THE RIDGE GRAPH OF THE METRIC POLYTOPE AND SOME RELATIVES

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Abstract. The metric polytope is a $\binom{n}{2}$-dimensional convex polytope defined by its $4\binom{n}{3}$ facets. The vertices of the metric polytope are known only up to $n = 6$, for $n = 7$ they number more than 60 000. The study of the metric polytope and its relatives (the metric cone, the cut polytope and the cut cone) is mainly motivated by their application to the maximum cut and multicommodity flow feasibility problems. We characterize the ridge graph of the metric polytope, i.e. the edge graph of its dual, and, as corollary, obtain that the diameter of the dual metric polytope is 2. For $n \geq 5$, the edge graph of the metric polytope restricted to its integral vertices called cuts, and to some $\{\frac{1}{3}, \frac{2}{3}\}$-valued vertices called anticuts, is, besides the clique on the cuts, the bipartite double of the complement of the folded n-cube. We also give similar results for the metric cone, the cut polytope and the cut cone.

Key words: Metric polytope, cut polytope, ridge graph, edge graph.

1. Introduction

We first recall the definitions of the *metric polytope* $MetP_n$ and its relatives, the *metric cone* Met_n, the *cut polytope* $CutP_n$ and the *cut cone* Cut_n. Then we present some applications to well known optimization problems and some combinatorial and geometric properties of those four polyhedra.

For all 3-sets $\{i, j, k\} \subset \{1, \ldots, n\}$, we consider the following inequalities:

$$x_{ij} - x_{ik} - x_{jk} \leq 0 \tag{1}$$
$$x_{ij} + x_{ik} + x_{jk} \leq 2. \tag{2}$$

The inequalities (1) define the metric cone Met_n and the metric polytope $MetP_n$ is obtained by bounding Met_n by the inequalities (2). The $3\binom{n}{3}$ facets defined by the inequalities (1), which can be seen as triangle inequalities for distance x_{ij} on $\{1, 2, \ldots, n\}$, are called *homogeneous triangle facets* and are denoted by $Tr_{ij,k}$. The $\binom{n}{3}$ facets defined by the inequalities (2) are called *non-homogeneous triangle facets* and are denoted by Tr_{ijk}.

Given a subset S of $V_n = \{1, 2, \ldots, n\}$, the cut determined by S consists of the pairs (i, j) of elements of V_n such that exactly one of i, j is in S. $\delta(S)$ denotes both the cut and its incidence vector in $\mathbb{R}^{\binom{n}{2}}$, i.e. $\delta(S)_{ij} = 1$ if exactly one of i, j is in S and 0 otherwise for $1 \leq i < j \leq n$. By abuse of language, we use the term cut for

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both the cut itself and its incidence vector, so $\delta(S)_{ij}$ are considered as coordinates of a point in $\mathbb{R}^{\binom{n}{2}}$. The cut polytope $CutP_n$ is the convex hull of all 2^{n-1} cuts, and the cut cone Cut_n is the conic hull of all $2^{n-1}-1$ nonzero cuts.

We have $CutP_n \subseteq MetP_n$ and $Cut_n \subseteq Met_n$ with equality only for $n \leq 4$. Any facet of the metric polytope contains a facet of the cut polytope and the vertices of the cut polytope are vertices of the metric polytope, in fact the cuts are precisely the integral vertices of the metric polytope. Actually the metric polytope $MetP_n$ wraps the cut polytope $CutP_n$ very tightly since, in addition to the vertices, all edges and 2-faces of $CutP_n$ are also faces of $MetP_n$ [12]. There is a $1-1$ correspondence between the elements of the metric cone Met_n and all the semi-metrics on n points, and the elements of the cut cone Cut_n correspond precisely to the semi-metrics on n points that are isometrically embeddable into some l_1^m, see[2], it is easy to see that $m \leq \binom{n}{2}$. Those polyhedra were considered by many authors, see for instance [1, 3, 6, 9, 10, 11, 12, 13, 16, 17] and references there.

One of the motivations for the study of these polyhedra comes from their applications in combinatorial optimization, see for instance [11]. Given a graph $G=(V_n,E)$ and nonnegative weights w_e, $e \in E$, assigned to its edges, the max-cut problem consists in finding a cut $\delta(S)$ whose weight $\sum_{e\in\delta(S)} w_e$ is as large as possible. By setting $w_e=0$ if e is not an edge of G, we can consider the complete graph on V_n. Then the max-cut problem can be stated as a linear programming problem over the cut polytope $CutP_n$ as follows:

$$\begin{cases} \max \ \ w^T \cdot x \\ x \in CutP_n. \end{cases}$$

Since $MetP_n$ is a relaxation of $CutP_n$, optimizing $w^T \cdot x$ over the metric polytope instead of the cut polytope provides an upper bound for the max-cut problem [3].

With E the set of edges of the complete graph on V_n, an instance of the multicommodity flow problem is given by two nonnegative vectors indexed by E: a capacity $c(e)$ and a requirement $r(e)$ for each $e \in E$. Let $U=\{e \in E : r(e)>0\}$. If T denotes the subset of V_n spanned by the edges in U, then we say that the graph $G=(T,U)$ denotes the *support* of r. For each edge $e=(s,t)$ in the support of r, we seek a flow of $r(e)$ units between s and t in the complete graph. The sum of all flows along any edge $e' \in E$ must not exceed $c(e')$. If such a flow exists, we call c, r *feasible.* A necessary and sufficient condition for feasibility is given by the Japanese theorem [15]: a pair c, r is feasible if and only if $(c-r)^T x \geq 0$ is valid over Met_n. For example, $Tr_{ij,k}$ can be seen as an elementary solvable flow problem with $c(ij)=r(ik)=r(jk)=1$ and $c(e)=r(e)=0$ otherwise, so the inequalities (1) correspond to $(c-r)^T x \geq 0$ for $x \in Met_n$. Therefore, the metric cone Met_n is the dual cone to the cone of feasible multicommodity flow problems.

