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Polytopes and arrangements: Diameter and curvature

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Abstract

We introduce a continuous analogue of the Hirsch conjecture and a discrete analogue of the result of Dedieu, Malajovich and Shub. We prove a continuous analogue of the result of Holt and Klee, namely, we construct a family of polytopes which attain the conjectured order of the largest total curvature.

Keywords: Polytopes; Arrangements; Diameter; Central path; Total curvature

1. Continuous analogue of the conjecture of Hirsch

By analogy with the conjecture of Hirsch, we conjecture that the order of the largest total curvature of the central path associated to a polytope is the number of inequalities defining the polytope. By analogy with a result of Dedieu, Malajovich and Shub, we conjecture that the average diameter of a bounded cell of a simple arrangement is less than the dimension. We prove a continuous analogue of the result of Holt–Klee, namely, we construct a family of polytopes which attain the conjectured order of the largest total curvature. We substantiate the conjectures in low dimensions and highlight additional links.

Let P be a full dimensional convex polyhedron defined by m inequalities in dimension n. The diameter $\delta(P)$ is the smallest number such that any two vertices of the polyhedron P can be connected by a path with at most $\delta(P)$ edges. The conjecture of Hirsch, formulated in 1957 and reported in [3], states that the diameter of a polyhedron defined by m inequalities in dimension n is not greater than $m-n$. The conjecture does not hold for unbounded polyhedra. A polytope is a bounded polyhedron. No polynomial bound is known for the diameter of a polytope.

Conjecture 1.1 (*Conjecture of Hirsch for polytopes*). *The diameter of a polytope defined by m inequalities in dimension n is not greater than* $m-n$.

Intuitively, the total curvature [16] is a measure of how far off a certain curve is from being a straight line. Let $\psi : [\alpha, \beta] \to \mathbb{R}^n$ be a $C^2(\alpha-\varepsilon, \beta+\varepsilon)$ map for some $\varepsilon > 0$ with a non-zero derivative in $[\alpha, \beta]$. Denote its arc length by $l(t) = \int_\alpha^t \|\dot{\psi}(\tau)\|\,\mathrm{d}\tau$, its parametrization by the arc length by $\psi_{\mathrm{arc}} = \psi \circ l^{-1} : [0, l(\beta)] \to \mathbb{R}^n$, and its curvature at the point t by $\kappa(t) = \ddot{\psi}_{\mathrm{arc}}(t)$.The total curvature is defined as $\int_0^{l(\beta)} \|\kappa(t)\|\,\mathrm{d}t$. The requirement $\dot{\psi} \neq 0$ insures that any given segment of the curve is traversed only once and allows to define a curvature at any point on the curve.

From now on we consider only polytopes, i.e., bounded polyhedra, and denote those by P. For a polytope $P = \{x : Ax \geqslant b\}$ with $A \in \mathbb{R}^{m\times n}$, denote $\lambda(P)$ the largest total curvature of the primal central path corresponding to the standard logarithmic barrier function, $-\sum_{i=1}^m \ln(A_i x - b_i)$, of the linear programming problem $\min\{c^{\mathrm{T}}x : x \in P\}$ over all possible c. Following the analogy with the diameter, let $\Lambda(m, n)$ be the largest total curvature $\lambda(P)$ of the primal central path over all polytopes P defined by m inequalities in dimension n.

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Conjecture 1.2 (*Continuous analogue of the conjecture of Hirsch*). *The order of the largest total curvature of the primal central path over all polytopes defined by m inequalities in dimension n is the number of inequalities defining the polytopes, i.e.*, $\Lambda(m, n) = \mathcal{O}(m)$.

Remark 1.1. In [6] the authors showed that a redundant Klee–Minty n-cube $\mathscr{C}$ satisfies $\lambda(\mathscr{C}) \geqslant (\frac{3}{2})^n$, providing a counterexample to the conjecture of Dedieu and Shub [5] that $\Lambda(m, n) = \mathcal{O}(n)$.

