rt} + z_{st} \text{ with } 1 \le r < s < t \le n\}.$$

The *Japanese theorem* of IRI [18] and ONAGA AND KAKUSHO [21] gives a necessary and sufficient condition for the feasibility of the fractional multicommodity flow problem.

Theorem 2 [18,21]. *A multiflow problem c,d is fractionally feasible if and only if $d - c \in F_n$.*

Facets of the flow cone are useful to show the infeasibility of multiflow problems. In the previous example, if we change the demand vector to $d = [\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ ]$, the problem becomes infeasible. This is demonstrated by the facet $a = [\ 2\ 2\ 1\ 1\ 2\ 1\ 1\ 1\ 1\ 2\ ]$. For each flow f^{ijk}, $a\,f^{ijk} \le 0$, but $a(d-c) = 2 > 0$ so c,d is infeasible.

The reader should compare this description of multiflows with the development of the solitaire cone in Sect. 1, and in particular equation 2 with 3, and Theorem 1 with Theorem 2. There are two obvious differences in the two problems. The first is that in the flow cone we consider all triples from a set of n elements, whereas in the solitaire cone the triples are constrained to be consecutive horizontal or vertical entries in a rectangular array. The second difference is that in the flow cone for a given triple r, s, t we include all three orderings of the non-negative entries $1,-1,-1$. In the solitaire cone we only allow the orderings $1,-1,-1$ and $-1,-1,1$. The third ordering, $-1,1,-1$, could be considered as an additional move in the peg game where two pegs surrounding an empty hole are replaced by a peg in the empty hole. The cone induced by this variation of the Solitaire game is called *the complete solitaire cone* $\mathcal{CS}_B$; see Sect. 5.3. With this additional move, we can in fact show a strong correspondence between the peg game and multicommodity flows.

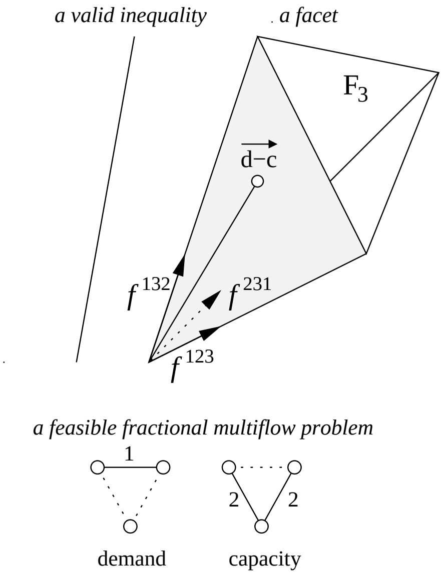

Fig. 10. All fractionally feasible multicommodity flow problems form the flow cone F_3

2.2. *Multicomomodity flows versus peg game*

Given a graph $G = (V, E)$, the *line graph* of G is a graph whose vertices correspond to edges in G. Two vertices are adjacent in the line graph if the corresponding edges share an endpoint in G. Let L_n denote the line graph of K_n. For given vectors c and d, we will construct a *generalized peg game* on $B = L_n$. Each vertex of L_n represents a hole of B. The moves on B derive from triangles in K_n. Let i, j, k be three vertices in G. Then the vertices ij, ik, jk in L_n form a *valid triangle* and a *valid move* on the board B. We allow all three possible moves on the triangle: for the integer game this means that the number of pegs on any two of the holes is decremented by one, and the number of pegs on the remaining the hole is incremented by one. We assign initially c_{ij} pegs to vertex ij of L_n. This is the starting configuration S. It is required to reach a finishing position with at least d_{ij} pegs at each vertex ij, using valid (respectively valid fractional) moves. We illustrate in Fig. 11 the correspondence for the problem presented in Fig. 9.

Theorem 3. *A multiflow problem c, d on K_n is integer (respectively fractional) feasible if and only if the corresponding generalized peg game on L_n is integer (respectively fractional) feasible. Therefore, the dual metric cone M_n^* equals the complete solitaire cone $\mathcal{C}S_{L_n}$ for a game played on the line graph of the complete graph on n nodes.*

Proof. First we show that an integer (respectively fractional) multiflow for c, d corresponds to an integral (respectively fractional) solution of the peg game. Suppose z is an integral combination of flows. Each of the flows is in one to one correspondence with a move on a valid triangle of L_n. Since $d \le c + z$, in the peg game we end up with at least d_{ij} pegs at each vertex ij. Note if the multiflow is saturated, we end up with

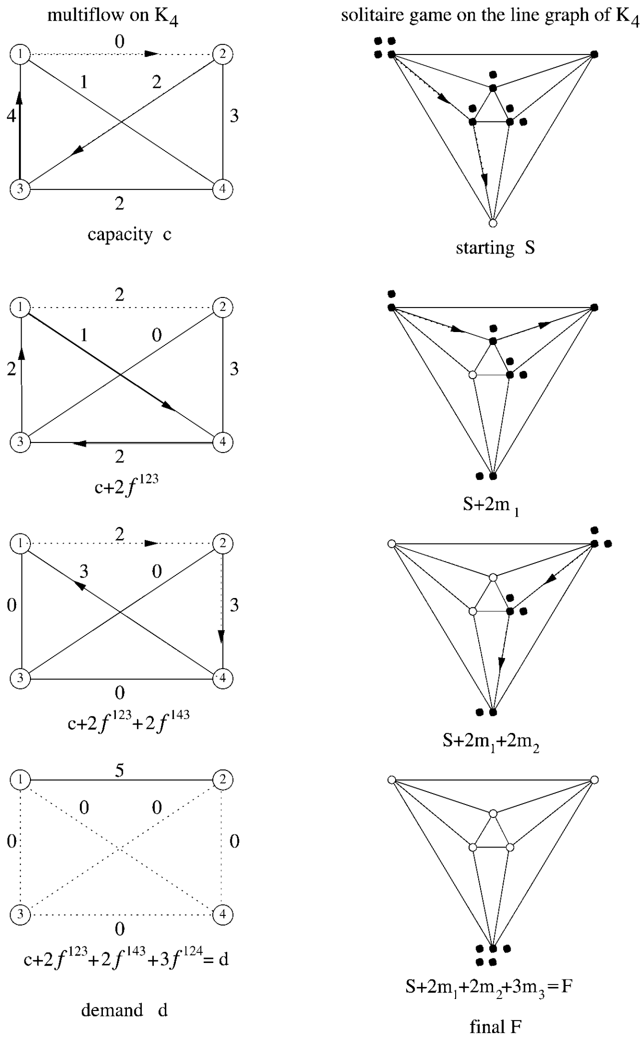

Fig. 11. Correspondence of multicommodity flows and solitaire peg game

exactly d_{ij} pegs at each vertex ij. In the same way a solution to the peg game gives the required vector z and its decomposition into flows. The same argument applies to the fractional game.

□

Together with Theorem 1 we get immediately the following corollary.