The metric polytope $MetP_n$ and the cut polytope $CutP_n$ share the same symmetry group induced by permutations on $V_n=\{1,\ldots,n\}$ and switching reflections [9, 16]. This group is isomorphic to $Aut(\square_n)$, see Remark 3.4 below. Given a cut $\delta(S)$, the switching reflection $r_{\delta(S)}$ is defined by $y=r_{\delta(S)}(x)$ where $y_{ij}=1-x_{ij}$ if $(i,j) \in \delta(S)$ and $y_{ij}=x_{ij}$ otherwise. These symmetries, which preserve adjacency, are widely used in the study of $MetP_n$ and its relatives.

$CutP_3$ and $MetP_3$ are combinatorially equivalent to the tetrahedron and $CutP_4$ and $MetP_4$ are combinatorially equivalent to the 6-dimensional cyclic polytope with 8 vertices. More generally the cut polytope is a 3-neighbourly polytope [12]. Any two cuts are adjacent both on $CutP_n$ [3] and on $MetP_n$ [18]; in other words $MetP_n$ is quasi-integral in terms of [19], i.e. the edge graph of the convex hull of its integral vertices, the edge graph of the cut polytope, is an induced subgraph of the edge graph of the metric polytope itself.

The paper is organized as follows. In Section 2 we characterize the ridge graph of the metric polytope $MetP_n$ and the metric cone Met_n. In Section 3, respectively Section 4, we give a partial result on the edge graph of the metric polytope $MetP_n$ and the ridge graph of the cut polytope $CutP_n$. Section 5 contains the proofs of Lemma 2.1 and Theorem 2.2. A general reference for the graph theory used in this paper is [4].

2. Ridge Graph of the Metric Polytope and the Metric Cone

2.1. Ridge graph of the Metric Polytope

The ridge graph G_n of the metric polytope $MetP_n$ is the edge graph of its dual. The nodes of G_n are the $4\binom{n}{3}$ triangle facets of $MetP_n$ and two facets are adjacent if and only if their intersection is a ridge, i.e. a face of codimension 2. We first determine the various intersections of two facets of $MetP_n$ by the following lemma:

Lemma 2.1 *For $n \geq 4$, the intersection of two facets of $MetP_n$ is either:*
(a) a face of codimension $n-1$, combinatorially equivalent to $MetP_{n-1}$, or
(b) a face of codimension 3, or
(c) a face of codimension 2, i.e. a ridge.

We call faces of type (a): *weak triangle faces.* From the proof of Lemma 2.1 given in Section 5, it is easy to check that all weak triangle faces belong to a facet of the cube $[0,1]^{\binom{n}{2}}$ and form one orbit of the symmetry group of $MetP_n$, i.e. are equivalent under permutation and switching. Since each weak triangle face is combinatorially equivalent to $MetP_{n-1}$, the metric polytope $MetP_n$ contains $2\binom{n}{2}$ copies of $MetP_{n-1}$. The same proof also shows that, for $n \geq 4$, two facets intersect in a ridge if and only if they are non-conflicting. Two facets are called *conflicting* if there exists a pair i,j such that the two facets have nonzero coordinates of distinct signs at the position i,j. For example, Tr_{123} and $Tr_{24,3}$ are conflicting at pair $2,3$. The notion of conflicting facets was introduced in [13, 16]. Using this property we are able to characterize G_n, the ridge graph of $MetP_n$. $G_3 = K_4$ and G_4 is the (4×4)-grid. For higher values of n, it is more convenient to consider $\bar{G}_n$, the complement of G_n, which has a smaller valency. For $i = 1,\ldots,n$, let $G_i \approx G$ be n isomorphic graphs and $\Gamma_i \approx \Gamma$ their isomorphic induced subgraphs, we call *bouquet of n graphs G with common* Γ the graph with vertex set $V = \cup_{i=1}^{n} V(G_i \setminus \Gamma_i) \cup V(\Gamma)$ and edge set $E = \cup_{i=1}^{n} E(G_i \setminus \Gamma_i) \cup \{(x,y) : x \in \Gamma, y \in G_i \setminus \Gamma_i, x \text{ and } y \text{ adjacent}\}$. For example, Fig. 1 and Fig. 2 represent a bouquet of 3 (3×3)-grids with common K_3 and a bouquet of 2 hexagons with common edge. With v denoting the number of nodes, k the valency of each node, λ the number of nodes adjacent to two adjacent nodes and μ the number of nodes adjacent to two non-adjacent nodes, we have:

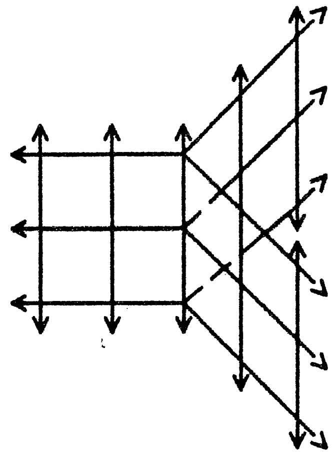

Fig. 1. The local graph of $\bar{G}_6$

Theorem 2.2 *For $n \geq 4$, $\bar{G}_n$ is locally the bouquet of $(n-3)$ (3×3)-grids with common K_3 having parameters: $v = 4\binom{n}{3}$, $k = 3(2n-5)$, $\lambda = 2(n-2)$ or 4, and $\mu = 6$, 4 for $n \geq 5$ or 0 for $n \geq 6$.*

Fig. 1 illustrates Theorem 2.2 for the case $n = 6$. From the parameters of $\bar{G}_n$ we can compute some parameters of G_n: the valency $v = \frac{2(n-3)(n-7)}{3}$ and $\mu = \frac{2(n-3)(n-13)}{3}$ or $\frac{2(n-3)(n-16)}{3} + 2$. This gives the two following corollaries:

