



A primal-simplex based Tardos' algorithm



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ABSTRACT

In the mid-eighties Tardos proposed a strongly polynomial algorithm for solving linear programming problems for which the size of the coefficient matrix is polynomially bounded in the dimension of the input. Combining Orlin's primal-based modification and Mizuno's use of the simplex method, we introduce a modification of Tardos' algorithm considering only the primal problem and using the simplex method to solve the auxiliary problems. The proposed algorithm is strongly polynomial if the coefficient matrix is totally unimodular and the auxiliary problems are non-degenerate.

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1. Introduction

In the mid-eighties Tardos [7,8] proposed a strongly polynomial algorithm for solving linear programming problems $\min\{\mathbf{c}^\top \mathbf{x} \mid \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ for which the size of A is polynomially bounded in the dimension of the input. Such instances include minimum cost flow, bipartite matching, multicommodity flow, and vertex packing in chordal graphs. The basic strategy of Tardos' algorithm is to identify the coordinates equal to zero at optimality. The algorithm involves solving several auxiliary dual problems by the ellipsoid or interior-point methods. By successively identifying such vanishing coordinates, the problem is made smaller and an optimal solution is obtained inductively. Considering only the primal problem, Orlin [5] proposed a modification of Tardos' algorithm which specifically identifies the coordinates strictly positive at optimality. He observed that the right-hand side coefficients of the auxiliary problems might be impractically large.

Mizuno [4] modified Tardos' algorithm by using a dual simplex method to solve the auxiliary problems. He observed that this approach is strongly polynomial if A is totally unimodular and the auxiliary problems are non-degenerate; that is, the basic variables are strictly positive for every basic feasible solution. The strong polynomiality is a consequence of the results of Kitahara and

Mizuno [2,3] which extend in part Ye's result [9] for Markov decision problems. The results of [2,3] bound the number of distinct basic feasible solutions generated by the simplex method, and thus bounds the number of pivots for non-degenerate instances.

Combining Orlin's and Mizuno's approaches, we introduce a modification of the algorithm proposed by Mizuno considering only the primal problem. The proposed algorithm is strongly polynomial if A is totally unimodular and the auxiliary problems are non-degenerate. As it involves only the primal and does not suffer from impractically large right-hand side coefficients, the proposed algorithm improves the implementability of the approach. While the proposed algorithm and the complexity analysis are focusing on the case where A is totally unimodular, the algorithm could be enhanced to handle any matrix. The enhanced algorithm would be strongly polynomial if A is integer and the absolute value of any subdeterminant of A is polynomially bounded in the dimension of the input.

2. A primal-simplex based Tardos' algorithm

2.1. Formulation and main result

Consider the following formulation:

$$\begin{aligned} & \text{minimize} && \mathbf{c}^\top \mathbf{x} \\ & \text{subject to} && \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \end{aligned} \quad (1)$$

where $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$ are given. We assume that A has full row rank m . The optimal solution of (1), if any, is assumed

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without loss of generality to be unique. Otherwise \mathbf{c} could be perturbed by $(\epsilon, \epsilon^2, \dots, \epsilon^n)$ for a sufficiently small $\epsilon > 0$. Since the results of [2,3] do not depend on the specific value of \mathbf{c} , such perturbation does not impact the analysis. Alternatively, in practice, the simplex method can be performed using a lexicographical order if a tie occurs when choosing an entering variable by Dantzig's rule.

Let $K^* \subseteq N = \{1, 2, \dots, n\}$ be the set of indices i such that $x_i^* > 0$ for the optimal solution \mathbf{x}^* of (1). The proposed algorithm inductively builds a subset $\bar{K} \subseteq K^*$ through solving an auxiliary problem. If $\bar{K} = K^*$, we obtained the optimal solution. Otherwise, we obtain a smaller yet equivalent problem by deleting the variables corresponding to \bar{K} . Thus, the optimal solution is obtained inductively. For clarity of the exposition of the algorithm and of the proof of Theorem 1, we assume in the remainder of the paper that A is totally unimodular; that is, all its subdeterminants are equal to either $-1, 0$ or 1 .

Theorem 1. *The primal-simplex based Tardos' algorithm is strongly polynomial if A is totally unimodular and all the auxiliary problems are non-degenerate; that is, all the basic variables are strictly positive for every basic feasible solution.*

Proof. See Section 3. \square

2.2. A primal-simplex based Tardos' algorithm

Step 0 (initialization): Let $\bar{K} := \emptyset$ and its complement $K := N$.

