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Geometric and Algebraic Combinatorics

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Abstract. The 2015 Oberwolfach meeting "Geometric and Algebraic Combinatorics" was organized by Gil Kalai (Jerusalem), Isabella Novik (Seattle), Francisco Santos (Santander), and Volkmar Welker (Marburg). It covered a wide variety of aspects of Discrete Geometry, Algebraic Combinatorics with geometric flavor, and Topological Combinatorics. Some of the highlights of the conference included (1) counterexamples to the topological Tverberg conjecture, and (2) the latest results around the Heron-Rota-Welsh conjecture.

Mathematics Subject Classification (2010): 05Exx, 52Bxx, 13Dxx, 13Fxx, 14Txx, 52Cxx, 55xxx, 57Qxx, 90Cxx.

Introduction by the Organisers

The 2015 Oberwolfach meeting "Geometric and Algebraic Combinatorics" was organized by Gil Kalai (Hebrew University, Jerusalem), Isabella Novik (University of Washington, Seattle), Francisco Santos (University of Cantabria, Santander), and Volkmar Welker (Philipps-Universität Marburg, Marburg).

The conference consisted of six one-hour lectures on outstanding recent developments in the field plus about twenty five shorter talks, ranging from 30 to 45 minutes. On Thursday evening there was a problem session and on Wednesday evening two informal working sessions organized by the participants: one on the face numbers of spheres and manifolds and another one on the topological methods in combinatorics. There were of course many other small group discussions. All together it was a very productive and enjoyable week.

The conference treated a broad spectrum of topics from Topological Combinatorics (such as very recent counterexamples to the topological Tverberg conjecture — this was one of the extremely exciting highlights of the conference — more on it below), Tropical Geometry (such as enumeration of curves, valuated matroids, tropical laplacians, etc.), Geometric Combinatorics (triangulated manifolds, random simplicial complexes, etc.), Discrete Geometry (diameters of polytopes, lattice points in rational polytopes, complexity, etc.), and Combinatorial Algebra (Stanley's partitioning conjecture, Gröbner bases of algebras associated with matroids, etc.). It is impossible to summarize in a one-page report the richness and depth of the work and the presentations. Instead we will concentrate here on some of the highlights.

The very first lecture on Monday was given by Karim Adiprasito and was devoted to the c-arrangements. An old conjecture (from late sixties) of Heron, Rota, and Welsh posits that a certain sequence of numbers associated with a matroid, called the Whitney numbers of the first kind, is log-concave. A few years ago this conjecture was solved by Huh and Katz for the case of realizable matroids utilizing the heavy machinery of algebraic geometry. In his talk, Adiprasito announced a far reaching generalization of the Huh–Katz's result (joint with Raman Sanyal) asserting that this conjecture holds for general linear c-arrangements. Perhaps the most stunning part of this presentation was the sketch of the proof: it relied on the Alexandrov-Fenchel inequality and on probabilistic methods such as the measure concentration techniques!

Tuesday's morning session was devoted to linear programming/optimization. One of the highlights of this session was a surprising connection between tropical geometry and optimization discussed in Michael Joswig's talk. Michael and his coauthors exploited this connection to disprove a continuous analog of the Hirsch conjecture proposed by Deza, Terlaky and Zinchenko.

Spectacular recent developments in topological combinatorics were presented in Tuesday afternoon's session. Specifically, Uli Wagner reported on his (joint with Isaac Mabillard) recent major theory that extends fundamental theorems from classical obstruction theory for embeddability to an obstruction theory for r-fold intersection of disjoint faces in maps from simplicial complexes to Euclidean spaces. This presentation was followed by Florian Frick's talk who building on Mabillard–Wagner's theory and on a result by Özaydin, presented his 3-page preprint refuting the topological Tverberg conjecture! This was a truly fascinating sequence of talks!

On Thursday, Roy Meshulam talked about the recent results on high-dimensional expansion. His lecture was followed by Nati Linial's account of the recent progress on random simplicial complexes.

Several talks were devoted to the recent progress on the face numbers of simplicial complexes. Afshin Goodarzi discussed his joint work with Karim Adiprasito and Anders Björner in which they characterized face numbers of sequentially Cohen-Macaulay complexes, thus providing an impressive generalization of the classical Macaulay-Stanley theorem to the nonpure case. Kalle Karu presented his recent work on the *cd*-index; specifically, he proved that the Murai–Nevo conjecture on the *cd*-index of Gorenstein* posets holds for the case of simplicial spheres. Satoshi Murai announced his (joint with Martina Juhnke-Kubitzke) very recent proof of the balanced generalized lower bound conjecture for simplicial polytopes posed by Klee and Novik. Ed Swartz closed the conference with the talk titled "What's next?" in which he surveyed recent developments on the face numbers of triangulated manifolds along with several old and new open problems, among them a recent conjecture by Bagchi and Datta. This conjecture posits bounds on the face numbers of triangulated manifolds in terms of the μ-numbers – certain invariants related to Morse theory, topological tightness, and commutative algebra.

It bears repeating that quite a few breakthrough results were announced and presented for the first time during the conference. These include the counterexamples to the topological Tverberg conjecture, the proof of the generalized lower bound conjecture for balanced polytopes, etc. In fact, several math arXiv preprints that are less than two months old and were presented at the workshop (and three of them were actually submitted to the arXiv during the workshop!), see

http://arxiv.org/abs/1502.01183 (Goodarzi's talk)
http://arxiv.org/abs/1412.6705 (Hähnle's talk)
http://arxiv.org/abs/1502.02975 (Ziegler's talk)
http://arxiv.org/abs/1502.00947 (Frick's talk)
http://arxiv.org/abs/1412.6048 (Karu's talk)
http://arxiv.org/abs/1412.8621 (Živaljević's talk).

Last but not least, there was a lively and incredible problem session: a large number of the problems/questions raised were answered on spot.

The collection of abstracts below presents an overview of the official program of the conference. It does not cover all the additional smaller presentations, group discussions and blackboard meetings, nor the lively interactions that occurred during the week. However, it does convey the manifold connections between the themes of the conference, refinements of well-established bridges, completely new links between seemingly distant themes, problems, methods, and theories, as well as demonstrates substantial progress on older problems. In short, it shows that the area is very much alive!