Corollary 2.3 *For $n \geq 4$, the diameter of G_n is 2.*

PROOF. For $n \geq 4$, $\mu > 0$, i.e. any two non-adjacent nodes of G_n share a common neighbour. The diameter of G_3 is obviously 1. □

Corollary 2.4 *The metric polytope has exactly $\frac{16(n-7)\binom{n}{4}}{3}$ ridges.*

PROOF. The number of faces of codimension 2 of a polytope is half of the total valency of its ridge graph. Since we know the common valency of all $4\binom{n}{3}$ nodes of G_n, the result is a straightforward calculation. □

G_4, the ridge graph of $MetP_4$, is the (4×4)-grid $= L(K_{4,4}) = L(\square_4)$, the line graph of the folded 4-cube. G_4 is a strongly regular graph with parameters $v = 16$, $k = 6$, $\lambda = 2$ and $\mu = 2$. There exists only one other strongly regular graph with the same parameters, namely the Shrikhande graph [4]. For $\bar{G}_5$, the complement of the ridge graph of $MetP_5$, both λ and μ take their values in $\{4, 6\}$, providing first example of an interesting generalization of strongly regular graph: regular graph of diameter 2 with $\lambda, \mu \in \{a, b\}$ with $b > a > 0$.

Remark 2.5 *As a direct consequence of the proof of Lemma 2.1, we obtain that two facets F_1 and F_2 of $CutP_n$ contained in two facets F_1' and F_2' of $MetP_n$ are adjacent in $CutP_n$ if and only if F_1' and F_2' are adjacent in $MetP_n$. This implies that any ridge of the metric polytope contains a ridge of the cut polytope.*

Before presenting another interesting consequence of the proof of Lemma 2.1, we recall the notion of *0-lifting* which was considered in[10]. Let v be a vector of length $\binom{n}{2}$ and $v' = (v, 0 \ldots, 0)$ of length $\binom{n+1}{2}$, then the inequality $v' \cdot x \leq a$ defines a facet F' of $CutP_{n+1}$, called 0-lifting of F, if and only if the inequality $v \cdot x \leq a$ defines a facet F of $CutP_n$. We extend this notion to a ridge of $CutP_n$, $W = F_1 \cap F_2$, by defining the 0-lifting of W: $W' = F_1' \cap F_2'$, where F_1', respectively F_2', is the 0-lifting of F_1', respectively F_2'. For example, the $(n-3)$-times 0-lifting of a ridge of $MetP_3$, i.e. of an edge, is a weak triangle face of $MetP_n$. By a proof similar to the one used for the Lemma 2.1 and by direct checking for $n = 4$, we have:

Lemma 2.6 *For $n \geq 4$, the 0-lifting of ridge of $CutP_n$ is a ridge of $CutP_{n+1}$.*

In the next theorem, we give some additional characteristics of the ridge graph of the metric polytope. With $\omega(G_n)$, respectively $\alpha(G_n)$, the size of the largest clique, respectively co-clique, of G_n and $k(G_n)$ the number of maximal cliques of full rank $\binom{n}{2}$, where the rank of a clique is the rank of the set of the $\{-1, 0, 1\}$-valued vectors representing the coefficients of the triangle facets belonging to the clique, we have:

Theorem 2.7 *For $n \geq 5$, the intersection of the facets belonging to a maximal clique C of G_n is a $(\binom{n}{2} - rank(C))$-face containing a unique $\{\frac{1}{3}, \frac{2}{3}\}$-valued point of $MetP_n$ which is a vertex if and only if $rank(C) = \binom{n}{2}$. $k(G_n)$ equals the number of $\{\frac{1}{3}, \frac{2}{3}\}$-valued vertices of $MetP_n$, $\omega(G_n) = \binom{n}{3}$ and $\alpha(G_n) = 4$.*

PROOF The *support* of a triangle facet $Tr_{ij,k}$ or Tr_{ijk} is the 3-set $\{i, j, k\}$. Since the 4 triangle facets sharing a same support obviously form the largest clique of $\bar{G}_n$, we have $\alpha(G_n) = 4$. Let C be a maximal clique of G_n and f be the face of $MetP_n$ which is the intersection of the facets belonging to C. We show that any 2-set $\{i, j\}$ belongs to the support of a facet of C by exhibiting, otherwise, a new facet T non-conflicting with any facet F of C (which contradicts the maximality of C). For any $k \in \{1, \ldots, n\} \setminus \{i, j\}$, the coefficients F_{ik} and F_{jk} of F can be $(\leq 0, \leq 0)$, $(\leq 0, \geq 0)$, $(\geq 0, \leq 0)$ or $(\geq 0, \geq 0)$; the corresponding facet T is $Tr_{ij,k}, Tr_{jk,i}, Tr_{ik,j}$ or Tr_{ijk}. Then the point y defined by $y_{ij} = \frac{1}{3}$ if $F_{ij} = -1$ and $y_{ij} = \frac{2}{3}$ if $F_{ij} = 1$ is the unique $\{\frac{1}{3}, \frac{2}{3}\}$-valued point of $MetP_n$ in f. Since clearly $dim(f) = \binom{n}{2} - rank(C)$, f is a vertex v if and only if $rank(C) = \binom{n}{2}$, then, by unicity, $v = y$. One can easily check that a $\{\frac{1}{3}, \frac{2}{3}\}$-valued vertex can belong to at most 1 of the 4 triangle facets sharing a same support (it holds for any vertex v such that $0 < v_{ij} < 1$), and that any *anticut* $\hat{\delta}(S) = \frac{2}{3}(1, \ldots, 1)$-$\frac{1}{3}\delta(S)$, which is a vertex of $MetP_n$ for $n \geq 5$ (see Section 3), belongs to exactly $\binom{n}{3}$ triangle facets. It implies that $\omega(G_n) = \binom{n}{3}$. For example, the 10 non-homogeneous triangle facets of $MetP_5$ intersect on the vertex $y = \frac{2}{3}(1, \ldots, 1)$ while the 6 facets $Tr_{12,3}, Tr_{12,4}, Tr_{12,5}, Tr_{45,1}, Tr_{45,2}$ and $Tr_{45,3}$ intersect on a 4-face f of $MetP_5$ generated by the vertices $\delta(S)$, $S = \emptyset, \{1,4\}, \{1,5\}, \{2,4\}$ and $\{2,5\}$. The unique $\{\frac{1}{3}, \frac{2}{3}\}$-valued point of f is $v = \frac{1}{6}(\delta(1,4) + \delta(1,5) + \delta(2,4) + \delta(2,5) + 2\delta(\emptyset))$, so $3v$ is the path metric of K_5 with the edges 1, 2 and 4, 5 deleted. □

