

# A counterexample to the dominating set conjecture

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**Abstract** The metric polytope  $\text{met}_n$  is the polyhedron associated with all semi-metrics on  $n$  nodes and defined by the triangle inequalities  $x_{ij} - x_{ik} - x_{jk} \leq 0$  and  $x_{ij} + x_{ik} + x_{jk} \leq 2$  for all triples  $i, j, k$  of  $\{1, \dots, n\}$ . In 1992 Monique Laurent and Svatopluk Poljak conjectured that every fractional vertex of the metric polytope is adjacent to some integral vertex. The conjecture holds for  $n \leq 8$  and, in particular, for the 1,550,825,600 vertices of  $\text{met}_8$ . While the overwhelming majority of the known vertices of  $\text{met}_9$  satisfy the conjecture, we exhibit a fractional vertex not adjacent to any integral vertex.

**Keywords** Dominating set conjecture · Metric polyhedra · Cut polyhedra

## 1 Introduction and notation

The  $\binom{n}{2}$ -dimensional *cut cone*  $\text{Cut}_n$  is usually introduced as the conic hull of the incidence vectors of all the cuts of the complete graph on  $n$  nodes. More precisely, given a subset  $S$  of  $V_n = \{1, \dots, n\}$ , the *cut* determined by  $S$  consists of the pairs  $(i, j)$  of elements of  $V_n$  such that exactly one of  $i, j$  is in  $S$ . By  $\delta(S)$  we denote both the cut and its incidence vector in  $\mathbb{R}^{\binom{n}{2}}$ , i.e.,  $\delta(S)_{ij} = 1$  if exactly one of  $i, j$  is in  $S$  and 0 otherwise for  $1 \leq i < j \leq n$ . We use the term *cut* for

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both the cut itself and its incidence vector, so  $\delta(S)_{ij}$  are coordinates of a point in  $\mathbb{R}^{\binom{n}{2}}$ .

The cut cone  $\text{Cut}_n$  is the conic hull of all  $2^{n-1} - 1$  nonzero cuts, and the *cut polytope*  $\text{cut}_n$  is the convex hull of all  $2^{n-1}$  cuts. The cut cone and a relaxation, the *metric cone*  $\text{Met}_n$ , can also be defined in terms of finite metric spaces in the following way. For all triples  $\{i, j, k\} \subset V_n$ , we consider the following inequalities.

$$x_{ij} - x_{ik} - x_{jk} \leq 0, \tag{1}$$

$$x_{ij} + x_{ik} + x_{jk} \leq 2. \tag{2}$$

Equation (1) specify the  $3\binom{n}{3}$  facets of the cone  $\text{Met}_n$  of *semimetrics* on  $V_n$ ; that is, of functions  $x : V_n \times V_n \rightarrow \mathbb{R}_+$  satisfying  $x_{ij} = x_{ji}$ ,  $x_{ii} = 0$ , and the *triangle inequalities* (1). While  $x$  is a *metric* only when  $x_{ij} > 0$  for all  $i \neq j$ , we will follow the usual convention and call  $\text{Met}_n$  the *metric cone*.

It is well-known that  $\text{Cut}_n$  is the conic hull of all, up to a constant multiple,  $\{0, 1\}$ -valued extreme rays of  $\text{Met}_n$ . The cuts satisfy the *perimeter inequalities* (2) which can also be obtained from (1) by the *switching* operation, see Sect. 4. Bounding  $\text{Met}_n$  by the  $\binom{n}{3}$  facets induced by (2), we obtain a natural relaxation of  $\text{cut}_n$ , the *metric polytope*  $\text{met}_n$ , so that  $\text{cut}_n$  is the convex hull of all  $\{0, 1\}$ -valued vertices of  $\text{met}_n$ .

One of the motivations for the study of these polyhedra comes from their applications in combinatorial optimization, the most important being the  $\text{MAX-CUT}$  and multicommodity flow problems. We refer to Deza and Laurent [8] and to Poljak and Tuza [13] for a detailed study of those polyhedra and their applications in combinatorial optimization.