We are extremely grateful to the Oberwolfach institute, its director and to all of its staff for providing a perfect setting for an inspiring, intensive week of "Geometric and Algebraic Combinatorics".

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Eran Nevo in the "Simons Visiting Professors" program at the MFO.

Gil Kalai, Isabella Novik, Francisco Santos, Volkmar Welker
Jerusalem/Seattle/Santander/Marburg, March 2015

Workshop: Geometric and Algebraic Combinatorics

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Abstracts

Miracles for c-arrangements

Karim Alexander Adiprasito

A c-arrangement is a finite collection of distinct affine subspaces of $\mathbb{R}^d$, all of codimension c, with the property that the codimension of the non-empty intersection of any subset of $\mathcal{A}$ is a multiple of c. c-arrangements, introduced by Goresky and MacPherson, are a generalization of real and complex hyperplane arrangements that are highly useful to test the limitations of algebraic results and methods versus more adaptable combinatorial techniques. In some important cases, this fails (such as the Arnold–Orlik–Solomon Theorem, which fails due to a counterexample of Ziegler). We present, specifically, three important cases that carry over from the algebraic setting:

- Firstly, we provide we provide the Lefschetz-type hyperplane theorem for complements of c-arrangements, and introduce Alexander duality for combinatorial Morse flows. Our results greatly generalize previous work by Hamm–Lê, Dimca–Papadima, Hattori, Randell, and Salvetti–Settepanella and others, and they demonstrate that in contrast to previous investigations, a purely combinatorial approach suffices to show minimality and the Lefschetz Hyperplane Theorem for complements of complex hyperplane arrangements.
- Secondly, we present a generalization of Hironaka's Theorem relating the boundary manifold of an arrangements to its complement. The proof is simple: the statement is simply the combinatorial Morse-theoretic dual of the Lefschetz theorem for complements of c-arrangements.
- Finally, we provide a simple proof of the Rota–Heron–Welsh conjecture for matroids realizable as c-arrangements in the sense of Goresky–MacPherson: we prove that the coefficients of the characteristic polynomial of the associated matroids form log-concave sequences, proving the conjecture for a family of matroids out of reach for all previous methods. To this end, we study the Lévy–Milman measure concentration phenomenon on natural pushforwards of uniform measures on the Grassmannian to realization spaces of arrangements under a certain extension procedure on matroids.

Balinski's theorem for arrangements of curves

Bruno Benedetti
(joint work with Barbara Bolognese, Matteo Varbaro)

A graph with at least k vertices is called *k-vertex-connected* (or simply *k-connected*) if the deletion of fewer than k vertices, however chosen, leaves it connected. Clearly 1-connected is the same as connected. By Menger's theorem, a graph with at least $k+1$ vertices is k-connected if and only if between any two vertices one can find k

vertex-disjoint paths [Men27, Whi32]. So k-connectivity can be thought of a way to quantify how connected a graph is.

Now, given a scheme $X \subset \mathbb{P}^n$, its *dual graph* $G(X)$ is the graph whose vertices are the irreducible components of X, and whose edges connect components that intersect in codimension 1. If I_X is the unique saturated ideal of $S = \mathbb{C}[x_0, \ldots, x_n]$ such that $X = \mathrm{Proj}(S/I_X)$, the scheme X is called *arithmetically Cohen–Macaulay* (resp. *arithmetically Gorenstein*) if S/I_X is a Cohen–Macaulay ring (resp. a Gorenstein ring). A famous result, known as *Hartshorne's connectedness theorem*, says that when X is arithmetically Cohen-Macaulay the dual graph of X is connected.

Can one give a quantitative version to Hartshorne's result?

Reduction to curves. The crucial case is when $\dim X = 1$, that is, when X is a curve. In fact, if $\dim X > 1$ we can intersect X with a generic hyperplane, thus obtaining an algebraic set $X' \subset \mathbb{P}^{n-1}$ with the same dual graph of X and with $\dim X' = \dim X - 1$. Repeating this trick, called *general hyperplane section* or *Bertini reduction*, we can always reduce to the case $\dim X = 1$. The advantage of reducing to curves is that many algebraic properties (like being arithmetically Cohen–Macaulay or Gorenstein) are maintained under general hyperplane sections; in addition, for curves "intersecting in codimension one" is the same as "intersecting".

Warning: Slightly different definitions of dual graph are in use. In the literature there is an alternative notion of "dual graph of a projective curve X". If $X_1, \ldots, X_s$ are the irreducible components of X, let us agree to call *dual multigraph of* X the cell complex $M(X)$ formed as follows: We take the vertex set $\{1, \ldots, s\}$, and we connect two vertices (not necessarily different) with k parallel edges if and only if the corresponding components intersect in k distinct points.

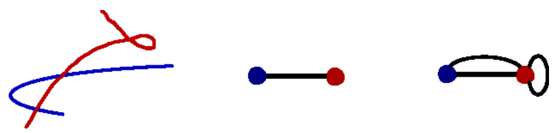

Figure 1. The dual graph (center) and the dual multigraph (right) of the curve arrangement on the left.

Obviously the dual graph $G(X)$ can be recovered from $M(X)$ by killing all loops, and then by consolidating parallel edges into single edges. So the multigraph $M(X)$ carries more information, such as the number of intersections or self-intersections. However, for the purposes of our study, looking at $M(X)$ or at $G(X)$ is the same. In fact, it is easy to show that $M(X)$ is k-connected if and only if $G(X)$ is.

Arrangements coming from simplicial complexes. There is a five-step combinatorial strategy to obtain many examples of arrangements of curves (in fact, of lines!) with desired algebraic properties.

(i) Start from a pure simplicial complex Δ, with vertices labeled by $0, \ldots, n$;
(ii) form the Stanley-Reisner ideal I_Δ in $\mathbb{C}[x_0, \ldots, x_n]$;

(iii) pass to the Zariski-corresponding set $Z(I_\Delta) \subset \mathbb{P}^n$;
(iv) perform general hyperplane sections until the dimension decreases to 2, and
(v) call $X = X(\Delta)$ the resulting line arrangement.

The following facts are easy to show:

Fact 1: The dual graph of Δ as simplicial complex, and the dual graph of $X = X(\Delta)$ as curve, are the same [BV14, Lemma 2.7].