Remark 2.8 *A vertex of the metric polytope belongs to at most $3\binom{n}{3}$ triangle facets, i.e. to 3/4 of the total number of facets of the metric polytope, and this value is attained only by the cuts. On the other hand, a facet of the cut polytope contains at most $3 \cdot 2^{n-3}$ cuts, i.e. 3/4 of the total number of vertices of the cut polytope, and this value is attained by the triangle facets, see [7].*

2.2. Ridge graph of the Metric Cone

The ridge graph G'_n of the metric cone Met_n can be easily deduced from G_n since, for $n \geq 4$, two facets of the metric cone intersect in a ridge if and only if they are non-conflicting. We have:

Theorem 2.9 *For $n \geq 4$, $\bar{G}'_n$ is locally the bouquet of $(n-3)$ hexagons with common edge having parameters: $v = 3\binom{n}{3}$, $k = 2(2n-5)$, $\lambda = n-2$ or 2, and $\mu = 4$, 3 or 2 or 0 for $n \geq 5$.*

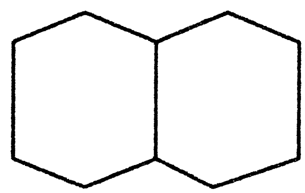

Fig. 2. The local graph of $\bar{G}'_5$ is the graph of the molecule of naphthalene

Fig. 2 illustrates Theorem 2.9 for the case $n = 5$, $\bar{G}'_5$ is locally the graph of the molecule of naphthalene. As for G_n, we can compute some parameters of G'_n: the valency $v = \frac{(n-3)(n-6)}{2}$ and $\mu = \frac{(n-3)(n-12)}{2}$ or $\frac{(n-3)(n-14)}{2} + 1$. This gives the two following corollaries:

Corollary 2.10 *For $n \geq 4$, the diameter of G'_n is 2.*

Corollary 2.11 *The metric cone has exactly $3(n^2-6)\binom{n}{4}$ ridges.*

G'_4, the ridge graph of Met_4, is the (4×3)-grid, i.e. $L(K_{4,3})$. G'_4 is a co-edge regular graph with parameters $v = 12$, $k = 5$, $\lambda = 2$ or 1 and $\mu = 2$.

In the same way as for the metric polytope and using a result of [13], we have: for $n \geq 5$, $k(G'_n)$, the number of maximal cliques of rank $\binom{n}{2}-1$ of the ridge graph of the metric cone, is equal to the number of the $\{1,2\}$-valued, up to multiple, extreme rays of the metric cone.

3. Edge Graph of the Metric Polytope

While the edge graph of the cut polytope $CutP_n$ is known to be a clique, even to find all the vertices of the metric polytope $MetP_n$ seems to be hopeless. Therefore, we first consider the restriction of Γ_n, the edge graph of $MetP_n$, to the 2^{n-1} cuts $\delta(S)$ and 2^{n-1} *anticuts* $\hat{\delta}(S) = \frac{2}{3}(1,\ldots,1)$-$\frac{1}{3}\delta(S)$ which are vertices of $MetP_n$ for $n \geq 5$. The cuts, respectively the anticuts, belong to the same orbit, i.e are equivalent under permutation and switching. We recall the definitions of the *folded n-cube* and the *bipartite double* of a graph, see[4]. The folded n-cube is the graph whose vertices are the partitions of V_n into two subsets, two partitions being adjacent when their common refinement contains a set of size one. The bipartite double of a graph G is the graph whose vertices are the symbols γ^+, γ^- $(\gamma \in G)$ and whose edges are the 2-sets $\{\gamma^\sigma, \delta^\tau\}$, with γ and δ adjacent in G and $\sigma \neq \tau$. We have:

Theorem 3.1 *For $n \geq 5$, the restriction of the edge graph of the metric polytope to the cuts and anticuts, is, besides the clique on the cuts, the bipartite double of the complement of the folded n-cube.*

PROOF. The cuts, respectively the anticuts, are known to form a clique [18], and a co-clique [16]. To find the adjacency relations between the cuts and the anticuts, we use the following result proved in [1]. An anticut $\hat{\delta}(S)$ lies on an extreme ray of the metric cone Met_n if and only if $1 < |S| < n-1$. In other words, $\delta(\emptyset)$ and $\hat{\delta}(S)$ are adjacent if and only if $\min(|S|, |\bar{S}|) > 1$. Using the fact that switching preserves adjacency and that $r_{\delta(T)}(\delta(S)) = \delta(S\Delta T)$, it implies that $\delta(S')$ and $\hat{\delta}(S)$ are adjacent if and only if $\min(|S\Delta S'|, |\bar{S}\Delta S'|) > 1$, i.e. $\delta(S')$ and $\hat{\delta}(S)$, seen as vertices of the folded n-cube, are not adjacent. □

Corollary 3.2 *The diameter of the restriction of the edge graph of the metric polytope to the cuts and anticuts is 2.*