Step 1 (reduction): If $\bar{K} \neq \emptyset$, remove the variables corresponding to \bar{K} in the following way.

Let $G \in \mathbb{R}^{m \times m}$ be a nonsingular submatrix of A such that its first $|\bar{K}|$ columns form $A_{\bar{K}}$ and $H = G^{-1}$. Let H_1 consist of the first $|\bar{K}|$ rows of H , H_2 denote the remainder, and consider the following reduced problem:

$$\begin{aligned} & \text{minimize} && \mathbf{c}'^\top \mathbf{x}' \\ & \text{subject to} && A' \mathbf{x}' = \mathbf{b}', \mathbf{x}' \geq \mathbf{0} \end{aligned} \quad (2)$$

where $A' = H_2 A_K$, $\mathbf{b}' = H_2 \mathbf{b}$, $\mathbf{c}' = \mathbf{c}_K - (H_1 A_K)^\top \mathbf{c}_{\bar{K}}$, and $\mathbf{x}' = \mathbf{x}_K$.

If $\bar{K} = \emptyset$, set $A' := A$, $\mathbf{b}' := \mathbf{b}$, and $\mathbf{c}' := \mathbf{c}$.

GO TO STEP 2.

Step 2 (scaling and rounding): Let $m' = m - |\bar{K}|$ and $n' = n - |\bar{K}|$. For a basis $L \subseteq K$ of A' and $\bar{L} = K \setminus L$, rewrite (2) as:

$$\begin{aligned} & \text{minimize} && \mathbf{c}'^\top \mathbf{x}' \\ & \text{subject to} && \mathbf{x}'_L + (A'_L)^{-1} A'_{\bar{L}} \mathbf{x}'_{\bar{L}} = (A'_L)^{-1} \mathbf{b}', \mathbf{x}' \geq \mathbf{0}. \end{aligned} \quad (3)$$

If $(A'_L)^{-1} \mathbf{b}' = \mathbf{0}$, stop. Otherwise, consider the following scaled problem:

$$\begin{aligned} & \text{minimize} && \mathbf{c}'^\top \mathbf{x}' \\ & \text{subject to} && \mathbf{x}'_L + (A'_L)^{-1} A'_{\bar{L}} \mathbf{x}'_{\bar{L}} = (A'_L)^{-1} \mathbf{b}'/k, \mathbf{x}' \geq \mathbf{0}, \end{aligned} \quad (4)$$

where $k = \|A'^\top (A' A'^\top)^{-1} \mathbf{b}'\|_2 / (m' + (n')^2)$. Then, consider the following rounded problem:

$$\begin{aligned} & \text{minimize} && \mathbf{c}'^\top \mathbf{x}' \\ & \text{subject to} && \mathbf{x}'_L + (A'_L)^{-1} A'_{\bar{L}} \mathbf{x}'_{\bar{L}} = \lceil (A'_L)^{-1} \mathbf{b}'/k \rceil, \mathbf{x}' \geq \mathbf{0}. \end{aligned} \quad (5)$$

If (5) is infeasible, stop. Otherwise, solve (5) using the simplex method with Dantzig's rule. If (5) is unbounded, stop. Otherwise, let \mathbf{x}'' denote the optimal solution and L'' the optimal basis. If $\bar{K} \cup L''$ is an optimal basis of the original problem (1), stop. Otherwise, GO TO STEP 3.

Step 3 (iteration): Set $\bar{K} := \bar{K} \cup J$ and $K := K \setminus J$ where $J = \{i \mid x''_i \geq n', i \in K\}$.

If $|K| = n - m$, stop. Otherwise, GO TO STEP 1.

2.3. Annotations of the proposed algorithm

Observation 1 points out that if $\bar{K} \subseteq K^*$ and the optimal solution of (1) is unique, then we can remove the non-negativity constraints for x_i for each $i \in \bar{K}$, and thus solve the reduced problem (2) instead of (1).