Fact 2: If Δ is Cohen–Macaulay (resp. Gorenstein), then X is arithmetically Cohen-Macaulay (resp. arithmetically Gorenstein).

Fact 3: If Δ is homeomorphic to the $(d-1)$-sphere, then X is an arithmetically Gorenstein scheme of Castelnuovo–Mumford regularity $d+1$ (see below).

Castelnuovo–Mumford regularity. The regularity for projective schemes is defined as follows. If $X = \mathrm{Proj}(S/I_X)$, with I_X saturated, let

$$\cdots \to F_j \to \cdots \to F_0 \to I_X \to 0$$

be a minimal graded free resolution of I_X. The *regularity of* X is the smallest integer r such that for each j, all minimal generators of F_j have degree $\leq r + j$.

Main results

In order to quantify Hartshorne's theorem, a first step is to look at arrangements of lines. A first result in this direction was obtained last year:

Theorem 1 (B.–Varbaro [BV14]). *If X is a Gorenstein line arrangement, then $G(X)$ is r-connected, where $r+1$ is the Castelnuovo–Mumford regularity of X.*

In case the line arrangement is of the type $X = X(\Delta)$, with Δ a triangulated $(d-1)$-sphere, by **Facts 2–3** above the scheme $X(\Delta)$ is a Gorenstein line arrangement of regularity $d+1$. Applying Theorem 1 we obtain that the dual graph $X = X(\Delta)$ (which by **Fact 1** coincides with the dual graph of Δ) is d-connected. So in this special case, Theorem 1 boils down to Balinski's theorem, "the graph of every $(d-1)$-sphere is d-connected". Theorem 1 is more general though: For example, most line arrangements are not of the form $X = X(\Delta)$ for any simplicial complex Δ. And even within simplicial complexes, not all Gorenstein complexes are triangulated spheres.

The proof of Theorem 1 relies on a subadditivity result for regularity by Derksen and Sidman [DS02], and on *liaison theory*, a classical method in algebraic geometry to study the properties shared by two schemes whose union is arithmetically Gorenstein, cf. [Mi98]. Very recently, we have obtained an extension to the case of projective curves (and more generally, of projective schemes of arbitrary dimension).

First of all, for reduced equidimensional curves, i.e. curves X whose corresponding saturated ideal I_X is radical and height-unmixed, one can prove that

$$\mathrm{reg}\, X \leq \deg X.$$

(The proof relies on the work by Gruson et al.[GLP83]; see [BBV].) We stress that this upper bound is at the moment available only for curves; a major open problem in commutative algebra is the *Eisenbud–Goto conjecture* that for any reduced, nondegenerate irreducible scheme X one has $\operatorname{reg} X \leq \deg X - \operatorname{codim} X + 1$. (Nondegenerate means essentially that n is the smallest integer such that $X \subset \mathbb{P}^n$.)

Theorem 2 (B.–Bolognese–Varbaro [BBV])**.** *Let X be an arithmetically Gorenstein projective scheme of regularity $r+1$.*

(A) *If every irreducible component of X has regularity $\leq R$, the dual graph of X is $\lfloor \frac{r+R-1}{R} \rfloor$-connected.*

(B) *If every irreducible component of X has degree $\leq D$, and X is reduced, the dual graph of X is $\lfloor \frac{r+D-1}{D} \rfloor$-connected.*

Note that when $R = 1$ (or equivalently when $D = 1$) and $\dim X = 1$ we are back in the line arrangement case, and the statement boils down to Theorem 1.

The new proof uses also liaison theory; the technical difficulty is that Derksen–Sidman's regularity result holds only for subspace arrangements, but we managed to replace it with a result by Caviglia [Ca07] by reducing ourselves to curves. (If the reduction is done via general hyperplane section, the degrees of the primary components are maintained; the regularities of such components need not be maintained, but they can only decrease, so they stay smaller or equal than R.) The bound of Theorem 2 is also sharp, and yields somewhat an interpolation between Balinski's theorem and Hartshorne's connectedness theorem.

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Face Numbers of Sequentially Cohen–Macaulay Complexes

Afshin Goodarzi

(joint work with Karim Adiprasito, Anders Björner)

The notion of sequentially Cohen–Macaulay complexes first arose in combinatorics: Motivated by questions concerning subspace arrangements, Björner & Wachs introduced the notion of nonpure shellability [3]. Stanley then introduced the sequentially Cohen–Macaulay property in order to have a ring-theoretic analogue

of nonpure shellability. Motivated by the Macaulay-Stanley theorem for Cohen-Macaulay complexes Björner & Wachs [3] posed the problem to characterize the possible face numbers of sequentially Cohen–Macaulay simplicial complexes.
Such a characterization, other than being interesting from the combinatorial point of view, has two numerical consequences in commutative algebra. Namely, characterizing the possible Betti tables of componentwise linear ideals and also characterizing the possible Hilbert series of local cohomology modules of sequentially Cohen-Macaulay standard graded algebras, see [1, 4, 5] for more details.
Recently [1], we gave a numerical characterization of the possible face numbers of sequentially Cohen–Macaulay complexes. In order to achieve the characterization, first we use properties of Kalai's algebraic shifting [6] to reduce the problem to the case of shifted complexes. Then we use a combinatorial correspondence between shifted multicomplexes and pure shifted simplicial complexes. Our combinatorial correspondence is a generalization of a bijection between multisets and sets provided by Björner, Frankl & Stanley [2].

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Tight 3-manifolds.

Bhaskar Bagchi

A finite connected simplicial complex X is said to be tight [5] with respect to a field $\mathbb{F}$ if the $\mathbb{F}$-linear map $H_*(Y) \to H_*(X)$ induced by the inclusion $Y \subseteq X$ is injective for all induced subcomplexes Y of X. Easy consequences of this definition:- (a) all induced subcomplexes and k-skeletons ($k \geq 1$) of an $\mathbb{F}$-tight complex are $\mathbb{F}$-tight; (b) all $\mathbb{F}$-tight complexes are 2-neighbourly; (c) if an $\mathbb{F}$-tight complex X triangulates an $\mathbb{F}$-homology closed manifold, then X is $\mathbb{F}$-orientable.