PROOF. Since the cuts, respectively the anticuts, form a clique and a co-clique, we shall exhibit a cut $\delta(T)$ adjacent to both anticuts $\hat{\delta}(S)$ and $\hat{\delta}(S')$ for any pair S, S'. Without loss of generality we can assume that $|S| \leq \frac{n}{2}$. Let $S' = \{1,2\}$, if $|S| > 1$, $\delta(T) = \delta(\emptyset)$ is obviously adjacent to both $\hat{\delta}(S)$ and $\hat{\delta}(S')$. If $|S| = 1$, $\delta(T) = \delta(I)$ with $I = \{i\}$ and $i \notin S \cup S'$. For $|S| = 0$, $\delta(T) = \delta(J)$, with $J = \{i,j\}$ and $i \notin S'$ and $j \in S'$, is adjacent to both $\hat{\delta}(S)$ and $\hat{\delta}(S')$. Then, using permutations and switching, we can exhibit a cut $\delta(T)$ adjacent to any pair of anticuts $\hat{\delta}(S)$ and $\hat{\delta}(S')$. □

The 32 vertices of $MetP_5$ are exactly the 16 cuts and the 16 anticuts, moreover, the complement of the folded 5-cube is isomorphic to the Clebsch graph [4], i.e. the skeleton $\frac{1}{2}H(5,2)$ of the 5-dimensional half-cube. Therefore, Γ_5 is, besides the clique K_{16} on the cuts, the bipartite double of the Clebsch graph.

Corollary 3.3 *$MetP_5$ has exactly 280 edges and its diameter is 2.*

PROOF. Since the valency of the Clebsch graph is 10, adjacency relations between the cuts and anticuts create $10 \times 16 = 160$ edges. The clique on the cuts creates $\binom{16}{2} = 120$ edges, and the co-clique on the anticuts none. Corollary 3.2 implies $\delta(\Gamma_5) = 2$. □

Remark 3.4 *[9] The symmetry group of the metric polytope $MetP_n$ and the cut polytope $CutP_n$ is isomorphic to the automorphism group of the folded n-cube, i.e. $Is(MetP_n) = Is(CutP_n) \approx Aut(\square_n)$.*

We consider for $n \geq 6$, in addition to the orbits of cuts and anticuts, the orbit formed by the $2^{n-1}\binom{n}{2}$ vertices called *trivial extension* of anticuts of $MetP_{n-1}$. The notion of trivial extension was introduced in [13, 16]. Given $S \subset V_n = \{1, 2, \ldots, n\}$ and a pair i, j with $1 \leq i < j \leq n$, the trivial extension of anticut $V(ij, S)$ determined by S and the pair i, j is given by:

$$\begin{aligned} V_{kl}(ij,S) &= \hat{\delta}_{kl}(S) && \text{for } 1 \le k < l \le n \text{ and } l \ne i \\ V_{il}(ij,S) &= \delta_{ij}(S) + (-1)^{\delta_{ij}(S)} V_{jl}(ij,S) && \text{for } l \notin \{i,j\}, \\ V_{ij}(ij,S) &= \delta_{ij}(S). \end{aligned}$$

For $n \geq 6$, the $2^{n-1}\binom{n}{2}$ trivial extensions of anticuts are vertices of the metric polytope. $V_{k,l}(ij,S)$ for $i \notin \{k,l\}$, the projection of $V(ij,S)$, is the anticut $\hat{\delta}(S \backslash \{i\})$ of $MetP_{n-1}$. The adjacency relations induced by trivial extensions of anticuts were given in [14]. Without loss of generality we can choose S' such as $|S \Delta S'| \leq |S \Delta \bar{S}'|$, otherwise we take $\bar{S}'$. Then the adjacency relations induced by the trivial extensions of anticuts are:

(a) $V(ij,S)$ and $V(i'j',S')$ are adjacent if and only if $|\{i,j\} \cap \{i',j'\}| = 1$ and $S \Delta S' \subset \{i,j\} \cap \{i',j'\}$.
(b) $V(ij,S)$ and $\delta(S')$ are adjacent if and only if, either $|S \Delta S'| > 2$, or $|S \Delta S'| = 2$ and $\{i,j\} \subset \overline{(S \Delta S')}$.
(c) $V(ij,S)$ and $\hat{\delta}(S')$ are adjacent if and only if $|S \Delta S'| \leq 1$ and $S \Delta S' \subset \{i,j\}$.

This leads to a generalization of Theorem 3.1, where $M_{ij} = \{V(ij,S) : S \subset V_n\}$ and $M_S = \{V(ij,S) : 1 \leq i < j \leq n\}$:

Theorem 3.5 *For $n \geq 6$, besides the clique on the cuts and the bipartite double of the complement of the folded n-cube on the cuts and anticuts, the restriction of the edge graph of the metric polytope to the cuts, anticuts and trivial extensions of anticuts, is given by:*
(1) $M_{ij} = \bar{K}_2\ _{-1}$.
(2) $M_S = L(K_n)$, the line graph of the complete graph.
(3) Adjacencies between M_S and $M_{S'}$ exist if and only if $\delta(S)$ and $\delta(S')$ are adjacent on the folded n-cube and, in this case, they form the bipartite double of $L(K_n)$.
(4) $V(ij,S)$ is adjacent to 3 anticuts, namely $\hat{\delta}(S \Delta \{i\})$, $\hat{\delta}(S \Delta \{j\})$ and $\hat{\delta}(S)$, and to $2^{n-1} - 3n + 2$ cuts.