## 2 A counterexample to the dominating set conjecture

Laurent and Poljak [12] conjectured that every fractional vertex of the metric polytope  $\text{met}_n$  is adjacent to some integral vertex, i.e., to a cut. Since we have  $\text{met}_3 = \text{cut}_3$  and  $\text{met}_4 = \text{cut}_4$ , the conjecture, which is commonly called the dominating set conjecture, is obviously true for the 4 vertices of  $\text{met}_3$  and for the 8 vertices of  $\text{met}_4$ . The conjecture holds for the 32 vertices of  $\text{met}_5$  and the 544 vertices of  $\text{met}_6$  as well as for several classes of vertices of  $\text{met}_n$ , see [10,11]. The conjecture was further substantiated by the computation of  $\text{met}_7$  and  $\text{met}_8$ . The 275,840 vertices of  $\text{met}_7$  and the 1,550,825,600 vertices of  $\text{met}_8$  are adjacent to a cut, see [3–5].

While the overwhelming majority of the known vertices of  $\text{met}_9$  satisfy the dominating set conjecture, few counterexamples were found. In particular, we exhibit the following fractional vertex not adjacent to any integral vertex.

**Proposition 1** *The neighbors of the fractional vertex  $\frac{1}{9}(2, 2, 3, 3, 4, 4, 5, 5, 4, 3, 5, 6, 6, 3, 3, 5, 5, 2, 4, 3, 5, 6, 3, 3, 6, 6, 5, 3, 2, 6, 6, 3, 3, 5, 3, 4)$  of the metric polytope  $\text{met}_9$  are all fractional.*

The vertex given in Proposition 1, as well as other vertices not adjacent to any cut, were found by an extensive computer search of the vertices of the 36-dimensional metric polytope  $\text{met}_9$ , see [2]. Note that while finding a vertex providing a counterexample to the dominating set conjecture is computationally challenging, to verify that a given vertex is indeed not adjacent to a cut is easy if the vertex is quasi-simple, i.e., if the incidence of the given vertex is equal to the dimension plus one. For example, one can easily check, see Sect. 4, that the vertex given in Proposition 1 satisfies with equalities 37 of the 336 inequalities defining  $\text{met}_9$  and is adjacent to 37 vertices which are all fractional.

### 3 Related questions

#### 3.1 The diameter of the metric polytope

Since any pair of cuts forms an edge of  $\text{met}_n$ , the dominating set conjecture would imply that the diameter  $\delta(\text{met}_n)$  of the metric polytope satisfies  $\delta(\text{met}_n) \leq 3$ . We recall that the diameter of a polytope  $P$  is the smallest number  $\delta(P)$  such that any two vertices of  $P$  can be connected by a path with at most  $\delta(P)$  edges. We have  $\delta(\text{met}_3) = \delta(\text{met}_4) = 1$ ,  $\delta(\text{met}_5) = \delta(\text{met}_6) = 2$  and  $\delta(\text{met}_7) = \delta(\text{met}_8) = 3$ . While the diameter of the restriction of  $\text{met}_9$  to its known vertices appears to be less than 3, it is not clear that the diameter of  $\text{met}_n$  is bounded by a constant.

#### 3.2 The no-cut set conjecture

*Conjecture 1* [5] For  $n \geq 6$ , the restriction of the metric polytope  $\text{met}_n$  to its fractional vertices is connected.

Conjecture 1 can be seen as complementary to the dominating set conjecture both graphically and computationally: for any pair of vertices, while the dominating set conjecture implies that there is a path made of cuts joining them, i.e., the cut vertices form a dominating set, Conjecture 1 means that there is a path made of non-cut vertices joining them, i.e., the cut vertices do not form a cut-set. On the other hand, while the dominating set conjecture means that the enumeration of the extreme rays of the metric cone  $\text{Met}_n$  is enough to obtain the vertices of the metric polytope  $\text{met}_n$ , Conjecture 1 means that we can obtain the vertices of  $\text{met}_n$  without enumerating the extreme rays of  $\text{Met}_n$ .