Corollary 1. *The integer generalized peg game is NP-complete even if in the starting position all holes contain at most one peg and the finishing position has positive requirements for exactly two holes.*

Remark 2. The integer generalized peg game can be solved in polynomial time if only one hole has positive finishing requirement. This is because in this case the corresponding multiflow problem is a single commodity multiflow for which many polynomial time algorithms exist. It is interesting that the 0-1 game is NP-complete on the n by n board, even if the final position contains exactly one peg; see UEHARA AND IWATA [25].

3. Facets of the solitaire cone

For simplicity we begin with rectangular boards. Most of the results can be applied to boards which are subsets of the square lattice in the plane, such as the original Peg solitaire board. We also give some results for their toric closures, which are simpler since they avoid many special situations caused by the boundary.

3.1. Rectangular boards

Let B be a rectangular m by n board, with $m \geq 4$ or $n \geq 4$. Using the notation described following equation 2, we will represent the coefficients of the facet inducing inequality

$$az \leq 0 \tag{4}$$

by the m by n array $a = [a_{i,j}]$. Inequality 4 holds for every $z \in S_B$. It is a convenient abuse of terminology to refer to a as a *facet* of S_B. A *corner* of a is a coefficient $a_{i,j}$ with $i \in \{1, m\}$ and $j \in \{1, n\}$.

We will frequently need to refer to three consecutive row or column elements of an m by n array. For this we use the notation $\mathcal{T} = (t_1, t_2, t_3)$ to refer to a *consecutive triple* of row or column indices. For example both $t_1 = i, j, t_2 = i, j+1, t_3 = i, j+2$ and $t_1 = i+2, j, t_2 = i+1, j, t_3 = i, j$ are consecutive triples. Using this notation we see that a move matrix for B is an m by n matrix that is all zero except for elements of some consecutive triple which take the values $1, -1, -1$. Each consecutive triple defines a *triangle inequality*

$$a_{t_1} \leq a_{t_2} + a_{t_3} \ . \tag{5}$$

A triangle inequality is *tight* if equality holds

$$a_{t_1} = a_{t_2} + a_{t_3} \ . \tag{6}$$

The following theorem summarizes known results on properties of valid inequalities (pagoda functions) for S_B; see BEASLEY [6]. We include a proof for completeness.

Theorem 4. *For each valid inequality $a = [a_{i,j}]$ for S_B*

1. *The triangle inequality 5 must hold for every consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$.*
2. *Negative coefficients of a can only occur in corners.*
3. *If $\mathcal{T} = (t_1, t_2, t_3)$ is a consecutive triple with $a_{t_2} = 0$ then $a_{t_1} = a_{t_3}$.*
4. *If two consecutive row (respectively, column) entries of a are zero the entire row (respectively, column) is zero.*

Proof. 1. Every move matrix is trivially in S_B and so inequality 4 with z equal to the move matrix corresponding to $\mathcal{T}$ gives the triangle inequality 5.

2. Let a_{t_2} be a coefficient of a that is not a corner. It must lie in two consecutive triples (t_1, t_2, t_3) and (t_3, t_2, t_1). The two triangle inequalities for these triples imply that $a_{t_2} \geq 0$.
3. Again this follows from the two triangle inequalities induced by the triples (t_1, t_2, t_3) and (t_3, t_2, t_1).
4. Follows by repeatedly applying part 3 of the theorem.

□

As mentioned in the introduction, it is not feasible to generate all facets for reasonably sized boards, and in general no characterization of facets is known. A large class of facets can, however, be generated by the following procedure.

GENFACET(B) /*procedure to generate a facet matrix a of S_B */

1. Choose a proper subset of coefficients of a satisfying:
   (a) If a corner is chosen, all coefficients in the row and/or column of length at least 4 containing the corner must also be chosen; and
   (b) If two consecutive coefficients are chosen, their entire row and column must also be chosen.
   Set these chosen coefficients to zero.
2. Choose any undefined coefficient that is not a corner and set it to one.
3. Choose a consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$ for which precisely two of the corresponding coefficients of a are defined. Define the remaining coefficient by equation 6 providing this does not violate any triangle inequality for a.
4. Repeat Step 3 until no further coefficient of a can be defined.

Theorem 5. *Given an m by n board B, with $m \geq 4$ or $n \geq 4$, if GENFACET(B) terminates with all elements of a defined, then a is a facet of S_B.*

Proof. By Lemma 3, under the conditions of the theorem, S_B is full dimensional. Let a be a matrix generated by GENFACET and let b be any other m by n matrix that is valid for S_B such that for every consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$,

$$a_{t_1} = a_{t_2} + a_{t_3} \Leftrightarrow b_{t_1} = b_{t_2} + b_{t_3}. \tag{7}$$

We will show that b is a positive scalar multiple of a, proving that a is facet inducing, since S_B is full-dimensional. By construction, a satisfies all triangle inequalities and so is valid for S_B, so we may apply results from Theorem 4.

First we show that for each $a_{i,j} = 0$ that is set in Step 1 of GENFACET we must also have $b_{i,j} = 0$. Initially assume that i, j is not a corner index, so there is some consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$ with $t_2 = i, j$. Suppose $a_{t_2} = 0$. It follows from Theorem 4 (3) that $a_{t_1} = a_{t_3}$. Applying relation 7 twice we obtain the equations

$$b_{t_1} = b_{t_2} + b_{t_3} \qquad\qquad b_{t_3} = b_{t_1} + b_{t_2}$$

and conclude that $b_{t_2} = 0$. An identical argument shows that $b_{i,j} = 0$ implies that $a_{i,j} = 0$. If i, j is a corner with $a_{i,j} = 0$, by construction of a there must be a consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$ with $t_1 = i, j$, $a_{t_2} = a_{t_3} = 0$, and such that a_{t_3} is not a corner coefficient. As we showed, this implies $b_{t_2} = b_{t_3} = 0$ and so applying Theorem 4 (4) to b we have $b_{t_1} = 0$.

The argument in the previous paragraph implies that all of the entries set to zero in Step 1 of GENFACET must be also zero in b. Suppose $a_{i,j}$ is the coefficient set to one in Step 2. Since it is not a corner element, Theorem 4 (2) implies that $b_{i,j} \geq 0$, and in fact it must be strictly positive since $a_{i,j}$ is non-zero. We may now scale b so that in fact $b_{i,j} = 1$.

We complete the proof with a simple inductive argument to show that at the end of each execution of Step 3 of GENFACET, if $a_{i,j}$ has been assigned then $b_{i,j} = a_{i,j}$. Indeed suppose this statement is true when Step 3 is about to be executed, and suppose some $a_{i,j}$ is defined by the equation

$$a_{t_1} = a_{t_2} + a_{t_3}.$$

One of the coefficients in this equation is $a_{i,j}$ and the other two coefficients have already been assigned. By induction, these two coefficients are equal to the corresponding coefficients of b. Now by condition 7 the equation

$$b_{t_1} = b_{t_2} + b_{t_3}$$

holds for b and so $b_{i,j} = a_{i,j}$. This completes the induction. At termination of GENFACET, all coefficients of a were assigned, so we have shown that b is equal to a up to multiplication by a positive scalar.