Observation 1. *Let $\bar{K} \subseteq K^*$. If $|\bar{K}| = m$; that is $|K| = n - m$, \bar{K} forms an optimal basis. Otherwise, let (1)' be the relaxation obtained from (1) by deleting the constraints $x_i \geq 0$ for all $i \in \bar{K}$. If (1) is feasible and bounded, and hence has a unique optimal solution \mathbf{x}^* , then \mathbf{x}^* is also the unique optimal solution of (1)'.*

Proof. Assume by contradiction, that there exists a feasible solution \mathbf{x}' of (1)' such that $\mathbf{x}' \neq \mathbf{x}^*$ and $\mathbf{c}'^\top \mathbf{x}' \leq \mathbf{c}'^\top \mathbf{x}^*$. By the definitions of K^* and of (1)', $x'_i > 0$ for any $i \in \bar{K}$ while x'_i can be negative for $i \in \bar{K}$. Let $\mathbf{y} = (1 - \epsilon)\mathbf{x}^* + \epsilon\mathbf{x}'$ with $\epsilon > 0$. For a sufficiently small ϵ , \mathbf{y} is non-negative and $A\mathbf{y} = (1 - \epsilon)A\mathbf{x}^* + \epsilon A\mathbf{x}' = \mathbf{b}$, while $\mathbf{c}'^\top \mathbf{x}^* \geq \mathbf{c}'^\top \mathbf{y} \geq \mathbf{c}'^\top \mathbf{x}'$ which contradicts the assumption that \mathbf{x}^* is the unique optimum of (1). \square

Note that the reduced problem (2) is a concise expression of the problem (1)' introduced in Observation 1. Specifically, (2) is obtained by expressing $\mathbf{x}_{\bar{K}}$ as $H_1 \mathbf{b} - H_1 A_K \mathbf{x}_K$ and substituting $H_1 \mathbf{b} - H_1 A_K \mathbf{x}_K$ for $\mathbf{x}_{\bar{K}}$ in the objective function. Therefore, the optimal solution for (2) yields the optimal solution for (1) via $\mathbf{x}_{\bar{K}} = H_1 \mathbf{b} - H_1 A_K \mathbf{x}_K$. The constant term in the objective function is removed for simplicity. Note that the matrices A' and $[I, (A'_L)^{-1} A'_{\bar{L}}]$ involved in (2), (3), (4), and (5) are totally unimodular if A is totally unimodular, see Theorem 19.5 in Schrijver [6].

In Step 2, the scaling factor k is strictly positive if $(A'_L)^{-1} \mathbf{b}' \neq \mathbf{0}$ and, see Lemma 2, $\lceil (A'_L)^{-1} \mathbf{b}'/k \rceil_\infty$ is polynomially bounded from above in m' and n' , which is a key fact for showing the strong polynomiality. Although the proposed algorithm builds the simplex tableau associated to (3) and the reduced problem (2) from scratch at each iteration, it is essentially for clarity of the exposition and can be ignored. In particular, one can observe that $L'' \setminus J$ forms the basis L for (3) at the next iteration, thus enabling a warm start. By performing Phase one of the two-phase simplex method for the rounded problem (5), we can check the feasibility of (5) and compute an initial basic feasible solution, unless it is infeasible.

In Step 3, $J \neq \emptyset$ by Lemma 1; that is, the size of K is strictly decreasing. Thus, the proposed algorithm terminates after at most m iterations. If (1) has an optimal solution, $\bar{K} \subseteq K^*$ by Corollary 1.

The stopping conditions of the proposed algorithm are:

- if $(A'_L)^{-1} \mathbf{b}' = \mathbf{0}$, the simplex tableau associated to (3) yields either the optimality of $\mathbf{x}' = \mathbf{0}$ or the unboundedness of the reduced problem (2).
- since the rounded problem (5) is a relaxation of the scaled problem (4),
 - the scaled problem (4) and the original problem (1) are both infeasible if (5) is infeasible
 - the scaled problem (4) is unbounded or infeasible if (5) is unbounded. In both cases, the original problem (1) has no optimal solution.
- if $|K| = n - m$ in Step 3, the problem (1) is infeasible as otherwise the algorithm finds an optimal basis in Step 2.

3. Proof of Theorem 1

Lemma 1 states that the set $J = \{i \mid x''_i \geq n', i \in K\}$ used in Step 3 is never empty and thus, the proposed algorithm solves the rounded problem (5) at most m times.

Lemma 1. *$J \neq \emptyset$ as any solution \mathbf{x}'' of the rounded problem (5) satisfies $\|\mathbf{x}''\|_\infty \geq n'$.*

Proof. We first remark that A' has full row rank. Indeed, if $\bar{K} = \emptyset$, then $A' = A$ and has full row rank. Otherwise, recall that G is a non-singular submatrix of A such that its first $|\bar{K}|$ columns form $A_{\bar{K}}$, that $H = G^{-1}$, and that the first $|\bar{K}|$, respectively, the last $m - |\bar{K}|$ rows of H , are denoted by H_1 , respectively H_2 . As H is invertible and A has full row rank, HA also has full row rank. Thus, H_2A , which consists of the last $m - |\bar{K}|$ rows of HA , has full row rank. On the other hand, we observe that $H_2A_{\bar{K}}$ is a zero matrix from the definition of G and H . Therefore, $A' = H_2A_{\bar{K}}$, a submatrix obtained from H_2A by dropping the zero matrix $H_2A_{\bar{K}}$, has full row rank. Thus A' has full row rank in either case; that is, $A'A'^T$ is positive definite.