For polytopes and arrangements, respectively central path and linear programming, we refer to the books of Grünbaum [9] and Ziegler [17], respectively Renegar [13] and Roos et al. [14].

2. Discrete analogue of the result of Dedieu, Malajovich, Shub

Let $\mathscr{A}$ be a simple arrangement formed by m hyperplanes in dimension n. We recall that an arrangement is called simple if $m \geqslant n + 1$ and any n hyperplanes intersect at a unique distinct point. Since $\mathscr{A}$ is simple, the number of bounded cells, i.e., bounded connected components of the complement to the hyperplanes, of $\mathscr{A}$ is $I = \binom{m-1}{n}$. Let $\lambda^c(P)$ denote the total curvature of the primal central path corresponding to $\min\{c^{\mathrm{T}}x : x \in P\}$. Following the approach of Dedieu et al. [4], let $\lambda^c(\mathscr{A})$ denote the average value of $\lambda^c(P_i)$ over the bounded cells P_i of $\mathscr{A}$; that is

$$\lambda^c(\mathscr{A}) = \frac{\sum_{i=1}^{i=I} \lambda^c(P_i)}{I}.$$

Note that each bounded cell P_i is defined by the same number m of inequalities, some being potentially redundant. Given an arrangement $\mathscr{A}$, the average total curvature of a bounded cell $\lambda(\mathscr{A})$ is the largest value of $\lambda^c(\mathscr{A})$ over all possible c. Similarly, $\Lambda_{\mathscr{A}}(m, n)$ is the largest possible average total curvature of a bounded cell of a simple arrangement defined by m inequalities in dimension n.

Proposition 2.1 (*Dedieu et al. [4]*). *The average total curvature of a bounded cell of a simple arrangement defined by m inequalities in dimension n is not greater than* $2\pi n$.

By analogy, let $\delta(\mathscr{A})$ denote the average diameter of a bounded cell of $\mathscr{A}$; that is

$$\delta(\mathscr{A}) = \frac{\sum_{i=1}^{i=I} \delta(P_i)}{I}.$$

Similarly, let $\Delta_{\mathscr{A}}(m, n)$ denote the largest possible average diameter of a bounded cell of a simple arrangement defined by m inequalities in dimension n.

Conjecture 2.1 (*Discrete analogue of the result of Dedieu, Malajovich and Shub*). *The average diameter of a bounded cell of a simple arrangement defined by m inequalities in dimension n is not greater than n.*

Haimovich's probabilistic analysis of the shadow-vertex simplex algorithm, see [2, Section 0.7], shows that the expected number of pivots is bounded by n. While the result and Conjecture 2.1 are similar in nature, they differ in some aspects: Haimovich considers the average over bounded and unbounded cells, and the number of pivots could be smaller than the diameter for some cells.

3. Additional links and low dimensions

3.1. Additional links

Proposition 3.1. *If the conjecture of Hirsch holds, then* $\Delta_{\mathscr{A}}(m, n) \leqslant n + \frac{2n}{m-1}$.

Proof. Let m_i denote the number of hyperplanes of $\mathscr{A}$ which are non-redundant for the description of a bounded cell P_i. If the conjecture of Hirsch holds, we have $\delta(P_i) \leqslant m_i - n$. It implies

$$\delta(\mathscr{A}) \leqslant \frac{\sum_{i=1}^{i=I}(m_i - n)}{I} = \frac{\sum_{i=1}^{i=I} m_i}{I} - n.$$

Since a facet belongs to at most 2 cells, the sum of m_i for $i = 1, \ldots, I$ is less than twice the number of bounded facets of $\mathscr{A}$. As a bounded facet induced by a hyperplane H of $\mathscr{A}$ corresponds to a bounded cell of the $(n - 1)$-dimensional simple arrangement

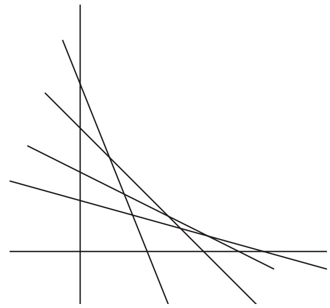

Fig. 1. The arrangement $\mathscr{A}^*_{6,2}$.