$\square$

Remark 3. If condition (a) in Step 1 is dropped, GENFACET could generate matrices that are not facets. For example, the matrices

$$a = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} -1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 1 \end{pmatrix}$$

have the same tight triangle inequalities. Since one is not a scalar multiple of the other, they are not facets. However a would be generated by GENFACET if condition (a) in Step 1 is dropped.

Remark 4.

1. Not all facets of rectangular boards are generated by GENFACET. For example the following facet (found by computer) of the 3 by 4 rectangular board cannot be generated by GENFACET.

$$\begin{pmatrix} -1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

2. GENFACET can easily be adapted to non-rectangular boards that are connected subsets of the square grid, such as the original peg solitaire game. The notion of corner generalizes in the obvious way to all holes that have exactly one horizontal and vertical neighbour. For example, the original English game has 8 corners and by GENFACET we can generate the two facets given in Fig. 12:

		2	5	3		
		1	3	2		
2	1	1	2	1	1	2
1	1	0	1	1	2	3
1	0	1	1	2	3	5
		1	0	1		
		2	1	1		

(i)

		1	1	1		
		0	0	0		
1	0	1	1	1	0	1
0	0	0	0	0	0	0
1	0	1	1	1	0	1
		0	0	0		
		1	1	1		

(ii)

Fig. 12. Two facets of the English solitaire cone

3. CONWAY [8] gives 14 valid inequalities (pagoda functions) for the English solitaire game. Of these, 11 described in his Figs. 21(d), 22(a)–(d) are facets of S_B and can be generated by GENFACET. The two pagoda functions in Figs. 22(h) and 22(v) – see [8] – are not facets since (h) is one half the sum of the inequality from (a) (interchanging the 0 and 1 in (a) as shown) and the same inequality with -1 replaced by +1; similarly, (v) is half the sum of 2 inequalities derived from (b) again with the indicated interchanges. The pagoda function in Fig. 21(c) – see Fig. 13 – is a facet of S_B, and is not generated by GENFACET.

		-1	0	-1		
		1	1	1		
-1	1	0	1	0	1	-1
0	1	1	2	1	1	0
-1	1	0	1	0	1	-1
		1	1	1		
		-1	0	-1		

Fig. 21 (c)

		3	0	3		
		2	0	2		
0	1	1	0	1	1	0
0	1	1	0	1	1	0
0	0	0	0	0	0	0
		1	0	1		
		-1	0	-1		

Fig. 21 (d)

Fig. 13. Two pagoda functions from CONWAY's book [8]

4. The facet of Fig. 14 given by BEASLEY [7] proves the infeasibility of the game of Fig. 6

To avoid the special effects created by the boundary of the rectangular board, we were motivated to study their toric closures. Some results on these are given in the Sect. 3.2 where we simply call them *toric boards*.

		11	8	3		
		7	5	2		
4	0	4	3	1	2	-1
3	0	3	2	1	1	0
1	0	1	1	0	1	-1
		2	1	1		
		-1	0	-1		

Fig. 14. A facet of the English board

3.2. Toric boards

From now on, the toric closure of a board is simply called a *toric board*. In other words, the toric m by n board for $m \geq 3$ or $n \geq 3$ is an m by n rectangular board with additional jumps which traverse the boundary. The associated toric solitaire cone is pointed and full-dimensional for $m \geq 3$ or $n \geq 3$. This can be proved along the lines of Lemma 3, by noting that the mn by mn matrix with -2 on the diagonal and 0 elsewhere is spanned by the move matrices expressed as mn-vectors (write each matrix row by row as a vector). Formally, we extend the definition of *consecutive triple* given in the last subsection by allowing row indices to be taken modulo m and column indices to be taken modulo n. For example, for a 4 by 4 toric board both $t_1 = 2, 3$, $t_2 = 2, 4$, $t_3 = 2, 1$ and $t_1 = 1, 3$, $t_2 = 4, 3$, $t_3 = 3, 3$ are consecutive triples. Similarly we extend the definition of a consecutive string of entries to include strings that traverse the boundary. All holes on a toric board are equivalent from the point of view of allowable jumps, so we say that the toric board has no corners.

The results of Sect. 3.1 can easily be adapted to toric boards. Theorem 4 applies, except that we get the stronger condition that all coefficients are non-negative since there are no corner coefficients. In GENFACET the condition on corner coefficients in Step 1 is not applied, since toric boards have no corners. Similarly, in Step 2 any undefined coefficient can be chosen. Theorem 5 is easily adapted to apply to toric boards B with $m \geq 3$ and $n \geq 3$, under which condition S_B is full-dimensional. Given any facet matrix, we may cyclically permute its rows and/or columns to obtain a possibly different matrix, which again defines a facet. We call such facets *isomorphic*. Observe that the 3 by 3 identity matrix and the matrix

$$a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

would be generated by GENFACET for the 3 by 3 toric board, and are facets. There are 6 facets isomorphic to the identity matrix and 9 facets isomorphic to a. These are all the facets for the solitaire cone of the 3 by 3 toric board.

We can use Theorems 4 and 5 to obtain a complete characterization of 0-1 facets of S_B when B is a toric board. Let a be an m by n 0-1 matrix. We define the 1-graph G_a

on a as follows: vertices of G_a correspond to non-zero coefficients, and two vertices are adjacent if the corresponding coefficients are in some consecutive triple where the remaining coefficient is zero. Note that in fact there must be at least two such triples since if (t_1, t_2, t_3) is such a triple then so is (t_3, t_2, t_1).

Theorem 6 (characterization of 0-1 valued facets).
Let B be the m by n toric board. A m by n 0-1 matrix b is a facet of S_B if and only if

1. *No non-zero row or column contains two consecutive zeroes, and*
2. *G_b is connected.*

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Fig. 15. Two pagoda functions of $S_{4\times 4}$, only the first one being a facet

Proof. To prove sufficiency we show that a matrix $a = b$ can be constructed by GENFACET and then apply Theorem 5 to show it is a facet. We begin by setting a_{ij} to zero if $b_{ij} = 0$. Since b is valid for S_B, Theorem 4 (4) implies the zeroes of a are a valid choice in Step 1 of GENFACET. We will show all other coefficients of a are set to one in GENFACET, so that $a = b$.

In Step 2, some non-zero coefficient $a_{i,j}$ of a is set to one, as required. Consider the first execution of Step 3. Let v be the vertex in G_b corresponding to $a_{i,j}$. Since G_b is connected, v is adjacent to some vertex w. By construction of G_b, the coefficients corresponding to v and w lie in some consecutive triple $\mathcal{T} = (t_1, t_2, t_3)$ where the remaining coefficient is zero. We may choose such a triple so that the zero coefficient is a_{t_2} or a_{t_3}. Then in the equation

$$a_{t_1} = a_{t_2} + a_{t_3}$$

one of the variables has value one and one of the right hand side variables has value zero, so the remaining variable must also have value one. Therefore in Step 3, the undefined coefficient, corresponding to vertex w, gets set to one.