Corollary 3.6 *For $n \geq 6$, the diameter of the restriction of the edge graph of the metric polytope to the cuts, anticuts and trivial extensions of anticuts is 2.*

PROOF. Since the diameter of the restriction of edge graph of the metric polytope to the cuts and anticuts is 2, we shall find a common neighbour to any pair of trivial extensions $V(ij,S)$ and $V(i'j',S')$. For $n \geq 7$, $2^{n-1} - 3n + 2 > 2^{n-2}$, i.e. the number of cuts adjacent to a given trivial extension $V(ij,S)$ is more than half the number of cuts. This implies that any pair of non-adjacent trivial extensions of anticuts is adjacent to a common cut; by same reasons any pair of non-adjacent anticut and trivial extension of anticut shares a common neighbour. For $n = 6$, one can directly check it. □

Corollary 3.7 *$MetP_6$ has exactly 14 256 edges and its diameter is 2.*

PROOF. The 544 vertices of $MetP_6$ are exactly the 32 cuts, 32 anticuts and 480 trivial extensions of anticuts. Since each trivial extension of anticut is adjacent to $4(n-2)$ trivial extensions, and to $2^{n-1} - 3n + 5$ cuts and anticuts, they create 12 960 edges. The valency of the complement of the folded 6-cube being 25, the cuts and anticuts create 1 296 edges. Corollary 3.6 gives $\delta(\Gamma_6) = 2$. □

Remark 3.8 *It was conjectured in [16] that the integral vertices of the metric polytope, i.e. the cuts, form a dominating clique in its edge graph Γ_n, i.e. that every vertex of the metric polytope is adjacent to a cut. It would imply that the diameter of the metric polytope satisfies $\delta(\Gamma_n) \leq 3$.*

4. Ridge Graph of the Cut Polytope

As for the vertices of the metric polytope $MetP_n$, the determination of all the facets of the cut polytope $CutP_n$ seems to be hopeless. For n odd, we consider the restriction of Ω_n, the ridge graph of the cut polytope, to the $4\binom{n}{3}$ triangle facets, which are all equivalent under permutations and switching, and the 2^{n-1} facets of the orbit of the equicut facet, where, for n odd, the equicut facet is defined by the following inequality:

$$\sum_{1 \leq i < j \leq n} x_{ij} \leq \frac{n^2 - 1}{4}.$$

All cuts contained in the equicut facet are equicuts, i.e. cuts $\delta(S)$ such that $|S| = \lfloor \frac{n}{2} \rfloor$ or $|S| = \lceil \frac{n}{2} \rceil$. We have:

Lemma 4.1 *For odd $n \geq 5$, the facets of the orbit of the equicut facet form the co-clique $\bar{K}_{2^{n-1}}$ in the ridge graph of the cut polytope.*

PROOF. Let F and F' be two facets of the orbit of the equicut facet, since switching preserves adjacency, we can assume that F is the equicut facet. Let $\delta(S)$ be the nonzero cut such that F' is the switching of F by $\delta(S)$, and let W_S, respectively $W_{\bar{S}}$, be the projection of $W = F \cap F'$ on S, respectively $\bar{S}$. To prove that F and F' are not adjacent in the ridge graph of the cut polytope, we shall show that $codim(W) > 2$. We have: $dim(W) \leq dim(W_S) + dim(W_{\bar{S}}) + |S| \cdot |\bar{S}|$, then, since W_S is in $\mathbb{R}^{\binom{|S|}{2}}$ and $W_{\bar{S}}$ is in in $\mathbb{R}^{\binom{|\bar{S}|}{2}}$, the last inequality can be rewritten: $codim(W) \geq codim(W_S) + codim(W_{\bar{S}})$. By construction W_S, respectively $W_{\bar{S}}$, is the face E_S defined by the inequality: $\sum_{i,j \in S, i<j} x_{ij} \leq \lfloor \frac{|S|}{2} \rfloor \cdot \lceil \frac{|S|}{2} \rceil$, respectively the face $E_{\bar{S}}$ defined by the inequality: $\sum_{i,j \in \bar{S}, i<j} x_{ij} \leq \lfloor \frac{|\bar{S}|}{2} \rfloor \cdot \lceil \frac{|\bar{S}|}{2} \rceil$. Thus we have: $codim(W) \geq codim(E_S) + codim(E_{\bar{S}})$. A result of [5] states that $codim(E_S) = 1$ if $|S|$ is odd and $|S|$ otherwise, then assuming that $|S|$ is even (otherwise we consider $\bar{S}$), we obtain $codim(W) \geq |S| + 1 > 2$ since $S \neq \emptyset$. □

Theorem 4.2 *For odd $n \geq 7$, besides G_n on the triangle facets, and the co-clique on the orbit of the equicut facet, the restriction of the ridge graph of the cut polytope to the triangle facets and the orbit of the equicut facet is the complete bipartite graph $K_{k,l}$ with $k = 4\binom{n}{3}$ and $l = 2^{n-1}$, between the k triangle facets and the l facets switching equivalent to the equicut facet.*

PROOF. We use the following result proved in [8]. Given $v \in R^{\binom{m}{2}}$, if the inequality $v \cdot x \leq 0$ defines a facet of Cut_m, then the inequality $v' \cdot x \leq 0$ defines a facet of the equicut polytope EP_n for all odd $n \geq 2m+1$, where $v' = (v, 0 \ldots, 0) \in R^{(\,)}$ and EP_n is the convex hull of all the equicuts of V_n. We apply this result for $m = 3$ to a triangle facet of Cut_3 defined by the inequality $v \cdot x \leq 0$, then any facet defined by $(v, 0 \ldots, 0) \cdot x \leq 0$, i.e. any homogeneous triangle facet of Cut_n, is a facet of EP_n for odd $n \geq 7$. In other words, any homogeneous triangle facet of Cut_n intersects with the equicut facet on a ridge for odd $n \geq 7$. Then, similarly, one can check that all triangle facets are ridge-adjacent to the equicut facet and, by switching, ridge-adjacent to all facets of the orbit of the equicut facet. □

The 56 facets of $CutP_5$ are exactly the 40 triangle facets and the 16 facets switching equivalent to the equicut facet. One can directly check that a pair of facets of $CutP_5$ are ridge-adjacent if and only if they are non-conflicting. It implies that the facets switching equivalent to the equicut facet form a co-clique and that each of these 16 facets is ridge-adjacent to 10 triangle facets. For example, the equicut facet is ridge-adjacent to the 10 non-homogeneous triangle facets.