$\mathscr{A} \cap H$, the sum of m_i is less than $2m\binom{m-2}{n-1}$. Therefore, we have, for any simple arrangement $\mathscr{A}$, $\delta(\mathscr{A}) \leqslant \frac{2m\binom{m-2}{n-1}}{\binom{m-1}{n}} - n = \frac{2mn}{m-1} - n = \frac{n(m+1)}{m-1}$. □

Remark 3.1. In the proof of Proposition 3.1, we overestimate the sum of m_i for $i = 1, \ldots, I$ as some bounded facets belong to exactly 1 bounded cell. Let us call such bounded facets external. We hypothesize that any simple arrangement has at least $n\binom{m-2}{n-1}$ external facets, in turn, this would strengthen Proposition 3.1 to: *If the conjecture of Hirsch holds, then* $\Delta_{\mathscr{A}}(m, n) \leqslant \frac{n(m-n+1)}{m-1}$.

Similarly to Proposition 3.1, the results of Kalai and Kleitman [11] and Barnette [1] which bounds the diameter of a polytope by, respectively, $2 \colon m^{\log(n)+1}$ and $\frac{2^{n-2}}{3}(m - n + \frac{5}{2})$, directly yield

Proposition 3.2. $\Delta_{\mathscr{A}}(m, n) \leqslant \frac{4mn\left(2m\binom{m-2}{n-1}\right)^{\log n}}{m-1}$ *and* $\Delta_{\mathscr{A}}(m, n) \leqslant n(\frac{m+1}{m-1} + \frac{5}{2n})\frac{2^{n-2}}{3}$.

The special case of $m = 2n$ of the conjecture of Hirsch is known as the *d*-step conjecture (as the dimension is often denoted by d in polyhedral theory). In particular, it has been shown by Klee and Walkup [12] that the special case $m = 2n$ for all n is equivalent to the conjecture of Hirsch. A continuous analogue would be: if $\Lambda(2n, n) = \mathcal{O}(n)$ for all n, then $\Lambda(m, n) = \mathcal{O}(m)$.

Remark 3.2. In contrast with Proposition 3.1, $\Lambda(m, n) = \mathcal{O}(m)$ does not imply that $\Lambda_{\mathscr{A}}(m, n) = \mathcal{O}(n)$ since all the m inequalities count for each $\lambda(P_i)$ while it is enough to consider the m_i non-redundant inequalities for each $\delta(P_i)$.

3.2. Low dimensions

In dimensions 2 and 3 we have, respectively, $\delta(P) \leqslant \lfloor \frac{m}{2} \rfloor$ and $\delta(P) \leqslant \lfloor \frac{2m}{3} \rfloor - 1$, implying:

Proposition 3.3. $\Delta_{\mathscr{A}}(m, 2) \leqslant 2 + \frac{2}{m-1}$ and $\Delta_{\mathscr{A}}(m, 3) \leqslant 3 + \frac{4}{m-1}$.

In dimension 2, let S_2 be a unit sphere centered at (1, 1) and consider the arrangement $P\mathscr{A}^*_{m,2}$ made of the 2 lines forming the nonnegative orthant and additional $m - 2$ lines tangent to S_2 and separating the origin from the center of the sphere. See Fig. 1 for an illustration of $\mathscr{A}^*_{6,2}$. Besides $m - 2$ triangles, the bounded cells of $\mathscr{A}^*_{m,2}$ are made of $\binom{m-2}{2}$ 4-gons. We have $\delta(\mathscr{A}^*_{m,2}) = \frac{2(m-2)}{m-1}$, and thus,

Proposition 3.4. $2 - \frac{2}{m-1} \leqslant \Delta_{\mathscr{A}}(m, 2) \leqslant 2 + \frac{2}{m-1}$.

Remark 3.3. The arrangement $\mathscr{A}^*_{m,2}$ was generalized in [7] to an arrangement with $\binom{m-n}{n}$ cubical cells yielding that the dimension n is an asymptotic lower bound for $\Delta_{\mathscr{A}}(m, n)$ for fixed n.