In general, every time we execute Step 3 we can select an undefined coefficient whose corresponding vertex in G_b is adjacent to some already defined coefficient. Since G_b is connected, this is always possible. In this way all of the coefficients that were not set to zero receive the value one. The conditions on the zeroes are sufficient to ensure that all triangle inequalities are satisfied by b. Therefore GENFACET has constructed a which is identical to b. It follows from Theorem 5 that b is a facet, concluding the proof of sufficiency.

For necessity, suppose (1) is violated, then b violates a triangle inequality and does not generate a valid inequality for S_B. Now suppose (2) is violated. G_b consists of two or more components. We form a new matrix a that is identical to b except that in one of the connected components of G_b we replace all coefficients that have value one with the value $\frac{3}{2}$. Clearly a is not a scalar multiple of b. We show a is valid for S_B and every tight triangle inequality satisfied by b is satisfied by a. This is a contradiction since it shows b is not a facet of S_B. To show validity of a, the only triangle inequalities that could fail for a are of the form

$$a_{t_1} > a_{t_2} + a_{t_3}$$

where $a_{t_1} = \frac{3}{2}$. Now the right hand side variables cannot both be zero or the triangle inequality would fail for b, so $\{a_{t_2}, a_{t_3}\} = \{0, 1\}$. But this is impossible because it would imply an edge in G_b between the vertices corresponding to non-zero coefficients, contradicting the fact that they lie in separate components. To show a and b have the same tight triangle inequalities, consider a consecutive triple for which the equation

$$a_{t_1} = a_{t_2} + a_{t_3}$$

holds. Since the coefficient values are chosen from $\{0, 1, \frac{3}{2}\}$ there is no way both 1 and $\frac{3}{2}$ can appear in the equation. Therefore non-zero coefficients lie in the same connected component, and the same equation holds for b. Similarly, a tight triangle inequality holding for b must have any non-zero coefficients in the same connected component, so the corresponding equation holds for a.

□

Theorem 6 is useful for proving large classes of 0-1 matrices are facets. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ be two vectors. We say the m by n matrix a is the *product* of x and y if for all $1 \leq i \leq m$ and $1 \leq j \leq n$ $a_{i,j} = x_i y_j$. A simple application of Theorem 6 gives the following:

Corollary 2.

1. *A 0-1 n-vector is a facet of the 1 by n toric board if and only if it has no pair of consecutive zeroes, no string of five or more ones, and at most one string of four ones.*
2. *The product of two 0-1 facets of the* 1 *by m and* 1 *by n toric boards gives a 0-1 facet of the m by n toric board.*

We end this section by remarking that a 0-1 vector a is valid over S_B if and only if the position $F = a$ is not reachable from any other position in the 0-1 peg game. To see this note that if $F = a$ is reachable, then the last jump must result in the configuration 0 0 1 in some row or column, violating the triangle inequality, so a is not valid. Conversely if a is not valid, it must contain the string 0 0 1 in some row or column. Replacing this by the string 1 1 0 gives a position leading to $F = a$. A similar statement is not true for the integer game: for example the valid $a = [\ 0\ 1\ 1\ ]$ for the 1 by 3 game can be reached from [1 2 0]. It would be interesting to see if the 0-1 facets of the 0-1 peg game have some natural interpretation in terms of the game itself.

3.3. Bounds on number of facets

As mentioned in the introduction, experimental evidence and the fact that the n by n game is NP-complete indicates that solitaire cones are likely to have a very large number of facets. In this section we give some additional theoretical evidence for this observation. We begin by a simple exponential upper bound applicable to a large number of different boards. For simplicity we restrict ourselves to boards which are a connected subset of the square lattice.

Lemma 5 (upper bound).
Let n_B be the number of moves on a board B with h_B holes. S_B has at most $\binom{n_B}{h_B-1} < 2^{n_B}$ facets. In particular the m by n toric board generates at most $\binom{4mn}{mn-1} < (\frac{4^4}{3^3})^{mn}$ facets.

Proof. This follows from the fact that S_B is a cone in h_B dimensions defined by n_B extreme rays. Each facet is defined by a set of $h_B - 1$ of these rays.

□

Theorem 7 (lower bound).

1. *There are at least $6^{\frac{m+n}{9}}$ 0-1 facets of the m by n toric board generated by products of facets of the 1 by m and 1 by n toric boards, for m and n both divisible by 9.*
2. *There are at least $2^{\frac{mn}{4}}$ 0-1 facets generated by the m by n toric board, for even values of $m \geq 4$, and n divisible by 4.*

Proof. 1. First consider the case $m = 1$ and set $n = 9k$ for any positive integer k. Consider vectors of length $9k$ with the following properties:
(a) the first component is zero;
(b) there are a total of $3k$ zeroes, no two of which are consecutive;
(c) there are $6k$ ones arranged in $3k$ blocks, with precisely k blocks each of length one, two and three, in any order.
Each such vector satisfies the conditions of Corollary 2 (1) and so generates a facet of the 1 by n toric board. There are

$$\binom{3k}{k}\binom{2k}{k} > 6^k = 6^{\frac{n}{9}}$$

such facets. For any m which is a multiple of nine, we get the stated bound by combining the result above with Corollary 2 (2).

2. Consider an m by n matrix defined by

$$a_{ij} = \begin{cases} 1 & i + j \equiv 0 \pmod 2 \\ 0 & i + j \equiv 1 \pmod 2, \textit{and } j \equiv 0, 1 \pmod 4 \\ x & i = 1,\ j = 2 \\ 1 - x & i = 3,\ j = 2 \\ * & \text{otherwise.} \end{cases}$$

In the above, x and $*$ denote elements than can be arbitrarily set to either zero or one. A generic form of the matrix for m=4, n=8 is:

$$a = \begin{pmatrix} 1 & x & 1 & 0 & 1 & * & 1 & 0 \\ 0 & 1 & * & 1 & 0 & 1 & * & 1 \\ 1 & 1-x & 1 & 0 & 1 & * & 1 & 0 \\ 0 & 1 & * & 1 & 0 & 1 & * & 1 \end{pmatrix}$$

For $x = 0$ or $x = 1$, the remaining $\frac{mn}{4} - 2$ entries of a may be chosen freely. Therefore there are $2^{\frac{mn}{4}-1}$ matrices with this generic form. We use Theorem 6 to show that they generate facets for the m by n toric board. Indeed, by construction, there can be no two consecutive zeroes. The vertices in G_a corresponding to any given row are connected, since no row has 4 consecutive ones. Column one connects all the odd numbered rows together, and column 4 connects all the even rows. Finally the consecutive triple $(x, 1, 1 - x)$ in column two connects rows one and two, so G_a is connected. To complete the proof, we note that we can obtain an additional distinct set of $2^{\frac{mn}{4}-1}$ matrices as follows. For each matrix generated previously, delete column one and append it after the last column. These matrices are all new because the bottom right entry changes from one to zero.