For vertices $x \neq y$, let's define X^x and X_y to be the antistar of x (resp. link of y) in X. Also put $X^x_y = (X^x)_y = (X_y)^x$. Let $c_X(x, y)$ denote the number of connected components K of X^x_y such that x is adjacent in X_y to some vertex of K. As an easy consequence of the Mayer-Vietoris theorem, we obtain the following structural restriction on (the 2-skeleton of) an $\mathbb{F}$-tight complex.

Lemma. For any two vertices $x \neq y$ of an $\mathbb{F}$-tight complex X, we have $c_X(x, y) = c_X(y, x)$.

It takes an elementary combinatorial argument to deduce the following consequence of this lemma.

Corollary. If some vertex link of an $\mathbb{F}$-tight complex X is a cycle then X triangulates a closed 2-manifold (which, of course, must be 2-neighbourly and $\mathbb{F}$-orientable).

Since the number of vertices of an $\mathbb{F}$-orientable 2-neighbourly closed 2-manifold is severely constrained (particularly when char $(\mathbb{F}) \neq 2$), we deduce:

Corollary. Let S be the link of a vertex in an $\mathbb{F}$-tight complex. Then S has no induced cycle of length n where (a) char $(\mathbb{F}) \neq 2$ and $n \equiv 0, 1, 4, 5, 7, 8, 9$ or $10 \pmod{12}$ or (b) char $(\mathbb{F}) = 2$ and $n \equiv 1 \pmod 3$.

In [4], Kalai gave a structural characterization of the 1-skeleton of a stacked sphere. In the case $d = 2$, it says that a triangulated 2-sphere is stacked iff it has no induced cycle of length ≥ 4. We observe that the following stronger result is easy to prove in this case: a triangulated 2-sphere is stacked iff it has no induced cycle of length 4 or 5. In conjunction with part (a) of the above Corollary, this observation yields the following characterization of tight 3-manifolds.

Theorem 1. A triangulation of a closed 3-manifold is $\mathbb{F}$-tight, char $(\mathbb{F}) \neq 2$, iff it is neighbourly, stacked and orientable.

(Recall: A triangulation of a closed 3-manifold is said to be stacked if it can be obtained as the boundary of a triangulated 4-manifold Δ such that all the 2-faces of Δ are in its boundary. The "if" part of the above theorem is a previous result due to Bagchi and Datta [2].)

Let S^2_4 and I^2_{12} denote the boundary complexes of the tetrahedron and the icosahedron, respectively. We also show that these are the only triangulations of S^2 with no induced cycle of length 3 and no induced cycle of length $1 \pmod 3$. In conjunction with part (b) of the Corollary, this yields:

Theorem 2. If X is an $\mathbb{F}$-tight triangulation of a closed 3-manifold, char$(\mathbb{F}) = 2$, then each vertex link of X is a connected sum of copies of S^2_4 and I^2_{12} (in some order).

We are able to compute the σ_0 of any such connected sum. In conjunction with results from [1] and Theorem 2, this yields the following restrictions on the parameters of an $\mathbb{F}$-tight closed 3-manifold.

Theorem 3. Let X be an $\mathbb{F}$-tight closed 3-manifold. Then $n := f_0(X)$ and $\beta_1 := \beta_1(X; \mathbb{F})$ satisfy:-

(a) $\binom{n-4}{2} \equiv 10\beta_1 \pmod{388}$. Also $\binom{n-4}{2} \geq 10\beta_1$, with equality iff X is stacked.

(b) $429\binom{n-4}{2} - 388n\left\lfloor \frac{n-4}{9} \right\rfloor \leq 4290\beta_1$. Equality holds here iff each vertex link of X is a connected sum of $\left\lfloor \frac{n-4}{9} \right\rfloor$ copies of I^2_{12} and $n - 4 - 9\left\lfloor \frac{n-4}{9} \right\rfloor$ copies of S^2_4 (in some order).

As an elementary consequence of this theorem, we find:

Corollary. If there is an $\mathbb{F}$-tight closed 3-manifold which is not stacked, then its parameters satisfy $n \geq 72$ and $\beta_1 \geq 189$. Also its Z-homology must have non-trivial 2-torsion in degree one. (The pair $(n, \beta_1) = (72, 189)$ is feasible for such a triangulation.)

The results presented here are from a joint work with Basudeb Datta and Jonathan Spreer [3].

I will like to pose:

Conjecture. Let X be a 2-dimensional simplicial complex such that for every induced subcomplex Y of X and any two vertices $y_1 \neq y_2$ of Y, we have $c_Y(y_1, y_2) = c_Y(y_2, y_1)$. Then X is Z_2-tight.

Note that, in view of the above lemma, this conjecture would imply that if a 2-complex X is $\mathbb{F}$-tight for some $\mathbb{F}$ then it is Z_2-tight. It would also be a structural characterization of the Z_2-tight complexes of dimension 2.

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Combinatorics of tropical curves in 3-space

Grigory Mikhalkin

Let Γ be a finite graph and $\partial\Gamma \subset \Gamma$ be the set of its 1-valent vertices, The *tropical structure* on Γ is a choice of complete inner metric on $\Gamma \setminus \partial\Gamma$. This means that the tropical structure specifies the length for all edges except those which are adjacent to 1-valent vertices. A continuous map

$$h : \Gamma \setminus \partial\Gamma \to \mathbb{R}^3$$

is called a tropical curve in a 3-space if the restriction of h to the interior of each edge $E \subset \Gamma$ is smooth and the following two conditions hold (cf. [2]).

(integrality): If $u(E)$ is a unit tangent vector to an edge E then we have $(dh)(u(E)) \in \mathbb{Z}^3$, where dh is the differential of h.

(balancing): We have $\sum_j (dh)(u(E_j)) = 0$ for each vertex $v \in \Gamma \setminus \partial\Gamma$. Here the sum is taken over all edges E_j adjacent to v and $u(E_j)$ is the unit tangent vector to E_j directed away from v.

Such tropical curves can be used for enumeration of classical (complex and real) tropical curves, particularly through the so-called *floor diagrams*, see [1]. Let $\mathrm{Def}(h)$ be the space of local deformation of a curve $h : \Gamma \setminus \partial\Gamma \to \mathbb{R}^3$. The space $\mathrm{Def}(h)$ is (locally) a union of convex cones. The tropical Riemann-Roch inequality implies that each maximal convex cone in $\mathrm{Def}(h)$ has dimension at least $\kappa = \#(\partial\Gamma)$. If Γ is a tree then this dimension is equal to κ.