In dimension 2, for $m \geqslant 4$, consider the polytope $P^*_{m,2}$ defined by the following m inequalities: $y \leqslant 1$, $x \leqslant \frac{y}{10} + \frac{1}{2}$, $-x \leqslant \frac{y}{3} + \frac{1}{3}$ and $(-1)^i x \leqslant \frac{10^{i-2}y}{11} + \frac{5}{11} - \frac{10^{-4}}{m}\frac{i}{m}$ for $i = 4, \ldots, m$. See Fig. 2 for an illustration of $P^*_{6,2}$ and Fig. 3 for the central path over $P^*_{34,2}$ with $c = (0, 1)$.

Proposition 3.5. *The total curvature of the central path of* $\min\{y : (x, y) \in P^*_{m,2}\}$ *satisfies*

$$\liminf_{m\to\infty} \frac{\lambda^{(0,1)}(P^*_{m,2})}{m} \geqslant \pi.$$

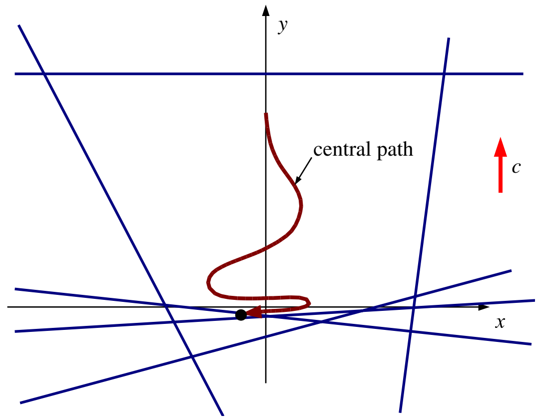

Fig. 2. The polytope $P^*_{6,2}$ and its central path.

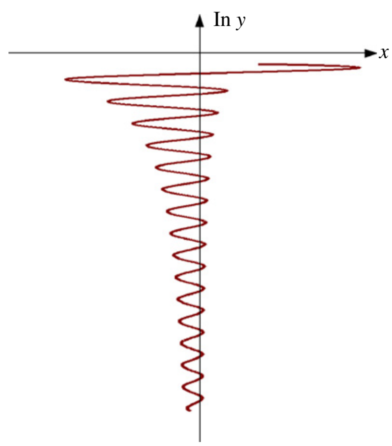

Fig. 3. The central path for $P^*_{34,2}$.

Proof. First we show that the central path $\mathscr{P}$ goes through a sequence of $m-2$ points $(x_j, \frac{10^{1-j}}{5})$ for $j=1,\ldots,m-2$ with $x_j \geqslant 0$ for odd j and $x_j \leqslant \dfrac{-10^{-4}}{m}$ for even j. For $i=2,\ldots,m$ and $j=1,\ldots,m-2$, denote z_i^j the first coordinate of the intersection of the line $y=\frac{10^{1-j}}{5}$ and the facet of $P^*_{m,2}$ induced by the ith inequality defining $P^*_{m,2}$, that is, $z_2^j=\frac{10^{-j}}{5}+\frac{1}{2}$, $z_3^j=-\frac{10^{-j+1}}{15}-\frac{1}{3}$, and $z_i^j=(-1)^i(\frac{10^{i-j-1}}{55}+\frac{5}{11}-\dfrac{10^{-4}}{m}\frac{i}{m})$ for $i=4,\ldots,m$. As the central path may be characterized as the set of minimizers of the barrier function over appropriate level sets of the objective function, the point $(x_j,\frac{10^{1-j}}{5})$ of $\mathscr{P}$ satisfies $x_j=\arg\max_x \sum_{i=2}^m \ln(-1)^i(z_i^j-x)$. Therefore, to show that $x_j \geqslant 0$ for odd j and that $x_j \leqslant \dfrac{-10^{-4}}{m}$ for even j, it is enough to prove that $g^j(0)>0$ for odd j and $g^j(\dfrac{-10^{-4}}{m})<0$ for even j where $g^j(x)=\sum_{i=2}^m \frac{\mathrm{d}}{\mathrm{d}x}\ln(-1)^i(z_i^j-x)$. For simplicity we assume that m is even. A similar argument applies for odd values of m. Since $(-1)^{k+1}\left(\frac{1}{x-z_k^j}+\frac{1}{x-z_{k+1}^j}\right)>0$ for $k \geqslant j+4$ and $\dfrac{-10^{-4}}{m} \leqslant x \leqslant 0$, we have