□

4. Skeletons and diameters of solitaire cones

In this section, after presenting in detail more than 50 small dimensional cases, we give some results and conjectures on the combinatorial and geometric properties of the solitaire cone. In particular, we investigate the diameter, edge connectivity, adjacency and incidence relationships of the solitaire cone and its dual. Two extreme rays (resp. facets) of a polyhedral cone are *adjacent* if they belong to a face of dimension (resp. codimension) two. The number of rays (resp. facets) adjacent to the ray r (resp. facet F) is denoted A_r (resp. A_F). A ray and a facet are *incident* if the ray belongs to the facet. We denote by I_r (resp. I_F) the number of facets (resp. rays) incident to the ray r (resp. facet F). The diameter of S_B (resp. its dual S_B^*), that is, the smallest number δ such that any two vertices can be connected by a path with at most δ edges, is $\delta(S_B)$ (resp. $\delta(S_B^*)$). We recall that for $n \geq 4$ or $m \geq 4$ $S_{m,n}$ is pointed and full-dimensional and that the moves are extreme rays (for the toric case this holds for $n \geq 3$ or $m \geq 3$).

As in previous sections, for a solitaire game played on a board B a black (respectively white) hole represents a peg (respectively an empty hole) as in Fig. 16 and 18. The coordinates of a ray or a facet of S_B are naturally indexed by B as in Fig. 17 and 20.

4.1. *Small dimensional solitaire cones*

We first consider a solitaire game played on a rectangular or triangular board as in Fig. 16. In Table 1 we give for each board B, the number of extreme rays and facets, the minimal and maximal adjacency and incidence of the extreme rays A_r, I_r and of the facets A_F, I_F of the solitaire cone S_B, its diameter $\delta(S_B)$ and the diameter of its dual $\delta(S_B^*)$.

For example, the last row in Table 1 means that each of the 36 extreme rays of the cone $S_{\Delta_{15}}$ belong to at least 6 920 and at most 10 905 of its 21 744 facets. These facets are of size at least 14, that is, simplices, and at most 30 (we recall that the size of a facet

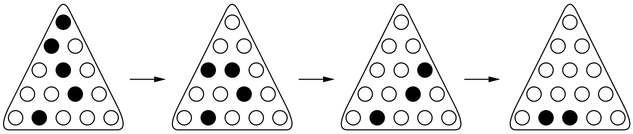

Fig. 16. Moves on the 15 triangular board $B = \Delta_{15}$

Table 1. Small rectangular and triangular boards

Board	#*rays*	I_r	A_r	# *facets*	I_F	A_F	$\delta(S_B)$	$\delta(S_B^*)$
3×3	12	11∼17	10∼11	18	8∼11	8∼17	2	2
4×4	32	1 584∼2 109	29∼31	3 531	15∼26	15∼695	2	≤ 3
3×4	20	50∼82	17∼19	107	11∼28	11∼79	2	3
3×5	28	421∼856	25∼27	1 277	14∼24	14∼429	2	3
Δ_{10}	18	87∼105	16∼17	182	9∼14	9∼79	2	3
Δ_{15}	36	6 920∼10 905	33∼35	21 744	14∼30	14∼4 750	2	≤ 3

is the number of extreme rays contained in the facet). The unique facet F_0 of maximal incidence $I_{F_0} = 30$ and maximal adjacency $A_{F_0} = 4\,750$ is induced by the inequality: $c \cdot x \leq 0$, where $c = [\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ ]$; see Fig. 17. More than half of the facets of $S_{\Delta_{15}}$ are simplices and its diameter $\delta(S_{\Delta_{15}}) = 2$.

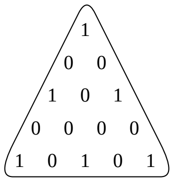

Fig. 17. The unique facet F_0 of maximal size and adjacency of $S_{\Delta_{15}}$

Then, we consider a solitaire game played on a toric rectangular or triangular board as in Fig. 18. The adjacency and incidence relationships and the diameters are given in Table 2. For example, the 16-dimensional cone generated by the 64 moves of the 4 by 4 toric board has 95 444 facets of which almost half are simplices.

4.2. Skeletons and diameters

The following results and conjectures are stated in terms of rectangular toric board but require only minor modifications for solitaire games played on toric or non-toric boards of any shapes; see Remark 6. We consider the solitaire cone $S_{m \times n}$ induced by a game played on an m by n rectangular toric board with $n \geq m \geq 1$ and $n \geq 3$. The number of extreme rays of $S_{m \times n}$, that is, the moves of the solitaire game is:

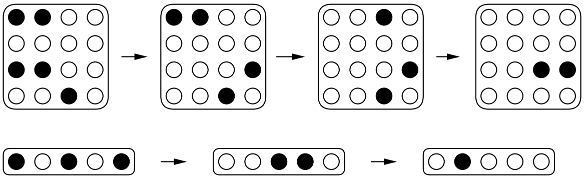

Fig. 18. Moves on the 4 by 4 and 1 by 5 toric boards

Table 2. Small rectangular toric boards

Board	*#rays*	I_r	A_r	# *facets*	I_F	A_F	$\delta(S_B)$	$\delta(S_B^*)$
1×3	3	2	2	3	2	2	1	1
1×4	8	3	3	6	4	4	3	2
1×5	10	7	6	15	4∼6	4∼6	2	3
1×6	12	9	9	17	6∼8	6∼12	2	2
1×7	14	32	12	70	6∼8	6∼13	2	5
1×8	16	42	14	86	7∼10	7∼21	2	4
1×9	18	119	16	255	8∼12	8∼36	2	5
1×10	20	214	18	447	9∼12	9∼34	2	5
1×11	22	508	20	1 078	10∼14	10∼51	2	6
1×12	24	964	22	2 013	11∼16	11∼84	2	6
3×3	18	11	15	15	12∼14	12∼14	2	2
4×4	64	25 348	58	95 444	15∼48	15∼8 195	2	?
3×4	36	190∼233	30∼33	498	13∼26	14∼166	2	3
3×5	45	12 963∼13 438	40∼42	39 060	14∼34	14∼3 404	2	?

$$\begin{array}{ll} f_o(S_{m\times 3}) = 3, 6, 18 & \text{for } m = 1, 2, 3. \\ f_o(S_{m\times n}) = 2mn & \text{for } n \geq 4 \text{ and } m \leq 2. \\ f_o(S_{3\times n}) = 3mn & \text{for } n \geq 4. \\ f_o(S_{m\times n}) = 4mn & \text{for } n \geq m \geq 4. \end{array}$$

The coordinates and coefficients of the extreme rays and facets of $S_{m\times n}$ are naturally indexed by the $m \times n$ board. For example, the extreme ray $r = [\ 0\ -1\ -1\ 1\ 0\ ]$ of $S_{1\times 5}$ and the corresponding start and finish positions are represented in Fig. 19. The *support* of r is the set $\sigma_r = \{2, 3, 4\}$ of nonzero coordinates of r.