Suppose that $\kappa = 2k$ is even and that Γ is a tree. Edges adjacent to $\partial\Gamma$ can be oriented towards $\partial\Gamma$, thus we may associate to each such edge an integer 3-vector. By the balancing condition the sum of all these vectors must be zero. A collection of κ vectors whose sum is zero is called *the topic degree* of the tropical curve h.

Let us fix the set $\mathcal{P}$ of $k = \frac{\kappa}{2}$ points in general position in $\mathbb{R}^3$. It may be shown that there exists only finitely many curves of given topic degree passing through the fixed k points. Furthermore, the combinatorics of these curves and positions of the fixed points there is quite restricted.

It is convenient to introduce the following convention. Note that every edge in a tree define a partition of the set $\partial\Gamma$ into two disjoint sets. The cardinality of these sets have the same parity as we assume $\kappa = 2k$. An edge E in a tree Γ (with even number $\kappa = 2k$ of leaves) is called *even* if both of these sets have even cardinality. Otherwise, E is called *odd*. Clearly any connected component of the union of all odd edges is homeomorphic to $\mathbb{R}$. We call such component a *worldline*. Note that we have exactly k worldlines which can be thought of as worldliness of particles. Similarly, connected components of the union of even edges can be interpreted as interactions among the particles influencing their worldlines.

Theorem 1 ([2])**.** *Suppose that $\mathcal{P} \subset \mathbb{R}^3$ is a collection of k points in general position and $h : \Gamma \setminus \partial\Gamma \to \mathbb{R}^3$ is a tropical curve such that $h(\Gamma \setminus \partial\Gamma) \supset \mathcal{P}$.*

Then each worldline of h contains a unique point of $\mathcal{P}$ and vice versa, each point of $\mathcal{P}$ is contained in a unique worldline of h, see Figure 1.

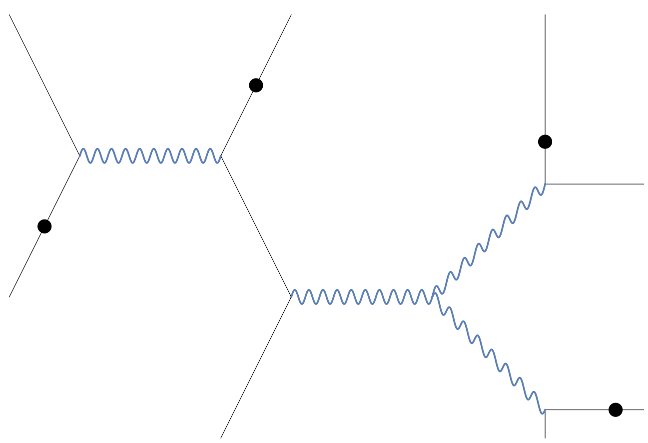

FIGURE 1. A tropical curve passing through a collection of generic points in 3-space and its decomposition into odd (solid) and even (wavy) parts.

Furthermore, the decomposition of tropical curve into worldlines can be used to compute the Welschinger signs (see [3]) of real algebraic curves approximating the tropical curve. This sign is defined as the self-linking number of the real algebraic curve enhanced with the framing which is the subbundle of the normal bundle of maximal degree. This framing is topologically unique in the case when the curve passes through k generic points. E.g. in the case of projective curves of degree d passing through generic $2d$ points in $\mathbb{RP}^3$ the degree of the framing is $2d-1$.

The framing can be computed tropically with the help of an infinitesimal tropical surfaces near $h(\Gamma \setminus \partial\Gamma)$ in $\mathbb{R}^3$. Combinatorially this subdivides the vertices of $\Gamma \setminus \partial\Gamma$ into two classes: *flat* and *branched* vertices, see [2]. The degree of the tropical framing is equal to the number of flat vertices.

Theorem 2 ([2]). *If h is a tropical curve in $\mathbb{R}^3$ that passes through $k = \frac{\kappa}{2}$ then the maximal degree of the tropical framing is $k-1$. Furthermore for such curve we can always choose the tropical framing of degree $k-1$ so that each worldline except for one contains a unique flat vertex.*

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Valuated matroids in tropical geometry

Diane Maclagan

(joint work with Felipe Rincón)

The main aim of this talk was to illustrate the uses of valuated matroids in tropical geometry, and to highlight the connections with an emerging theory of tropical schemes.

We first recall the definition of a valuated matroid. A subset $\mathcal{B} \subset \binom{[n]}{r}$ is the set of bases of a matroid M if and only if the matroid polytope

$$P_M = \operatorname{conv}(\sum_{i \in B} \mathbf{e}_i : B \in \mathcal{B}) \subset \mathbb{R}^n$$

has all edges parallel to $\mathbf{e}_i - \mathbf{e}_j$. Valuated matroids, introduced by Dress and Wenzel in the early 90s [DW92] are given by a function

$$w \colon \binom{[n]}{r} \to \mathbb{R} \cup \{\infty\}$$

with the property that the regular subdivision of $\operatorname{conv}(\sum_{i \in B} \mathbf{e}_i : w(B) < \infty)$ induced by w is a matroid subdivision (each polytope is a matroid polytope).

Equivalently, the function w defines a valuated matroid if for all $B_1, B_2 \in \binom{[n]}{r}$, and $i \in B_1 \setminus B_2$, there exists $j \in B_2 \setminus B_1$ with

$$w(B_1) + w(B_2) \geq w(B_1 \setminus i \cup j) + w(B_2 \setminus j \cup i).$$

As with regular matroids, the motivating example of a valuated matroid is a *realizable* valuated matroid. To describe these, we fix a field K with a valuation. This is a function $\mathrm{val}\colon K \to \mathbb{R}\cup\{\infty\}$ satisfying $\mathrm{val}(ab) = \mathrm{val}(a)+\mathrm{val}(b)$, $\mathrm{val}(a+b) \geq \min(\mathrm{val}(a), \mathrm{val}(b))$, and $\mathrm{val}(a) = \infty$ if and only if $a = 0$. One example is given by the p-adic valuation on $\mathbb{Q}$: $\mathrm{val}(p^n a/b) = n$ if p does not divide a or b. A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in K^r$ gives a valuated matroid by setting, for $B = \{i_1, \ldots, i_r\}$, V to be the $r \times r$ matrix with columns given by $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_r}$, and