$$\sum_{i=j+4}^{i=m} \frac{1}{x-z_i^j} \begin{cases} \geqslant 0, & j \text{ odd}, \ x=0, \\ \leqslant 0, & j \text{ even}, \ x=\dfrac{-10^{-4}}{m}. \end{cases} \tag{1}$$

This yields $g^1(0) \geqslant \frac{-1}{\frac{1}{2}+\frac{1}{50}}+\frac{1}{\frac{1}{3}+\frac{1}{15}}-\frac{1}{\frac{100}{55}+\frac{5}{11}-10^{-4}}=\frac{772}{5639}>0$. For $j \geqslant 2$, rewrite

$$g^j(x)=\left(\frac{1}{x-z_2^j}+\frac{1}{x-z_3^j}\right)+\sum_{i=4}^{i<j+2}\frac{1}{x-z_i^j}+\sum_{i=j+2}^{i<j+4}\frac{1}{x-z_i^j}+\sum_{i=j+4}^{i=m}\frac{1}{x-z_i^j}.$$

Observe

$$\frac{1}{x-z_2^j}+\frac{1}{x-z_3^j}=\begin{cases}\dfrac{-1}{\dfrac{1}{2}+\dfrac{10^{-j}}{5}}+\dfrac{1}{\dfrac{1}{3}+\dfrac{10^{-j+1}}{15}} & \text{for } x=0,\\[2ex] \dfrac{-1}{\dfrac{1}{2}+\dfrac{10^{-j}}{5}+\dfrac{10^{-4}}{m}}+\dfrac{1}{\dfrac{1}{3}+\dfrac{10^{-j+1}}{15}-\dfrac{10^{-4}}{m}} & \text{for } x=\dfrac{-10^{-4}}{m},\end{cases} \tag{2}$$

and

$$\sum_{i=j+2}^{i<j+4}\frac{1}{x-z_i^j}\begin{cases}\geqslant \dfrac{1}{\dfrac{10}{55}+\dfrac{5}{11}}-\dfrac{1}{\dfrac{100}{55}+\dfrac{5}{11}-\dfrac{10^{-4}}{m}}, & j\geqslant 3 \text{ odd},\ x=0,\\[2ex] \leqslant \dfrac{-1}{\dfrac{10}{55}+\dfrac{5}{11}+\dfrac{10^{-4}}{m}}+\dfrac{1}{\dfrac{100}{55}+\dfrac{5}{11}-2\dfrac{10^{-4}}{m}}, & j\leqslant m-4 \text{ even},\ x=\dfrac{-10^{-4}}{m},\\[2ex] \leqslant \dfrac{-1}{\dfrac{10}{55}+\dfrac{5}{11}+\dfrac{10^{-4}}{m}}, & j=m-2,\ x=\dfrac{-10^{-4}}{m}.\end{cases} \tag{3}$$