In the next lemma we give a characterization of the adjacency of the extreme rays of the solitaire cone induced by an 1 by n board. As a corollary, we obtain the adjacency, edge-connectivity, diameter and the number of 2-faces of this cone.

Lemma 6. *Any pair of extreme rays of $S_{1\times 3}$ are adjacent and, for $n \geq 4$, two distinct extreme rays u and v of $S_{1\times n}$ are non-adjacent if and only if:*

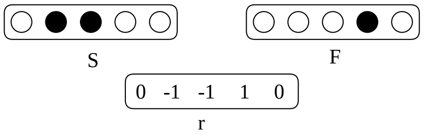

Fig. 19. The extreme ray r of $S_{1\times 5}$ corresponding to the move from S to F

1. $u \cdot v \in \{0, -2\}$ *for* $n = 4$*;*
2. $u \cdot v = -2$ *or* $u \cdot v = -1$ *and* $|\sigma_u \cap \sigma_v| = 1$ *for* $n = 5$*;*
3. $u \cdot v = -2$ *or* $u_{i+3\mathrm{mod}6} = v_i \quad i : 1, \dots, 6$ *for* $n = 6$*;*
4. $u \cdot v = -2$ *for* $n \geq 7$.

Proof. Given in the Appendix. □

Corollary 3. *The skeleton of* $S_{1\times n}$ *satisfies*

1. *the adjacency of an extreme ray is* $k_{1\times n} = 2, 3, 6, 9, 2(n-1)$ *for* $n = 3, 4, 5, 6, \geq 7$*;*
2. *the diameter* $\delta(S_{1\times n}) = 1, 3, 2$ *for* $n = 3, 4, \geq 5$*;*
3. $S_{1\times n}$ *has exactly* $3, 12, 30, 54, 2n(n-1)$ *2-faces for* $n = 3, 4, 5, 6, \geq 7$*;*
4. *the edge connectivity* $c_e(S_{1\times n}) = 2, 3, 6, 9, 2(n-1)$ *for* $n = 3, 4, 5, 6, \geq 7$.

Proof. (1), (2) The adjacency and the diameter are straightforward. (3) The number of 2-faces of a cone is half the total adjacency of its skeleton. (4) We recall the following result of PLESNÍK [23]: the edge connectivity of a graph of diameter 2 equals its minimum degree. Then, since for $n \geq 5$ the cone $S_{1\times n}$ has diameter 2 and since $c_e(S_{1\times n}) = 2, 3$ for $n = 3, 4$, we have $c_e(S_{1\times n}) = k_{1\times n}$ for $n \geq 3$. □

Clearly, for the solitaire cone induced by a 2 by n board, two rays with supports lying in the same row have the same adjacency relationships as in $S_{1\times n}$ and two rays with supports lying in different rows are always adjacent. In Theorem 8 we generalize the adjacency relationships of $S_{1\times n}$ and $S_{2\times n}$ to $S_{m\times n}$ for $n \geq 4$ or $m \geq 4$.

Theorem 8 (characterization of extreme rays adjacency).
Any pair of extreme rays of $S_{m\times n}$ *with distinct support are adjacent and, for* $m \geq 3$*, the adjacency relationship of any pair* u, v *of extreme rays of* $S_{m\times n}$ *is given by:*

1. *either* σ_u *and* σ_v *do not belong to the same row or column, then* u *and* v *are adjacent;*
2. *or* σ_u *and* σ_v *belong to the same row or column of length* n*, then* u *and* v *have the same adjacency relationship as in* $S_{1\times n}$*, except that* u *and* v *are not adjacent if* $\sigma_u = \sigma_v$.

Proof. Given in the Appendix. □

Corollary 4. *For $n \geq 7$ and $m \geq 7$, the skeleton of a $S_{m\times n}$ satisfies*

1. *the adjacency of an extreme ray is $k_{m\times n} = 4mn - 3$;*
2. *the diameter $\delta(S_{m\times n}) = 2$;*
3. *$S_{m\times n}$ has exactly $2mn(4mn-3)$ 2-faces;*
4. *the edge connectivity $c_e(S_{m\times n}) = 4mn - 3$.*

Proof. Same as for Corollary 3.

□

Two rays are called *strongly conflicting* (respectively *conflicting*) if there exist 2 pairs i, j and k, l (respectively a pair i, j) such that the two rays have nonzero coordinates of distinct signs at positions i, j and k, l (respectively i, j). For example, the two adjacent rays of $S_{4\times 4}$ given in Fig. 20 are conflicting at the position 2, 2 but not strongly conflicting.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Fig. 20. Two rays conflicting as the position 2, 2

Remark 5. While for $n \geq 7$ and $m \geq 7$ a pair of extreme rays of the solitaire cone $S_{m\times n}$ are adjacent if and only if they are not strongly conflicting, for $n \geq 4$ two extreme rays of the dual metric cone M_n^* are adjacent if and only if they are not conflicting; see [9].

The following conjectures are based on Remark 5 and other similarities between the solitaire cone and the dual metric cone investigated in Sect. 2.

Conjecture 1.

1. For $n \geq 3$ and $m \geq 3$, the $\{0,1\}$-valued facets of the solitaire cone form a dominating set in the skeleton of its dual, that is, each facet of $S_{m\times n}$ is adjacent to a $\{0,1\}$-valued facet.
2. For m, n large enough, at least one facet of $S_{n\times m}$ is a simplex (that is, the number of rays contained in the facet equals the dimension of the cone minus one).

Item (1) of Conjecture 1 holds for $S_{3\times 4}$ but is false for $m \leq 2$. The smallest 1 by n toric board for which the conjecture fails is the 1 by 7 board. If true, item (2) would imply that the edge connectivity, the minimal incidence and the minimal adjacency of the skeleton of $S^*_{m\times n}$ are equal to $mn - 1$. This holds for the cones presented in Table 2 except for $S_{3\times i} : i = 3, 4$ and $S_{1\times i} : i = 4, 6$.

Remark 6. For the non-toric boards given in Table 1, a pair of extreme rays of the solitaire cone S_B are adjacent if and only if they are not strongly conflicting and their supports do not lie entirely on the boundary of the board.

5. The binary solitaire cone and other relatives

The link with the dual metric cone – see Theorem 3 – and the similarities between their combinatorial structures – see Remark 5 – leads to the study of a dual cut cone analogue; that is, the cone generated by the $\{0, 1\}$-valued facets of the solitaire cone, when this cone is full dimensional.

5.1. The binary solitaire cone

The dual cut cone is generated by the $\{0, 1\}$-valued facets of the dual metric cone. Similarly, we consider the cone generated by the $\{0, 1\}$-valued facets of the solitaire cone. This cone is called the *binary solitaire cone*, denoted $\mathcal{BS}_B$, and is studied in [4]. The following two results are contained there.

Theorem 9 [4]. *The extreme rays of the solitaire cone, that is, the moves, are extreme rays of the binary solitaire cone.*

Conjecture 2 [4]. The incidence of the moves is maximal in the skeleton of $\mathcal{BS}_{m\times n}$.