We then remark that, for any \mathbf{g} , $A'^T(A'A'^T)^{-1}\mathbf{g}$ is the minimal l_2 -norm point satisfying $A'\mathbf{x}' = \mathbf{g}$. Indeed, consider the following minimization problem where B is a matrix such that BB^T is positive definite: $\min\{\mathbf{x}^T\mathbf{x}/2 \mid B\mathbf{x} = \mathbf{g}\}$. The Lagrangian multiplier method – see [1] for example – yields that the minimum point satisfies $\mathbf{x} + B^T\lambda = \mathbf{0}$ and $\mathbf{g} - B\mathbf{x} = \mathbf{0}$ for some λ .

Let \mathbf{x}'' be a solution of the rounded problem (5). We have $A'\mathbf{x}'' = A'_l[(A'_l)^{-1}\mathbf{b}'/k]$, $A'A'^T$ is positive definite, and, for any \mathbf{g} , $A'^T(A'A'^T)^{-1}\mathbf{g}$ is the minimal l_2 -norm point satisfying $A'\mathbf{x}' = \mathbf{g}$. Thus,

$$\begin{aligned} \|\mathbf{x}''\|_2 &\geq \|A'^T(A'A'^T)^{-1}A'_l[(A'_l)^{-1}\mathbf{b}'/k]\|_2 \\ &\geq \|A'^T(A'A'^T)^{-1}\mathbf{b}'/k\|_2 - \|A'^T(A'A'^T)^{-1}A'_l\mathbf{d}\|_2 \\ &= (m' + (n')^2) - \left\| A'^T(A'A'^T)^{-1}A'_l \begin{pmatrix} \mathbf{d} \\ \mathbf{0}_{\bar{l}} \end{pmatrix} \right\|_2 \\ &\geq (n')^2 + m' - \|\mathbf{d}\|_2 \end{aligned}$$

where $k = \|A'^T(A'A'^T)^{-1}\mathbf{b}'\|_2/(m' + (n')^2)$ and $\mathbf{d} = (A'_l)^{-1}\mathbf{b}'/k - [(A'_l)^{-1}\mathbf{b}'/k]$. Since $\|\mathbf{d}\|_\infty < 1$, we obtain that

$$\|\mathbf{x}''\|_\infty \geq \|\mathbf{x}''\|_2/n' > ((n')^2 + m' - m')/n' = n'. \quad \square$$

Corollary 1 ensures the validity of the reduction performed in Step 4, that is, $\bar{K} \subseteq K^*$, whose proof is a direct consequence of **Theorem 2**.

Theorem 2 (Theorem 10.5 in Schrijver [6]). *Let A be an $m \times n$ -matrix, and let Δ^* be such that for each nonsingular submatrix B of A all entries of B^{-1} are at most Δ^* in absolute value. Let \mathbf{c} be a column n -vector, and let \mathbf{b}' and \mathbf{b}^* be column m -vectors such that $P'' : \max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}'\}$ and $P^* : \max\{\mathbf{c}^T\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}^*\}$ are finite. Then, for each optimal solution \mathbf{x}'' of P'' , there exists an optimal solution \mathbf{x}^* of P^* with $\|\mathbf{x}'' - \mathbf{x}^*\|_\infty \leq n\Delta^*\|\mathbf{b}' - \mathbf{b}^*\|_\infty$.*

Corollary 1. *Let \mathbf{x}'' be an optimal solution of the rounded problem (5), and $J = \{i \mid x''_i \geq n', i \in K\}$ as defined in Step 3 of the proposed algorithm. If the scaled problem (4) is feasible, the i th coordinate of the optimal solution of the scaled problem (4) is strictly positive for $i \in J$. Furthermore, the same holds for the reduced problem (2) and the original problem (1) as the scaling factor k is strictly positive.*