Corollary 4.3 *The diameter of Ω_5 is 2.*

PROOF. Since the diameter of G_5 is 2, we shall exhibit a facet adjacent to any pair of facets of the orbit of the equicut facet. Let F denote the equicut facet and $F_{\delta(S)}$ denote the facet obtained by the switching of F by the nonzero cut $\delta(S)$. If $|S| = 1$, respectively $|S| = 2$, $Tr_{\bar{S}\setminus\{i\}}$ for $i \in \bar{S}$, respectively $Tr_{\bar{S}}$, are adjacent to F and $F_{\delta(S)}$. Then, by switching, any pair of facets of the orbit of F has 1 or 4 common neighbours. □

Corollary 4.4 *$CutP_5$ has exactly 640 ridges.*

PROOF. Ridges of $CutP_5$ which are not among the 480 ridges of $MetP_5$, see Corollary 2.4, arise from the 160 adjacency relations between the orbit of the triangle facets and the orbit of the equicut facet. □

Remark 4.5 *With $f_i(P)$ denoting the number of i-faces of a polytope P, we have:*

$$f_i(MetP_5) + f_{9-i}(MetP_5) = f_i(CutP_5) + f_{9-i}(CutP_5) \quad \textit{for } i = 0 \textit{ or } 1.$$

Corollary 4.6 *For odd $n \geq 5$, the diameter of the restriction of the edge graph of the cut polytope to the triangle facets and the orbit of the equicut facet is 2.*

PROOF. Since the diameter of G_n is 2, it is a direct consequence of Theorem 4.2 and Corollary 4.3. □

Conjecture 4.7 *The triangle facets form a dominating set in the ridge graph of the cut polytope, i.e. every facet of the cut polytope is adjacent to a triangle facet. Since the diameter of the restriction of Ω_n to the triangle facets is 2, it would imply that the diameter of the dual cut polytope satisfies $\delta(\Omega_n) \leq 4$. This conjecture holds for $n \leq 7$, moreover, $\delta(\Omega_6) = 3$, see[7].*

Remark 4.8 *One can check that, with $d(G)$ the dominating number of a graph G, i.e. the size of the smallest dominating set in G, we have, for the ridge graph of the cut polytope, $d(\Omega_3) = 1$, $d(\Omega_4) = 4$ and $d(\Omega_5) = 10$ (take the 10 non-homogeneous facets), and for the ridge graph of the metric polytope $d(\Gamma_3) = 1$, $d(\Gamma_4) = 1$ and $d(\Gamma_5) = 6$ (take the cuts $\delta(\{i\}), i = 1, \ldots, 5$ and the cut $\delta(\{1, 2\})$).*

5. Proofs

5.1. Proof of Lemma 2.1

The proof of Lemma 2.1 is based on a case by case analysis of the different intersections of pairs of triangle facets. The common support σ of two triangle facets $Tr_{ij,k}$ (or Tr_{ijk}) and $Tr_{i'j',k'}$ (or $Tr_{i'j'k'}$) is defined by $\sigma = \{i,j,k\} \cup \{i',j',k'\}$. Then we consider the 4 cases corresponding to the 4 possible values taken by σ.

Case $\sigma = 3$
We first recall the definition of trivial extension of a vertex of $MetP_{n-1}$ given in [16], which generalizes the trivial extension of anticuts of $MetP_{n-1}$ given in Section 3. For $\delta = 0$ or 1, we consider the map ϕ_δ from $R^{\binom{n-1}{2}}$ to $R^{\binom{n}{2}}$ defined as follows. For $v \in MetP_{n-1}$, the metric polytope on the $(n-1)$ points of $\{1,\ldots,n\}\backslash\{i\}$,

$$\begin{array}{ll} \phi_\delta(v)_{kl} = v_{kl} & \text{for } 1 \le k < l \le n \text{ and } l \ne i \\ \phi_\delta(v)_{il} = \delta + (-1)^\delta v_{jl} & \text{for } l \notin \{i,j\}, \\ \phi_\delta(v)_{ij} = \delta & \end{array}$$

$\phi_\delta(v)$ is called a trivial extension of v. Since permutations and switching reflections preserve the adjacency, without loss of generality, we can consider the following intersection of a pair of triangle facets: $F = Tr_{12,3} \cap Tr_{13,2}$. We have:

$$v \in F = Tr_{12,3} \cap Tr_{13,2}$$

$$\Rightarrow \begin{cases} v_{12} - v_{13} - v_{23} \le 0 \\ v_{13} - v_{12} - v_{23} \le 0 \end{cases}$$

$$\Rightarrow \begin{cases} v_{23} = 0 \\ v_{13} = v_{12} \end{cases}$$

using $0 \le v_{ij} \le 1$ for $1 \le i < j \le n$

$$\Rightarrow \begin{cases} v_{23} = 0 \\ v_{i3} = v_{i2} \text{ for } i \notin \{2,3\} \end{cases}$$

$$\Rightarrow \begin{cases} v \in \phi_0(MetP_{n-1}) \\ v \in (T_{2i,3} \cap Tr_{3i,2}) \text{ for } i \notin \{2,3\} \end{cases}$$

On the other hand, we obviously have: $\phi_0(MetP_{n-1}) \subset (Tr_{12,3} \cap Tr_{13,2})$. Then, since the map ϕ_δ preserves the faces of $MetP_{n-1}$ as well as their dimension [16], $F = \phi_0(MetP_{n-1})$ implies that F, which is called a weak triangle face, is a face of dimension $\binom{n-1}{2}$ combinatorially equivalent to $MetP_{n-1}$.

Case $\sigma = 4$
We first recall the definitions of the intersection vector $\pi(S)$ and the characteristic vector $\chi(S)$. For $S \subset V_n$, $\pi(S) \in \{0,1\}^{\binom{n+1}{2}}$ is defined by: $\pi(S)_{ij} = 1$ if $\{i,j\} \subset S$

and 0 otherwise, and $\chi(S) \in \{0,1\}^n$ is defined by: $\chi(S)_i = 1$ if $\{i\} \subset S$ and 0 otherwise. We also recall that the linear independence of the characteristic vectors $\chi(S_i)$ for $i = 1, \ldots, n$ implies the linear independence of the cuts $\delta(S_i)$ for $i = 1, \ldots, n$, and that linear independence of the intersection vectors $\pi(S_i)$ for $i = 1, \ldots, n$ is equivalent to the linear independence of the cuts $\delta(S_i)$ for $i = 1, \ldots, n$.