This strengthens the analogy with the dual metric cone, for which the extreme rays are also extreme rays of the dual cut cone.

5.2. The trellis solitaire cone

The $\{0, 1\}$-valued facets of the solitaire cone have much less structure than the set of cut metrics. In fact, the cut metrics are related to products of vectors of length n. This motivates the next definition. Let f and g be $\{0, 1\}$-valued vectors of length m and n respectively, and let $c_{ij} = f_i \cdot g_j$ for $i = 1, \ldots m,\ j = 1, \ldots n$. If $c \cdot x \leq 0$ defines a facet of $\mathcal{BS}_{m\times n}$, we call it a *trellis facet*. The *trellis solitaire cone* $\mathcal{TS}_B$ is generated by all of the *trellis facets* of the binary solitaire cone $\mathcal{BS}_B$. See item 2 of Corollary 2 for an easy construction of trellis facets. For example, among the two following facets of $\mathcal{BS}_{3\times 5}$, only the right one is a trellis facet.

$$\boxed{\begin{array}{ccccc} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{array}} \qquad \boxed{\begin{array}{ccccc} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{array}}$$

Fig. 21. A facet and a trellis facet of $\mathcal{BS}_{3\times 5}$

5.3. The complete solitaire cone

The *complete solitaire cone* $\mathcal{CS}_B$ is induced by a variation of the Solitaire game. To the classical moves we add the moves which consist of removing two pegs surrounding an empty hole and placing one peg in this empty hole as shown in Fig. 22.

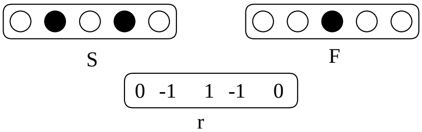

Fig. 22. The extreme ray r of $\mathcal{C}S_{1\times 5}$ corresponding to the move from S to F

6. Conclusions

The solitaire cone shares many similar combinatorial properties with the dual metric cone M_n^*. In particular:

1. their extreme rays have similar adjacency relationships; see Remark 5;
2. both cones have diameter 2; see DEZA AND DEZA [9] and Corollary 4;
3. their numbers of facets are bounded above and below by an exponential in the dimension; see Lemma 5 and Theorem 7 and, for the metric cone, AVIS [2] (lower bound) and GRAHAM, YAO AND YAO [16] (upper bound);
4. their extreme rays are also extreme rays of the cones generated by their $\{0, 1\}$-valued facets; see Theorem 9.
5. while the extreme rays of the solitaire cone are conjectured to be of maximum incidence in the cone generated by its $\{0, 1\}$-valued facets, the corresponding result is proved for the dual metric cone; see Conjecture 2 and DEZA AND DEZA [10];
6. we have $M_n^* = CS_{L_n}$ where L_n is the line graph of the complete graph on n nodes; see Theorem 3;
7. the $\{0, 1\}$-valued facets of the flow cone are the incidence vectors of cuts in the complete graph. The cone generated by these facets is the dual of the well studied cut cone; see DEZA AND LAURENT [13]. PAPERNOV [22] gave a complete characterization of multiflow problems for which the flow cone $F_n = M_n^*$ in Theorem 2 can be replaced by the dual of the cut cone. For example, single commodity flow problems are in this class, and the corresponding theorem is the celebrated max flow/min cut theorem. It would be interesting to see if any analogous relaxation of Theorem 1 can be found;
8. so far we have not yet found an analogue of the *hypermetric* facets of the metric cone M_n, that is, a "nice" family of $\{0, -1, 1\}$-valued extreme rays of the binary solitaire cone $\mathcal{B}S_B$. Another open question is the determination of a tighter relaxation of the solitaire cone S_B by some *cuts analogue*. The trellis solitaire cone $\mathcal{T}S_B$ is a candidate as well as the cone generated by the $\{0, 1\}$-valued facets with the minimal number of ones. For $S_{4\times 4}$ and $S_{3\times i}$: $i = 3, 4, 5$, these facets have maximal incidence and adjacency in the skeleton of $S_{m\times n}^*$.

Appendix

In this section, the dual problem being easier to state, we always consider the dual solitaire cone S_B^* whose extreme rays (respectively facets) are the r_i (respectively induced by $c_j \cdot x \le 0$) where $r_i \cdot x \le 0$ induces a facet of S_B (respectively c_j is an extreme ray of S_B). Clearly, a pair of extreme rays r', r'' are adjacent in S_B if and only if their corresponding facets $F_{r'}, F_{r''}$ are adjacent in S^*_B, that is, if the codimension of $F_{r'} \cap F_{r''}$ is two.

Proof of Lemma 6 and Theorem 8

Proof of Lemma 6. In the following we identify the extreme ray r of $S_{1\times n}$ with the facet F_r of $S^*_{1\times n}$. All facets of $S^*_{1\times n}$ being equivalent up to a scrolling and a reversing of the board, it is enough to find all the neighbours of the facet $F_u : [\,-1\ -1\ 1\ 0 \ldots\ 0\,]$. The adjacency relationships were checked by computer up to $n = 8$, so we can assume that $n \ge 9$. We first prove that the facet $F_v : [\,0\ 1\ -1\ -1\ 0 \ldots\ 0\,]$ is not adjacent to F_u and then that all other facets are adjacent to F_u. Any extreme ray r belonging to $F_u \cap F_v$ satisfies:

$$\begin{cases} r_1 + r_2 - r_3 = 0 \\ r_2 - r_3 - r_4 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} r_1 = r_4 = 0 \\ r_2 = r_3 \end{cases}$$

(since $0 \le r_i$ for $1 \le i \le n$)

$$\Rightarrow \begin{cases} r_3 = r_5 \\ r_4 = 0 \end{cases}$$

(since $r_3 - r_4 - r_5 \le 0$ and $-r_3 - r_4 + r_5 \le 0$)

which implies that $F_u \cap F_v \subset F'_v$ where $v' = [\,0\ 0\ 1\ -1\ -1\ 0 \ldots\ 0\,]$ and therefore $F_u \cap F_v$, being an intersection of more than 3 facets, is of codimension at least 3, that is, F_u and F_v are not adjacent.