### 4 Counterexamples generation and verification

One important feature of the metric and cut polyhedra is their very large symmetry group. We recall that the symmetry group  $\text{Is}(P)$  of a polyhedron  $P$  is the group of isometries preserving  $P$  and that an isometry is a linear transformation preserving the Euclidean distance. For  $n \geq 5$ , the symmetry groups of the polytopes  $\text{met}_n$  and  $\text{cut}_n$  are isomorphic and induced by permutations on  $V_n$  and *switching reflections by a cut*, see [7], and the symmetry groups of the cones

$\text{Met}_n$  and  $\text{Cut}_n$  are isomorphic to  $\text{Sym}(n)$ , see [6]. Given a cut  $\delta(S)$ , the switching reflection  $r_{\delta(S)}$  is defined by  $y = r_{\delta(S)}(x)$  where  $y_{ij} = 1 - x_{ij}$  if  $(i, j) \in \delta(S)$  and  $y_{ij} = x_{ij}$  otherwise.

#### 4.1 Counterexamples generation

The vertices of  $\text{met}_n$  are partitioned into orbits under the action of the symmetry group  $\text{Is}(\text{met}_n)$ . Using a parallel implementation of an orbitwise enumeration algorithm, 1,056,368 orbits of vertices of  $\text{met}_9$  were computed on Shared Hierarchical Academic Research Computing Network (SHARCNET) clusters. Among these 1,056,368 orbits, 1,221 provide counterexamples to the dominating set conjecture, including 483 made of quasi-simplicial vertices. As the dimension of  $\text{met}_9$  is 36, these quasi-simplicial counterexamples satisfy with equality exactly 37 inequalities. The remaining 738 orbits providing counterexamples to the dominating set conjecture are made of vertices satisfying with equality at least 38 inequalities. Some of these counterexamples have a relatively large incidence and adjacency: For example, the vertex  $\frac{1}{9}(1, 2, 3, 3, 4, 4, 4, 6, 3, 4, 4, 3, 3, 5, 7, 5, 5, 6, 6, 2, 4, 6, 3, 3, 3, 3, 3, 3, 3, 3, 6, 6, 4, 6, 4, 2)$  satisfies with equality 44 inequalities and has 84 fractional adjacent vertices, and the vertex  $\frac{1}{12}(3, 3, 3, 6, 7, 7, 7, 7, 6, 6, 3, 4, 4, 4, 4, 6, 3, 4, 4, 4, 4, 9, 4, 4, 8, 8, 7, 7, 7, 7, 8, 4, 4, 4, 4, 8)$  satisfies with equality 43 inequalities and has 202 fractional adjacent vertices. See [2] for a complete list of the known counterexamples.

#### 4.2 Counterexamples verification

For a given quasi-simple vertex, one can easily verify that all the adjacent vertices are fractional by performing 3 simple computations which we illustrate using the vertex given in Proposition 1.

- (i) Check which of the 336 inequalities of  $\text{met}_9$  are satisfied with equality by the given vertex. For the vertex given in Proposition 1, we obtain the 37 inequalities given in Sect. 4.3.1.
- (ii) Compute the pointed cone formed by the active inequalities identified in (i). For the vertex given in Proposition 1, we obtain a quasi-simplicial cone with 37 extreme rays.
- (iii) From the given vertex, follow each extreme ray of the cone computed in (ii) until at least one of the inequalities defining  $\text{met}_9$  is violated. The obtained points are the adjacent vertices. For the vertex given in Proposition 1, we obtain the 37 adjacent vertices given in Sect. 4.3.2. As these 37 adjacent vertices are all fractional, the vertex given in Proposition 1 is indeed a counterexample.

Note that, while the computation (ii) can be extremely expensive for a highly degenerate vertex in high dimension, it can be done efficiently if the given vertex is quasi-simple. It takes less than a second of CPU time for the vertex given in Proposition 1 using enumeration packages such as *lrs* [1] or *cdd* [9].