For odd $j\geqslant 3$ and $x=0$, we have

$$\begin{aligned}
\sum_{i=4}^{i<j+2}\frac{1}{x-z_i^j} &\geqslant -\frac{1}{\dfrac{10^{3-j}}{55}+\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)}+\frac{1}{\dfrac{10^{4-j}}{55}+\dfrac{5}{11}}+\cdots-\frac{1}{\dfrac{1}{55}+\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)}\\
&=\frac{-1}{\dfrac{5}{11}-\dfrac{10^{-4}}{m}}\left(\frac{1}{1+\dfrac{10^{3-j}}{55\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)}}+\frac{1}{1+\dfrac{10^{5-j}}{55\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)}}+\cdots+\frac{1}{1+\dfrac{1}{55\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)}}\right)\\
&\quad+\frac{11}{5}\left(\frac{1}{1+\dfrac{11\cdot 10^{4-j}}{5\cdot 55}}+\frac{1}{1+\dfrac{11\cdot 10^{6-j}}{5\cdot 55}}+\cdots+\frac{1}{1+\dfrac{11\cdot 10^{-1}}{5\cdot 55}}\right)\\
&\geqslant \frac{-1}{\dfrac{5}{11}-\dfrac{10^{-4}}{m}}\left(1-\frac{10^{3-j}}{55\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)}+\left(\frac{10^{3-j}}{55\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)}\right)^2\right.\\
&\quad\left.+1-\frac{10^{5-j}}{55\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)}+\left(\frac{10^{5-j}}{55\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)}\right)^2+\cdots+1-\frac{1}{55\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)}+\left(\frac{1}{55\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)}\right)^2\right)\\
&\quad+\frac{11}{5}\left(1-\frac{11\cdot 10^{4-j}}{5\cdot 55}+\cdots+1-\frac{11\cdot 10^{-1}}{5\cdot 55}\right)\\
&=\frac{-1}{\dfrac{5}{11}-\dfrac{10^{-4}}{m}}\left(\left\lfloor\frac{j}{2}\right\rfloor-\frac{1}{55\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)}\cdot\frac{1-.01^{\lfloor j/2\rfloor}}{1-.01}+\left(\frac{1}{55\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)}\right)^2\frac{1-.0001^{\lfloor j/2\rfloor}}{1-.0001}\right)\\
&\quad+\frac{11}{5}\left(\left\lfloor\frac{j}{2}\right\rfloor-1-\frac{11}{550}\cdot\frac{1-.01^{\lfloor j/2\rfloor-1}}{1-.01}\right)\geqslant\frac{-\left\lfloor\dfrac{j}{2}\right\rfloor\dfrac{10^{-4}}{m}}{\left(\dfrac{5}{11}\right)^2-\dfrac{5}{11}\dfrac{10^{-4}}{m}}+\frac{1}{55\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)^2}-\frac{1}{55^2\left(\dfrac{5}{11}-\dfrac{10^{-4}}{m}\right)^3.9999}-\frac{11}{5}-\left(\frac{11}{5}\right)^2\frac{1}{550\cdot .9999},
\end{aligned}$$

where the second inequality is based on $1-v \leqslant \frac{1}{1+v} \leqslant 1-v+v^2$, $v \geqslant 0$ and the last equality is obtained by summing up the terms in three resulting geometric series. This, combined with observations (1)–(3), gives, for odd $j \geqslant 3$,

$$g^j(0) \geqslant \left(-2+\frac{1}{\frac{1}{3}+\frac{1}{1500}}\right)+\left(\frac{1}{\frac{10}{55}+\frac{5}{11}}-\frac{1}{\frac{100}{55}+\frac{5}{11}-.0001}\right)+\left(\frac{-.00005}{\left(\frac{5}{11}\right)^2-\frac{5}{11}\cdot .0001}+\frac{1}{55\left(\frac{5}{11}\right)^2}\right.$$
$$\left.-\frac{1}{55^2\left(\frac{5}{11}-.0001\right)^3 .9999}-\frac{11}{5}-\left(\frac{11}{5}\right)^2\frac{1}{550\cdot .9999}\right)=\frac{49}{63838}>0.$$