Then, to prove that all other facets are adjacent to F_u, we consider any facet $F_a \ne F_v$ and show that for any third facet F_b we can find an extreme ray r of $S^*_{1\times m}$ satisfying both $r \in F_u \cap F_a$ and $r \notin F_b$, that is, $codim(F_u \cap F_a) = 2$. First, let assume that the supports of F_u, F_a and F_b do not overlap and that the gaps between σ_b and σ_u and between σ_b and σ_a are not equal to 0 or 2, as in the following example:

$$\underbrace{\overbrace{-1,-1,1}^{\sigma_u},0,\ldots,0,\overbrace{1,-1,-1}^{\sigma_a},0,\ldots,0,\overbrace{1,-1,-1}^{\sigma_b},0,\ldots,0}_{n}$$

Consider the following $\{0, 1\}$-valued ray:

$$r = \underbrace{\overbrace{1,0,1}^{\sigma_u},0,1,0\ldots,0,1,0\overbrace{1,0,1}^{\sigma_a},0,1,0\ldots,1,0,\overbrace{1,1,1}^{\sigma_b},0,1\ldots,1,0}_{n}\ .$$

This ray r is an extreme ray of $S^*_{1\times n}$ if and only if its 1-graph G_r is connected; see Theorem 6. According to the parity of the gaps between the supports σ_u and σ_a, σ_a and σ_b and σ_b and σ_u, we can fill these gaps by 0, 1-valued strings: 0 1 0 1 ... 0 1 0 or 0 1 1 0 1 0 1 ... 0 1 0 such that the graph G_r obtained is connected. Therefore r is an extreme ray of $S^*_{1\times n}$ satisfying both $r \in F_u \cap F_a$ and $r \notin F_b$, that is, F_u and F_a are adjacent.

If F_u, F_a and F_b do not overlap but the gaps between σ_b and σ_u or between σ_b and σ_a equal 0 or 2, then we can use the same technique considering one of the following $\{0, 1\}$-valued rays:

$$r^1 = \underbrace{\overbrace{1, 0, 1}^{\sigma_u}, 0, 1, 0 \ldots, 0, 1, 0 \overbrace{1, 0, 1}^{\sigma_a}, \overbrace{0, 1, 0}^{\sigma_b}}_{n}$$

$$r^2 = \underbrace{\overbrace{1, 0, 1}^{\sigma_u}, 0, 1, 0 \ldots, 0, 1, 0 \overbrace{1, 0, 1}^{\sigma_a}, 0, 1, \overbrace{0, 1, 1}^{\sigma_b}, 0, 1 \ldots, 1, 0}_{n} \;.$$

When the supports of F_u, F_a and F_b overlap, we can, by essentially projecting on the joint support $\sigma_u \cup \sigma_a \cup \sigma_b$, use the same technique and generalize what happens for $n \leq 8$). For example, take $n = 11$ and F_u, F_a and F_b given by:

$$\overbrace{-1, -1, 1}^{\sigma_u}, 0, 0, 0, 0, 0, 0, 0, 0 \qquad \overbrace{1, -1, -1}^{\sigma_a}, 0, 0, 0, 0, 0, 0, 0, 0$$
$$0, 0, \overbrace{1, -1, -1}^{\sigma_b}, 0, 0, 0, 0, 0, 0$$

the desired extreme ray of $S^*_{1\times 11}$ satisfying both $r \in F_u \cap F_a$ and $r \notin F_b$ is:

$$r = \overbrace{1, 0, 1, 1, 1}^{\sigma_u \cup \sigma_a \cup \sigma_b}, 0, 1, 0, 1, 0, 1$$

which means that F_u and F_a are adjacent. Note that for some cases, the desired extreme ray is not $\{0, 1\}$-valued. For example, take $n = 11$ and F_u, F_a and F_b given by:

$$\overbrace{-1, -1, 1}^{\sigma_u}, 0, 0, 0, 0, 0, 0, 0, 0 \qquad 0, \overbrace{-1, -1, 1}^{\sigma_a}, 0, 0, 0, 0, 0, 0, 0$$
$$\overbrace{1, -1, -1}^{\sigma_b}, 0, 0, 0, 0, 0, 0, 0, 0 \;.$$

Projecting on $B_{1\times 5}$, we get the extreme ray $r_{1\times 5} = 1, 1, 2, 3, 2$ satisfying both $r \in F_{u_{1\times 5}} \cap F_{a_{1\times 5}}$ and $r \notin F_{b_{1\times 5}}$ and the desired extreme ray of $S^*_{1\times 11}$ satisfying both $r \in F_u \cap F_a$ and $r \notin F_b$ is derived from $r_{1\times 5}$ using GENFACET; see Theorem 5: $r = 1, 1, 2, 3, 2, 1, 1, 0, 1, 0, 1$.

Proof of Theorem 8. We first consider a pair of facets F_u and F_v such that σ_u and σ_v belong to the same row or column of length d. For $d = 3$, we have $\sigma_u = \sigma_v$ and one can easily check that F_u and F_v are never adjacent. For $d \geq 4$ and $\sigma_u \neq \sigma_v$, we can in the same way as for the proof of Lemma 6, find a third facet F'_v containing $F_u \cap F_v$ if F_u and F_v, seen as facets of $S_{1\times n}$, are non-adjacent. For example, with:

$$F_u : \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix} \qquad F_v : \begin{bmatrix} 0 & 1 & -1 & -1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

the following facet:

$$F_{v'} : \begin{bmatrix} 0 & 0 & 1 & -1 & -1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

satisfies $F_u \cap F_v \subset F_{v'}$, that is, F_u and F_v are non-adjacent. The only difference between $S_{m\times n}$ and $S_{1\times n}$ is that F_u and the facet F_w with $\sigma_u = \sigma_w$ are not adjacent. We have:

$$F_u : \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix} \qquad F_w : \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

and the following facet:

$$F_{w'} : \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$$

satisfies $F_u \cap F_w \subset F_{w'}$, that is, F_u and F_w are non-adjacent. Then, to prove that all other facets which support belongs to the same row or column as σ_u are adjacent to F_u, we can apply the same technique as for the proof of Lemma 6.

We then consider F_u and F_a with σ_u and σ_a not in the same row or column. As for Lemma 6, we show that for any third facet F_b, we can find an extreme ray r of $S^*_{m\times n}$ satisfying both $r \in F_u \cap F_a$ and $r \notin F_b$, that is, $codim(F_u \cap F_a) = 2$. First let assume that the supports of F_u, F_a and F_b do not overlap as in the following example:

$$F_a : \begin{bmatrix} 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \dots & 0 & -1 & -1 & 1 & 0 & \dots \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix} \qquad F_b : \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}.$$

Consider the following $\{0, 1\}$-valued ray

$$r : \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix}.$$

This ray r is an extreme ray of $S^*_{m\times n}$ if and only if its 1-graph G_r is connected; see Theorem 6. According to the parity of the gaps between the supports σ_u and σ_a, σ_a and σ_b and σ_b and σ_u, we can fill these gaps by the following 0, 1-valued matrices (or their transposes) such that the graph G_r obtained is connected.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{bmatrix}$$

Therefore r is an extreme ray of $S^*_{m\times n}$ satisfying both $r \in F_u \cap F_a$ and $r \notin F_b$, that is, F_u and F_a are adjacent.

When the supports of F_u, F_a and F_b overlap, once again, we use the same technique which amounts to a tedious but easy case by case study and completes the proof. For example, take the following facets of $S^*_{4\times 4}$:

$$F_u : \begin{bmatrix} -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad F_a : \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad F_b : \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the desired extreme ray of $S^*_{4\times 4}$ satisfying both $r \in F_u \cap F_a$ and $r \notin F_b$ is:

$$r : \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

which means that F_u and F_a are adjacent.

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