Computations (i) and (iii) are straightforward and take less than a second of CPU time.

### 4.3 Given counterexample incidence and adjacency lists

#### 4.3.1 Given counterexample incidence list

The vertex given in Proposition 1 satisfies with equalities the following 37 inequalities of mety:  $\Delta_{6,7,9}, \Delta_{5,8,9}, \Delta_{5,7,9}, \Delta_{5,7,8}, \Delta_{5,6,8}, \Delta_{4,7,9}, \Delta_{4,6,9}, \Delta_{4,6,8}, \Delta_{4,6,7}, \Delta_{4,5,9}, \Delta_{4,5,7}, \Delta_{3,6,9}, \Delta_{3,6,7}, \Delta_{3,5,8}, \Delta_{3,4,6}, \Delta_{2,7,9}, \Delta_{2,6,9}, \Delta_{2,6,8}, \Delta_{2,6,7}, \Delta_{2,5,8}, \Delta_{2,4,9}, \Delta_{2,4,8}, \Delta_{2,4,7}, \Delta_{2,4,6}, \Delta_{2,3,6}, \Delta_{1,5,8}, \Delta_{1,4,5}, \Delta_{1,3,8}, \Delta_{1,3,6}, \Delta_{1,3,5}, \Delta_{1,3,4}, \Delta_{1,2,9}, \Delta_{1,2,8}, \Delta_{1,2,7}, \Delta_{1,2,6}, \Delta_{1,2,5}, \Delta_{1,2,3}$  where the triangle inequality (1) and the perimeter inequality (2) are respectively denoted by  $\Delta_{i,j,\bar{k}}$  and  $\Delta_{i,j,k}$ .

#### 4.3.2 Given counterexample adjacency list

The vertex given in Proposition 1 is adjacent to the following 37 fractional vertices of mety:

- $\frac{1}{3}(0, 1, 1, 1, 2, 2, 1, 1, 1, 1, 1, 2, 2, 1, 1, 2, 0, 1, 1, 0, 2, 2, 1, 1, 2, 2, 1, 1, 0, 2, 2, 1, 1, 1, 1, 2)$
- $\frac{1}{3}(0, 1, 1, 1, 2, 2, 1, 1, 1, 1, 1, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 0, 2, 2, 1, 1, 1, 1, 2)$
- $\frac{1}{3}(1, 0, 2, 0, 1, 1, 0, 2, 1, 1, 1, 2, 2, 1, 1, 2, 0, 1, 1, 0, 2, 2, 1, 1, 2, 2, 1, 1, 0, 2, 2, 1, 1, 1, 1, 2)$
- $\frac{1}{3}(1, 1, 0, 1, 1, 1, 2, 2, 2, 1, 2, 2, 2, 1, 1, 1, 2, 0, 2, 1, 1, 1, 1, 1, 2, 2, 2, 0, 1, 1, 2, 1, 1, 1, 1, 0)$
- $\frac{1}{3}(1, 1, 0, 1, 1, 1, 2, 2, 2, 1, 2, 2, 2, 1, 1, 1, 2, 0, 2, 1, 1, 1, 1, 1, 2, 2, 2, 2, 1, 3, 2, 1, 1, 3, 1, 2)$