$$w(B) = \mathrm{val}(\det(V)).$$

Valuations also play a key role in tropical geometry. The *tropical semiring* is $(\overline{\mathbb{R}}, \oplus, \otimes)$, where $\overline{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$, the tropical addition $\oplus$ is minimum, and the tropical multiplication $\otimes$ is standard addition. For example, $5 \otimes (3 \oplus 7) = 5 + \min(3, 7) = 8$. The semiring of tropical polynomials, $S = \overline{\mathbb{R}}[x_0, \ldots, x_n]$, is the semiring of polynomials with coefficients in $\overline{\mathbb{R}}$ where addition and multiplication are tropical addition and multiplication. For example, as a function $xy \oplus 5 \otimes x \oplus 4 \otimes y \oplus 2$ equals $\min(x + y, x + 5, y + 4, 2)$. Note, however, that different polynomials can give rise to the same function; for example, $x^2 \oplus 1$ and $x^2 \oplus 3 \otimes x \oplus 1$ give rise to the same function, as $\min(2x, 1) = \min(2x, x + 3, 1)$ for all $x \in \overline{\mathbb{R}}$.

For $f \in S$ the hypersurface of f is

$$V(f) = \{\mathbf{w} \in \mathbb{R}^n : \text{ the minimum in } f \text{ is achieved at least twice}\}.$$

This is the $(n-1)$-skeleton of the complex dual to the regular subdivision of the Newton polytope of f induced by the coefficients of f. For an ideal $J \subset S$, the variety of J is $V(J) = \cap_{f \in J} V(f)$.

Given a standard polynomial $f \in R := K[x_0, \ldots, x_n]$, where K is a field with a valuation, we can form its tropicalization by replacing the coefficients by their valuations, and all additions and multiplications by tropical operations. For example, when $K = \mathbb{Q}$ with the 2-adic valuation, and $f = 7xy + 32x + 48y + 20$, we have $\mathrm{trop}(f) = xy \oplus 5 \otimes x \oplus 4 \otimes y \oplus 2$.

For a homogeneous ideal $I \subset R$, the tropicalization is $\mathrm{trop}(I) = \langle \mathrm{trop}(f) : f \in I \rangle$. The tropicalization of the variety defined by I is $V(\mathrm{trop}(I))$. This is a polyhedral complex satisfying certain combinatorial constraints (facet directions are rational, and the maximal cells can be given a positive integer weight that makes the complex *balanced*); see [MS15, §3.4]. A goal of tropical geometry is to recover information about the original variety $V(I)$ from its tropicalization.

As subschemes of $\mathbb{P}^n$ can be described by a homogeneous ideal in R, it would be tempting to define a tropical subscheme of tropical projective space to be given by an homogeneous ideal in S. However this gives too large a class of ideals. For example, the ideal generated by $x \oplus y$ in $\overline{\mathbb{R}}[x, y]$ is not of the form $\mathrm{trop}(I)$ for *any* ideal $I \subset K[x, y]$ for *any* choice of valued field K. In fact, if J is an arbitrary ideal in S, then $V(J)$ can be a fairly arbitrary union of convex sets, and not a polyhedral complex.

A solution to this problem is to place extra constraints on the ideals $J \subset S$ that we consider. If I is a homogeneous ideal in R, then I_d is a subspace of $\mathrm{span}(\mathrm{Mon}_d)$, where $\mathrm{Mon}_d = \{x^u : \deg(x^u) = d\}$. Under the valuation map a subspace of a vector space is taken to a valuated matroid. This motivates the following definition.

Definition 1. A homogeneous ideal J of S is a *tropical ideal* if J_d determines a valuated matroid on Mon_d. Equivalently, a tropical ideal is a compatible collection of valuated matroids, one for each degree d, where the ground set of the dth matroid is Mon_d.

The tropicalization $\mathrm{trop}(I)$ of a homogeneous ideal $I \subset R$ is a tropical ideal. The tropical ideal constraint suffices to guarantee that $V(J)$ "looks like" the tropicalization of a variety. A polyhedral complex is $\mathbb{R}$-rational if every polyhedron in the complex has rational facet directions (but not necessarily rational vertices).

Theorem 2 (M,Rincón)**.** *If I is a tropical ideal, then $V(I)$ is a finite $\mathbb{R}$-rational polyhedral complex.*

The definition of tropical ideals is inspired by work of J. and N. Giansiracusa on tropicalizing schemes [GG13], and follow-up work of the author with Rincón [MR14] making the connection of the Giansiracusa theory to valuated matroids. In several ongoing projects these authors and collaborators are in the process of developing more of the basic theory.

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Tropical Laplacians and Curve Arrangements on Surfaces

Eric Katz

(joint work with June Huh)

Let X be a smooth projective complex surface. Let $C_1, C_2, \dots C_n$ be a set of distinct smooth curves on X. Let Γ be the incidence graph whose vertices v_i correspond to the curves C_i and where an edge with multiplicity k is drawn between v_i and v_j if and only if there are k points of $C_i \cap C_j$ (counted with multiplicity). One can ask which finite graphs occur as incidence graphs. The answer is any by an observation of Kollár [3] by starting with planar curves in general position and then blowing up intersection points.

A better question is to consider curve arrangements satisfying the following condition: a curve arrangement $C_1, \ldots, C_n \subset X$ is said to be *sufficient* if some integer combination $\sum a_i C_i$ is ample. One obtains strong constraints on the curve arrangement once one also incorporates the data of the self-intersection numbers, C_i^2 of the curves. One then consider the vector space of divisors, $V \subset \mathrm{Div}(X) \otimes \mathbb{R}$ consisting of all linear combinations of the C_i's. By the Hodge index theorem, the intersection product on V has a single positive eigenvalue.