We consider the intersection of a pair of non-conflicting triangle facets, for example: $F = Tr_{23,1} \cap Tr_{23,4}$. Since F is the intersection of 2 facets, its codimension, $codim(F)$, is at least 2. We show that $codim(F)$ is exactly equal to 2 by exhibiting, besides the cut $\delta(\emptyset)$, a family of $\binom{n}{2}-2$ linearly independent cuts of the metric polytope $MetP_n$ belonging to F. For $n = 4$, the vertices of F are $\delta(\emptyset), \delta(2), \delta(3), \delta(1,2)$ and $\delta(1,3)$, and one can easily check that the last 4 cuts are linearly independent. Then, assume that F contains $\binom{n}{2}-2$ linearly independent nonzero cuts of $MetP_n$: $\delta(S_i)$ for $i = 1, \ldots, \binom{n}{2}-2$. Then, in addition to the $\binom{n}{2}-2$ following nonzero cuts of $MetP_{n+1}$: $\delta(S_i)$ for $i = 1, \ldots, \binom{n}{2}-2$, we consider the n cuts of $MetP_{n+1}$: $\delta(S_i \cup \{n+1\})$ for $i = 1, \ldots, n$. To check the linear independence of these $\binom{n}{2}-2+n = \binom{n+1}{2}-2$ cuts of $MetP_{n+1}$, we consider the matrix I_{n+1} of their associated intersection vectors. With the last $(n+1)$ columns corresponding to the coordinates $\delta_{i,n+1}$ for $i = 1, \ldots, n+1$ and the last n rows corresponding to $\pi(S_i \cup \{n+1\})$ for $i = 1, \ldots, n$, we have:

$$I_{n+1} = \begin{pmatrix} I_n & 0 \\ A & X_n \end{pmatrix}$$

where I_n is the matrix of the intersection vectors associated to the cuts of the metric polytope $MetP_n$: $\delta(S_i)$ for $i = 1, \ldots, n$ and X_n the matrix of the characteristic vectors associated to the cuts of $MetP_{n+1}$: $\delta(S_i \cup \{n+1\})$ for $i = 1, \ldots, n$. Since $rank(I_n) = \binom{n}{2}-2$ and $rank(X_n) = n$, we have $rank(I_{n+1}) = \binom{n}{2}-2+n = \binom{n+1}{2}-2$. So, for $n \geq 4$ the codimension of the intersection of a pair of non-conflicting triangle facets is 2.

Then, we consider the intersection of a pair of conflicting triangle facets, for example: $F = Tr_{23,1} \cap Tr_{24,3}$. Exactly in the same way as we did for the non-conflicting case, we can show that the codimension of F cannot increase with n. Since for $n = 4$, the vertices of F are $\delta(\emptyset), \delta(2), \delta(4)$ and $\delta(1,2)$, the codimension of F is 3. It implies that for $n \geq 4$, $codim(F) \leq 3$. One can easily check that $F = Tr_{23,1} \cap Tr_{24,3} = Tr_{24,1} \cap Tr_{14,3}$, which implies that F, being an intersection of more than 3 facets, can not be a ridge, i.e. $codim(F) = 3$.

Case $\sigma = 5$ or 6

For $\sigma = 5$ or 6, any pair of triangle facets is non-conflicting. As for the case of a non-conflicting pair for $\sigma = 4$, we first determine the dimension of F for $n = 5$ or 6. It turns out that F is always a ridge. For example, $F = Tr_{23,1} \cap Tr_{34,5}$ contains the cuts: $\delta(\emptyset)$, $\delta(2), \delta(3), \delta(4), \delta(1,2), \delta(1,3), \delta(2,4), \delta(4,5)$ and $\delta(3,5)$ and one can check that the last 8 cuts are linearly independent. Then, using once again the fact that the codimension of F cannot increase with n, we obtain that, for $n \geq 5$ or 6, F is a ridge.

Remark 5.1 *From the proof of Lemma 2.1, one can easily check that a weak triangle face contains* 2^{n-2} *cuts, a face resulting from the intersection of a pair of conflicting facets with* $\sigma = 4$ *contains* 2^{n-2} *cuts, and that a ridge contains either* $5 \cdot 2^{n-4}$ *cuts for* $\sigma = 4$*, or* $9 \cdot 2^{n-5}$ *for* $\sigma = 5$ *or* 6*.*

5.2. Proof of Theorem 2.2

Theorem 2.2 is a direct consequence of the fact that, for $n \geq 4$, a pair of triangle facets are not ridge-adjacent if and only if they are conflicting. For example, the 3 neighbours of the facet Tr_{123} in $\bar{G}_n$, the complement of the ridge graph of the metric polytope, corresponding to the case $\sigma = 3$, i.e. $Tr_{12,3}, Tr_{13,2}$ and $Tr_{23,1}$ form the common K_3 of a bouquet of $(n-3)$ (3×3)-grids. The $6(n-3)$ other neighbours, corresponding to the case $\sigma = 4$, belong to $(n-3)$ (3×3)-grids. For $i = 4, \ldots, n$, each grid is:

$$\begin{array}{ccc}
| & | & | \\
-Tr_{3i,2}- & Tr_{2i,3}- & Tr_{12,3}- \\
| & | & | \\
-Tr_{1i,3}- & Tr_{3i,1}- & Tr_{13,2}- \\
| & | & | \\
-Tr_{2i,1}- & Tr_{1i,2}- & Tr_{23,1}- \\
| & | & |
\end{array}$$

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