- $\frac{1}{3}(1, 1, 1, 1, 1, 1, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 1, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 1, 1)$
- $\frac{1}{3}(1, 1, 1, 1, 1, 1, 2, 2, 2, 1, 2, 2, 2, 1, 1, 1, 2, 0, 2, 1, 1, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 1, 1)$
- $\frac{1}{3}(1, 1, 1, 1, 1, 1, 2, 2, 2, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 1, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 1, 1)$
- $\frac{1}{3}(1, 1, 1, 1, 2, 1, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 1, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 1, 1)$
- $\frac{1}{3}(1, 1, 1, 1, 2, 1, 2, 2, 2, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 1, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 1, 1)$
- $\frac{1}{4}(1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 3, 1, 1, 2, 2, 0, 3, 1, 1, 2, 2, 1, 3, 3, 2, 1, 1, 3, 3, 1, 1, 2, 2, 2)$
- $\frac{1}{4}(1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 3, 3, 1, 1, 2, 2, 1, 1, 1, 3, 2, 1, 1, 3, 3, 3, 1, 1, 3, 2, 2, 2, 2, 2, 2)$
- $\frac{1}{6}(1, 1, 2, 2, 3, 3, 3, 3, 2, 2, 3, 4, 4, 2, 2, 3, 3, 2, 2, 2, 4, 4, 2, 2, 4, 4, 3, 2, 1, 4, 4, 2, 2, 3, 2, 3)$
- $\frac{1}{6}(1, 1, 3, 1, 3, 3, 2, 3, 2, 2, 2, 4, 4, 1, 2, 4, 2, 2, 2, 1, 4, 4, 2, 2, 3, 4, 4, 2, 1, 4, 4, 3, 2, 3, 2, 3)$
- $\frac{1}{6}(1, 1, 3, 2, 3, 3, 3, 3, 2, 2, 3, 4, 4, 2, 2, 4, 3, 2, 2, 2, 4, 4, 2, 2, 4, 4, 3, 2, 1, 4, 4, 2, 2, 3, 2, 3)$
- $\frac{1}{6}(1, 1, 3, 2, 3, 3, 3, 3, 2, 2, 3, 4, 4, 2, 2, 4, 3, 2, 2, 2, 4, 5, 2, 2, 4, 4, 3, 3, 1, 3, 4, 2, 2, 4, 2, 2)$
- $\frac{1}{6}(2, 1, 2, 2, 2, 2, 4, 4, 3, 2, 4, 4, 4, 2, 2, 3, 3, 1, 3, 3, 3, 4, 2, 2, 4, 4, 4, 2, 2, 4, 4, 2, 2, 4, 2, 2)$
- $\frac{1}{7}(1, 1, 3, 2, 3, 3, 3, 3, 2, 2, 3, 4, 4, 2, 2, 4, 3, 2, 2, 2, 4, 5, 2, 2, 4, 4, 3, 3, 1, 5, 4, 2, 2, 4, 2, 4)$