Similarly for even $j \geqslant 2$ and $x=\dfrac{-10^{-4}}{m}$ we have

$$\sum_{i=4}^{i<j+2}\frac{1}{x-z_i^j} \leqslant \frac{-1}{\frac{5}{11}+\frac{10^{-4}}{m}}\left(\left\lfloor\frac{j}{2}\right\rfloor-1-\frac{1}{550\left(\frac{5}{11}+\frac{10^{-4}}{m}\right)}\frac{1-.01^{\lfloor j/2\rfloor-1}}{1-.01}\right)$$
$$+\frac{1}{\frac{5}{11}-2\frac{10^{-4}}{m}}\left(\left\lfloor\frac{j}{2}\right\rfloor-1-\frac{1}{55\left(\frac{5}{11}-2\frac{10^{-4}}{m}\right)}\frac{1-.01^{\lfloor j/2\rfloor-1}}{1-.01}+\left(\frac{1}{55\left(\frac{5}{11}-2\frac{10^{-4}}{m}\right)}\right)^2\frac{1-.0001^{\lfloor j/2\rfloor-1}}{1-.0001}\right)$$
$$\leqslant\left(\left\lfloor\frac{j}{2}\right\rfloor-1\right)\frac{2\frac{10^{-4}}{m}+\frac{10^{-4}}{m}}{\left(\frac{5}{11}\right)^2-\left(\frac{-10^{-4}}{m}\right)^2-\frac{10^{-4}}{m}\left(\frac{5}{11}+\frac{10^{-4}}{m}\right)}$$
$$+\frac{1}{550\left(\frac{5}{11}+\frac{10^{-4}}{m}\right)^2}\cdot\frac{1}{1-.01}$$
$$-\frac{1}{55\left(\frac{5}{11}-2\frac{10^{-4}}{m}\right)^2}+\frac{1}{55^2\left(\frac{5}{11}-2\frac{10^{-4}}{m}\right)^3}\cdot\frac{1}{1-.0001}.$$

Thus, for even $j \geqslant 2$.

$$g^j\left(\frac{-10^{-4}}{m}\right) \leqslant \left(\frac{-1}{\frac{1}{2}+\frac{1}{500}+.0001}+\frac{1}{\frac{1}{3}-.0001}\right)+\left(\frac{-1}{\frac{10}{55}+\frac{5}{11}+.0001}+\frac{1}{\frac{100}{55}+\frac{5}{11}-.0002}\right)$$
$$+\left(\frac{.00015}{\left(\frac{5}{11}\right)^2-.0001^2-.0001\left(\frac{5}{11}+.0001\right)}-\frac{89}{99}\frac{1}{55\left(\frac{5}{11}\right)^2}+\frac{1}{55^2 \cdot .999}\cdot\frac{1}{\left(\frac{5}{11}-.0002\right)^3}\right)=\frac{-784}{3985}<0.$$

Therefore, the central path $\mathscr{P}$ goes through a sequence of $m-2$ points (x_j, y_j) with $y_j=\frac{10^{1-j}}{5}$ and $x_j \geqslant 0$ for odd j, $x_j \leqslant \frac{-10^{-4}}{m}$ for even j. One can easily check that $(x_j, y_j) \in \mathscr{P}$ for $j=1,\ldots,m-2$ by verifying that the analytic center χ is above the line $y=\frac{1}{5}$.

We have

$$\chi=(\chi_1,\chi_2)=\arg\max_{(x,y)\in P^*_{m,2}}\left(\ln(1-y)+\ln\left(-x+\frac{y}{10}+\frac{1}{2}\right)+\ln\left(x+\frac{y}{3}+\frac{1}{3}\right)\right.$$
$$\left.+\sum_{i=4}^{m}\ln\left((-1)^{i+1}x+\frac{10^{i-2}y}{11}+\frac{5}{11}-\frac{10^{-4}}{m}\frac{i}{m}\right)\right).$$

Therefore, to show that $\chi_2>\frac{1}{5}$, it is enough to prove that the derivative with respect to y of the log-barrier function is negative for $(x,y)\in P^*_{m,2}$ and $y\leqslant\frac{1}{5}$, that is

$$\frac{-1}{1-y}+\frac{1}{-10x+y+5}+\frac{1}{3x+y+1}+\sum_{i=4}^{m}\frac{10^{i-2}}{\left((-1)^{i+1}11x+10^{i-2}y+5-11\cdot\frac{10^{-4}}{m}\frac{i}{m}\right)}>0,$$

which is implied by $\dfrac{-1}{1-y}+\dfrac{100}{-11x+100y+5-11\cdot\dfrac{.0001}{m}\dfrac{4}{m}}>\dfrac{-5}{4}+\dfrac{100}{\dfrac{100}{5}+5+\dfrac{66}{15}}=\dfrac{1265}{588}>0.$