We define the tropical Laplacian of the curve arrangement to be the opposite of the intersection matrix:

$$L = \begin{bmatrix} -C_1^2 & -C_1 \cdot C_2 & \ldots & -C_1 \cdot C_n \\ -C_1 \cdot C_2 & -C_2^2 & \ldots & -C_2 \cdot C_n \\ \vdots & & \ddots & \vdots \\ -C_n \cdot C_1 & -C_n \cdot C_2 & \ldots & -C_n \cdot C_n \end{bmatrix}.$$

This matrix should be thought of as analogous to the graph Laplacian of Γ in that the off-diagonal entries are non-zero if and only if there is an edge between the corresponding vertices. The off-diagonal entries are all non-positive. However, the diagonal entries are not the degrees of the vertices.

Before we discuss the combinatorial constraints imposed by the Hodge index theorem, we need to introduce some terminology: A divisor in V is said to be *irrelevant* if it is in the kernel of L. For a divisor $D = \sum a_i C_i$, write $D(v_i) = a_i$. Let K be subspace of irrelevant divisors. Let $C\Gamma$ be the cone over Γ, considered as a 2-dimensional simplicial complex, called a fan. We weight each 2-dimensional cone with the multiplicity $m(v_i, v_j) = C_i \cdot C_j$. We identify v_i with a point on its corresponding ray. There is a piecewise linear map:

$$u : C\Gamma \to K^\vee$$

given by

$$v_i \mapsto (D \mapsto D(v_i)).$$

The map obeys the balancing condition from tropical geometry:

$$-C_i^2 u(v_i) = \sum_{v_j \sim v_i} m(v_i, v_j) u(v_j)$$

which is just the irrelevancy condition. As an example, for four lines in $\mathbb{P}^2$, the image of u is the much-celebrated tropical plane in $\mathbb{R}^3$. If the map u is an embedding of $C\Gamma$ in $K^\vee$, from the multiplicities and $\mathrm{im}(u)$, we can reconstruct the tropical Laplacians up to a subdivision equivalence relation. It is often preferable to work with the weighted fan $\mathcal{F} = \mathrm{im}(u)$.

The map u most naturally comes from tropical geometry. Let X be a 2-dimensional subvariety of a smooth toric variety $X(\Delta)$. Suppose that every intersection of X with a toric stratum is transverse. Then the intersection of X with the toric divisors gives a curve arrangement on X. The image of u naturally projects onto the tropicalization $\mathrm{Trop}(X)$.

We can view $D \in K$ as cutting out a hyperplane

$$H_D = \{\ell \in K^\vee | D(\ell) = 0\}$$

with open half-spaces H_D^+, H_D^-. Note here that $v_i \in H_D^+$ means $D(v_i) > 0$. Write $\Gamma_D^+, \Gamma_D^-, \Gamma_D^0$ for the subgraphs induced by vertices lying in H_D^+, H_D^-, H_D. Linear algebraic results coming out of the theory of Colin de Verdiere numbers constrain the way that hyperplanes can cut up the image of u. One has the linear algebraic lemma of van der Holst [2]:

Lemma 1. *If X_1 and X_2 are disjoint components of Γ_D^+ then there is $D' \in K$ such that $X_1 = \Gamma_{D'}^+, X_2 = \Gamma_{D'}^-$.*

This lemma implies the following result of Lovasz and Schrijver [4]

Proposition 2. *Let $D \in K$. If Γ_D^+ is disconnected then*

(1) there are no edges connecting Γ_D^+ and Γ_D^-,
(2) any component Y of Γ_D^+ or Γ_D^- satisfies $N(Y) = N(\Gamma_D^+ \cup \Gamma_D^-)$.

Here, N denotes the set of neighbours. This puts strong conditions on the graph if one side of a hyperplane is disconnected.

June Huh was able to cook up a balanced weighted fan $\mathcal{F}$ in $\mathbb{R}^4$ that violated the above proposition. Therefore, it cannot come from a curve arrangement on a surface. This fan is not the tropicalization of any surface in $(\mathbb{C}^*)^4$. Moreover, Babaee and Huh have used this fan to produce a counterexample to Demailly's strongly positive Hodge conjecture [1].

These results show how the Hodge index theorem constrains curve arrangements on surfaces and surfaces in toric varieties. We would really like a tropical Hodge index theorem. That is, given a 2-dimensional fan $\mathcal{F}$ in $\mathbb{R}^n$, we would like conditions guaranteeing that the tropical Laplacian has a single positive eigenvalue. Results of Lovasz, Schrijver, and van der Holst give necessary conditions. The sufficient conditions are still mysterious although they seem to have to do with the connectedness properties of the fan. A tropical Hodge index theorem with sufficient weak conditions would imply the Rota-Heron-Welsh conjecture on the log-concavity of characteristic polynomials of matroids.

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Diameter and Curvature – Pivot and Interior Point Methods

Antoine Deza and Tamás Terlaky

Linear optimization has long been a source of applications for geometers. Conversely, discrete and convex geometry have provided the foundations for many optimization techniques. The purpose of the talk is to further stimulate the interactions between geometers and optimizers. The simplex and primal-dual path-following interior point methods are currently the most computationally successful algorithms for linear optimization. While pivot algorithms follow an edge path, interior point method follows the central path. The algorithmic issues are closely related to the combinations and geometrical structure of the feasible region. Within this framework, the curvature of a polytope, defined as the largest possible total curvature of the associated central path, can be regarded as the continuous analogue of its diameter.

We recall the main theoretical and computational features of broadly used simplex methods and interior point methods, as well as criss–cross pivoting [8] and volumetric center interior point method. Pros and cons of the presented algorithms are contrasted [11, 14, 18, 19]. We highlight links between the edge and central paths, and between the diameter and curvature of a polytope. We recall continuous results of Dedieu, Malajovich, and Shub [2], and discrete results of Holt and Klee [10] and of Klee and Walkup [13], as well as related results [12, 17] and conjectures such as the Hirsch conjecture that was disproved by Santos [16].

We present analogous results dealing with average and worst-case behaviour of the curvature and diameter of polytopes [3, 4, 5, 6, 7], including a result of Allamigeon, Benchimol, Gaubert, and Joswig [1] who constructed a counterexample to the continuous analogue of the polynomial Hirsch conjecture. Finally, we recall similar results for the volumetric path [15] and the existence of short admissible pivot sequences [9] for both linear feasibility and linear optimization problems.