- $\frac{1}{7}(1, 1, 3, 2, 3, 4, 3, 3, 2, 2, 3, 4, 5, 2, 2, 4, 3, 2, 3, 2, 4, 5, 2, 3, 4, 4, 3, 2, 1, 5, 5, 2, 2, 3, 3, 4)$   
 $\frac{1}{7}(1, 1, 3, 2, 3, 4, 3, 4, 2, 2, 3, 4, 5, 2, 3, 4, 3, 2, 3, 2, 5, 5, 2, 3, 4, 5, 3, 2, 1, 4, 5, 2, 3, 3, 2, 3)$   
 $\frac{1}{7}(2, 2, 2, 3, 3, 2, 5, 4, 4, 2, 5, 5, 4, 3, 2, 4, 5, 1, 4, 3, 4, 5, 3, 2, 5, 4, 4, 3, 2, 5, 5, 2, 3, 5, 2, 3)$   
 $\frac{1}{7}(2, 2, 2, 3, 3, 3, 5, 5, 4, 2, 5, 5, 5, 3, 3, 4, 5, 1, 3, 3, 3, 5, 3, 3, 5, 5, 4, 2, 2, 4, 4, 2, 2, 4, 2, 2)$   
 $\frac{1}{9}(1, 2, 4, 2, 5, 5, 4, 4, 3, 3, 3, 6, 6, 3, 3, 6, 4, 3, 3, 2, 6, 6, 3, 3, 6, 6, 5, 3, 2, 6, 6, 3, 3, 5, 3, 4)$   
 $\frac{1}{9}(2, 2, 3, 3, 4, 4, 5, 3, 4, 3, 5, 6, 6, 3, 3, 5, 5, 2, 4, 3, 5, 6, 3, 3, 6, 6, 5, 3, 2, 6, 6, 3, 3, 5, 3, 4)$   
 $\frac{1}{9}(2, 2, 3, 3, 4, 4, 5, 5, 4, 3, 5, 6, 6, 3, 3, 5, 5, 2, 4, 3, 3, 6, 3, 3, 6, 6, 5, 3, 2, 6, 6, 3, 3, 5, 3, 4)$   
 $\frac{1}{9}(2, 2, 3, 3, 4, 4, 5, 5, 4, 3, 5, 6, 6, 3, 3, 5, 5, 2, 4, 3, 5, 6, 3, 3, 6, 6, 5, 3, 2, 6, 6, 3, 3, 5, 3, 6)$   
 $\frac{1}{9}(2, 2, 3, 3, 4, 4, 5, 5, 4, 3, 5, 6, 6, 3, 3, 5, 5, 2, 4, 3, 5, 6, 3, 3, 6, 6, 3, 3, 2, 6, 6, 3, 3, 5, 3, 4)$   
 $\frac{1}{9}(2, 2, 3, 3, 4, 4, 5, 5, 4, 3, 5, 6, 6, 3, 3, 5, 5, 2, 4, 3, 5, 6, 3, 3, 6, 6, 5, 3, 2, 6, 6, 3, 3, 3, 3, 4)$   
 $\frac{1}{9}(2, 2, 3, 3, 4, 4, 5, 5, 4, 3, 5, 6, 6, 3, 3, 5, 5, 2, 6, 3, 5, 6, 3, 3, 6, 6, 5, 3, 2, 6, 6, 3, 3, 5, 3, 4)$   
 $\frac{1}{9}(2, 2, 3, 3, 4, 6, 5, 5, 4, 3, 5, 6, 6, 3, 3, 5, 5, 2, 4, 3, 5, 6, 3, 3, 6, 6, 5, 3, 2, 6, 6, 3, 3, 5, 3, 4)$   
 $\frac{1}{9}(3, 2, 2, 4, 3, 3, 6, 6, 5, 3, 5, 6, 6, 3, 3, 4, 6, 1, 5, 4, 4, 6, 3, 3, 6, 6, 5, 3, 2, 6, 6, 3, 3, 5, 3, 4)$   
 $\frac{1}{10}(2, 2, 4, 3, 5, 4, 5, 5, 4, 4, 5, 7, 6, 3, 3, 6, 5, 3, 4, 3, 7, 7, 3, 4, 7, 7, 6, 3, 2, 6, 7, 4, 4, 5, 3, 4)$   
 $\frac{1}{10}(2, 2, 4, 3, 5, 5, 5, 6, 4, 4, 5, 7, 7, 3, 4, 6, 5, 3, 3, 3, 6, 7, 3, 3, 7, 6, 6, 4, 2, 7, 6, 4, 3, 6, 3, 5)$   
 $\frac{1}{10}(3, 3, 2, 4, 4, 3, 7, 7, 6, 3, 7, 7, 6, 4, 4, 5, 7, 1, 6, 4, 4, 6, 4, 3, 7, 7, 6, 3, 3, 7, 7, 3, 3, 6, 4, 4)$   
 $\frac{1}{12}(3, 3, 3, 5, 5, 5, 7, 7, 6, 4, 8, 8, 8, 4, 4, 6, 8, 2, 6, 4, 6, 8, 4, 4, 8, 8, 8, 4, 4, 8, 8, 4, 4, 8, 4, 4)$   
 $\frac{1}{12}(3, 3, 3, 5, 5, 5, 8, 7, 6, 4, 8, 8, 8, 5, 4, 6, 8, 2, 6, 5, 6, 8, 4, 4, 7, 8, 6, 4, 3, 8, 8, 3, 4, 7, 4, 5)$   
 $\frac{1}{12}(3, 3, 3, 5, 5, 5, 8, 7, 6, 4, 8, 8, 8, 5, 4, 6, 8, 2, 6, 5, 6, 8, 4, 4, 9, 8, 8, 4, 3, 8, 8, 5, 4, 7, 4, 5)$

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