To show that $\liminf_{m\to\infty}\frac{\lambda^{(0,1)^{\mathrm{T}}}(P^*_{m,2})}{m}\geqslant\pi$, consider three consecutive points from this sequence, say (x_{j-1},y_{j-1}), (x_j,y_j), (x_{j+1},y_{j+1}), and observe that for any $\varepsilon>0$ we can choose m so that for all $\varepsilon m\leqslant j<m-2$ we have $\frac{|y_j-y_{j-1}|}{|x_j-x_{j-1}|}<\varepsilon$, $\frac{|y_{j+1}-y_j|}{|x_{j+1}-x_j|}<\varepsilon$. Let m be such a value and $j\geqslant\varepsilon m$. Without loss off generality j might be assumed odd and let $\tau_{j-1},\tau_j,\tau_{j+1}\in\mathbb{R}$ be such that $\mathscr{P}_{\text{arc}}(\tau_k)=(x_k,y_k)$, $k=j-1,\ j,\ j+1$. We show by contradiction that there is a t_1 such that the first coordinate $\left(\dot{\mathscr{P}}_{\text{arc}}(t_1)\right)_1>\sqrt{1-\varepsilon^2}$. Suppose that for all $t\in[\tau_{j-1},\tau_j]$ we have $\left(\dot{\mathscr{P}}_{\text{arc}}(t)\right)_1\leqslant\sqrt{1-\varepsilon^2}$, then $\left(\dot{\mathscr{P}}_{\text{arc}}(t)\right)_2\leqslant-\varepsilon$ since $\|\dot{\mathscr{P}}_{\text{arc}}(t)\|=1$ and $(\mathscr{P}_{\text{arc}}(t))_2$ is monotone-decreasing with respect to t. By the Mean-Value Theorem it follows that $\tau_j-\tau_{j-1}>x_j-x_{j-1}$, and thus, by the same theorem, we must have $\left(\mathscr{P}_{\text{arc}}(\tau_j)\right)_2-\left(\mathscr{P}_{\text{arc}}(\tau_{j-1})\right)_2=y_j-y_{j-1}<-\varepsilon(x_j-x_{j-1})$, a contradiction. Similarly, there is a t_2 such that $\left(\dot{\mathscr{P}}_{\text{arc}}(t_2)\right)_1<-\sqrt{1-\varepsilon^2}$. Since the total curvature K_j of the segment of $\mathscr{P}_{\text{arc}}$ connecting the points (x_{j-1},y_{j-1}), (x_j,y_j), (x_{j+1},y_{j+1}) corresponds to the length of the curve $\dot{\mathscr{P}}_{\text{arc}}$ connecting the corresponding derivative points on a unit 2-sphere, K_j may be bounded below by the length of the geodesic between the points $\dot{\mathscr{P}}_{\text{arc}}(t_1)$ and $\dot{\mathscr{P}}_{\text{arc}}(t_2)$, that is, bounded below by a constant arbitrarily close to π. Now simply add all K_j for all $\varepsilon m\leqslant j<m-2$. $\square$

Holt and Klee [10] showed that, for $m>n\geqslant 13$, the conjecture of Hirsch is tight. Fritzsche and Holt [8] extended the result to $m>n\geqslant 8$. Since the polytope $P^*_{m,2}$ can be generalized to higher dimensions by adding the box constraints $0\leqslant x_i\leqslant 1$ for $i\geqslant 3$, we have

Corollary 3.1 (*Continuous analogue of the result of Holt and Klee*). $\liminf_{m\to\infty}\frac{\Lambda(m,n)}{m}\geqslant\pi$, *that is*, $\Lambda(m,n)$ *is bounded below by a constant times* m.

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