Proof. Define $\tilde{A} \in \mathbb{R}^{(2m'+n') \times n'}$, $\tilde{\mathbf{b}}''$, and $\tilde{\mathbf{b}}^* \in \mathbb{R}^{2m'+n'}$ as:

$$\tilde{A} = \begin{bmatrix} E \\ -E \\ -I \end{bmatrix}, \quad \tilde{\mathbf{b}}'' = \begin{bmatrix} (A'_l)^{-1}\mathbf{b}'/k \\ -(A'_l)^{-1}\mathbf{b}'/k \\ \mathbf{0} \end{bmatrix}, \quad \text{and} \quad \tilde{\mathbf{b}}^* = \lceil \tilde{\mathbf{b}}^* \rceil$$

where $E = [I, (A'_l)^{-1}A'_l]$. With this notation, the rounded problem (5), respectively the scaled problem (4), can be restated as $P'' : \max\{-\mathbf{c}'^T\mathbf{x} \mid \tilde{A}\mathbf{x} \leq \tilde{\mathbf{b}}''\}$, respectively $P^* : \max\{-\mathbf{c}'^T\mathbf{x} \mid \tilde{A}\mathbf{x} \leq \tilde{\mathbf{b}}^*\}$. Since E is totally unimodular, \tilde{A} is totally unimodular, and thus $\Delta^* = 1$ in **Theorem 2**. In addition, note that $\|\tilde{\mathbf{b}}'' - \tilde{\mathbf{b}}^*\|_\infty < 1$. Recall that the scaled problem (4) and P^* share the same unique optimal

solution \mathbf{x}^* as the optimal solution of the original problem (1) is assumed to be unique. Therefore, since \mathbf{x}'' is an optimal solution of P'' , we observe that $\|\mathbf{x}'' - \mathbf{x}^*\|_\infty < n'$ by **Theorem 2** and thus, $x''_i > 0$ for $i \in J$. \square

Finally, we show the strong polynomiality of the proposed algorithm using the results of Kitahara and Mizuno [2,3] showing that the number of different basic feasible solutions generated by the primal simplex method with the most negative pivoting rule – Dantzig’s rule – or the best improvement pivoting rule is bounded by:

$$n \left\lceil m \frac{\gamma}{\delta} \log \left(m \frac{\gamma}{\delta} \right) \right\rceil$$

where m is the number of constraints, n is the number of variables, and δ and γ are, respectively, the minimum and the maximum values of all the positive elements of the primal basic feasible solutions. Thus, we need to estimate the values δ and γ for the introduced auxiliary problems.

Since the coefficient matrices used in the proposed algorithm are totally unimodular and the right hand side vector of the rounded problem (5) is integer, we have $\delta \geq 1$. For γ , we use **Lemma 2**.

Lemma 2. *For the auxiliary problem (5), we have $\gamma \leq \gamma^* = m(mn + n^2) + 1$.*

Proof. Note that the right-hand side vector for (5) is $[(A'_l)^{-1}\mathbf{b}'/k]$. By the total unimodularity of A'_l , we observe that

$$\|[(A'_l)^{-1}\mathbf{b}'/k]\|_\infty \leq \|(A'_l)^{-1}\mathbf{b}'/k\|_\infty + 1 \leq m'\|\mathbf{b}'\|_\infty/k + 1.$$

The numerator $\|A'^T(A'A'^T)^{-1}\mathbf{b}'\|_2$ of k is bounded from below by $\|\mathbf{b}'\|_\infty/n'$ implying $\|[(A'_l)^{-1}\mathbf{b}'/k]\|_\infty \leq m'n'(m' + (n')^2) + 1$. Thus, by Cramer’s rule and the total unimodularity of the coefficient matrix $[I, (A'_l)^{-1}A'_l]$ of (5), the l_∞ -norm of a basic solution of (5) is bounded from above by $m'(m'n'(m' + (n')^2) + 1)$. \square

The two-phase simplex algorithm is called at most m times. Thus, the number of auxiliary problems solved by the proposed algorithm is bounded from above by $2m$ as each call corresponds to 2 auxiliary problems : one for each phase. Thus, if all the auxiliary problems are non-degenerate, the total number of basic solutions generated by the algorithm is bounded from above by $2m[n\lceil m\gamma^* \log(m\gamma^*) \rceil]$; that is by

$$2mn \lceil (m^4n + m^3n^3 + m^2) \log(m^4n + m^3n^3 + m^2) \rceil$$

which completes the proof of **Theorem 1**. Alternatively, since $m \leq n$, this bound can be restated as $O(m^4n^4 \log n)$. While assuming the non-degeneracy of the auxiliary problems is needed to use Kitahara–Mizuno’s bound, the number of degenerate updates of bases at a single basic solution is typically not too large in practice.

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