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Long and winding central paths

Michael Joswig
(joint work with Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert)

The interior point method for linear programming is widely used in practice. Further, it is also theoretically successful as Karmarkar established its worst-case polynomial running time in the bit model [7]. The interior point method traces an analytical curve, called the *central path*, in the interior of the feasible region towards the optimal face. Deza, Terlaky and Zinchenko proposed a continuous analog of the Hirsch conjecture [6]: They conjectured that the total curvature of the central path of a linear program in dimension d with $n > d$ constraints should be bounded by $O(n)$.

Dedieu, Malajovich and Shub proved [5] that this bound holds "on the average" in the following sense. The n linear constraints define an affine hyperplane arrangement which induces a polyhedral decomposition of $\mathbb{R}^d$. Each maximal cell corresponds to a choice of orientations of the n hyperplanes. The Zariski closure of the central path is a real algebraic curve, called the *central curve*, which is the union of the central paths in all maximal cells. The result in [5] establishes an upper bound which is linear in n times the number of maximal cells for the total curvature of the central curve. A similar result has been obtained by De Loera, Sturmfels and Vinzant using matroid theory [4].

Here we disprove the conjecture. More precisely, we have the following result.

Theorem. ([1]) Let $r \geq 1$ and let $t \gg 0$ be a real parameter which is sufficiently large. Then the total curvature of the central path of the linear program

$$
\begin{aligned}
\min \quad & v_0 \\
\text{s.t.} \quad & u_0 \leq t \\
& v_0 \leq t^2 \\
& v_i \leq t^{1-\frac{1}{2^i}}(u_{i-1} + v_{i-1}) && \text{for } i \in [r] \\
& u_i \leq t u_{i-1} && \text{for } i \in [r] \\
& u_i \leq t v_{i-1} && \text{for } i \in [r] \\
& u_r \geq 0, \ v_r \geq 0
\end{aligned}
$$

with $3r + 4$ inequalities in the $2r + 2$ variables $u_0, v_0, u_1, v_1, \dots, u_r, v_r$ is bounded from below by $\Omega(2^r/r)$. The feasible region is contained in the positive orthant and bounded; all inequalities define facets.

Our method is to tropicalize the central path in linear programming. The *tropical central path* is the piecewise-linear limit of the central paths of parameterized families of classical linear programs viewed through logarithmic glasses. We show in particular that the tropical analog of the analytic center is nothing but the tropical barycenter, that is, the maximum of a tropical polyhedron. It follows that unlike in the classical case, the tropical central path may lie on the boundary of the tropicalization of the feasible set, and may even coincide with a path of the tropical simplex method. Finally, our counter-example is obtained as a deformation of a family of tropical linear programs which are based on a construction by Bezem, Nieuwenhuis and Rodríguez-Carbonell [3].

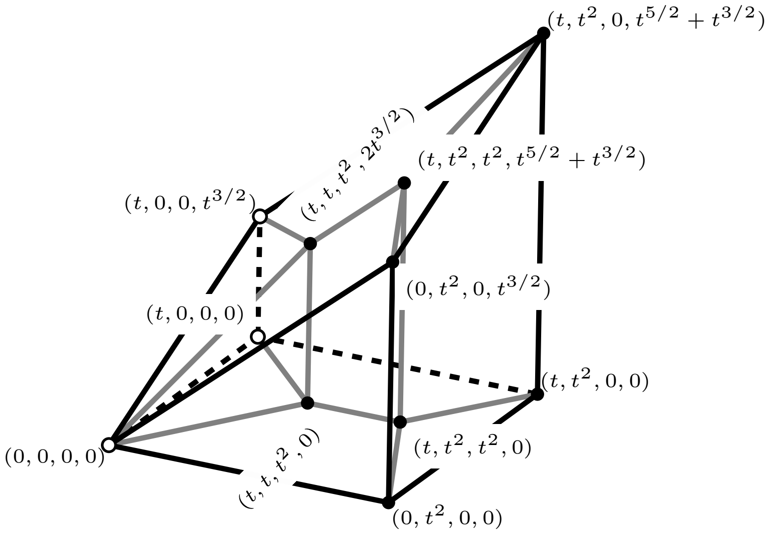

FIGURE 1. Schlegel diagram of the construction for $r = 1$ and $t \geq 2$, projected onto the facet $u_1 = 0$. The three vertices on the optimal face $v_0 = 0$ are marked

Example. Figure 1 shows the polytope for $r = 2$ and $t \geq 2$, which is sufficiently large in this case. The central path is very close to the boundary. In fact, it nearly traces a path in the vertex-edge-graph. This means that the interior point method behaves much like the simplex method on this example. The analytic center is near the vertex $(u_0, v_0, u_1, v_1) = (t, t^2, t^2, t^{5/2} + t^{3/2})$. From there the central path passes near the vertex $(t, t, t^2, 2t^{3/2})$ until it finally hits the optimal face $v_0 = 0$ close to the vertex $(t, 0, 0, t^{3/2})$.

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Geometric random edge

Friedrich Eisenbrand

(joint work with Santosh Vempala)

We study the complexity of the *simplex algorithm* to solve linear programs

$$\begin{aligned} &\max c^T x \\ &Ax \leq b \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{m \times n}$ is of full column rank, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ that satisfy a certain geometric property. The rows of A are denoted by a_i, $1 \leq i \leq m$. We assume without loss of generality that $\|a_i\| = 1$ holds for $i = 1, ..., m$. They shall have the following *δ-distance property* (we use $\langle . \rangle$ to denote linear span):

> For any $I \subseteq [m]$, and $j \in [m]$, if $a_j \notin \langle a_i : i \in I \rangle$ then $d(a_j, \langle a_i : i \in I \rangle) \geq \delta$. In other words, if a_j is not in the span of the $a_i, i \in I$, then the distance of a_j to the subspace that is generated by the a_i, $i \in I$ is at least δ.

The δ-distance property is a geometric generalization of the algebraic property of a matrix being integral to having small sub-determinants. Suppose, for example, that A is *totally unimodular*, i.e., all $k \times k$ sub-determinants are bounded by one in absolute value and let $a_1, \ldots, a_n$ be n linearly independent rows of A. The *adjoint* matrix $\widetilde{A} = (b_1, \ldots, b_n)$ of the matrix with rows $a_1, \ldots, a_n$ is an integer matrix with all components in $\{0, \pm 1\}$. The vector b_1 is orthogonal to $a_2, \ldots, a_n$ and $|a_1^T b_1| \geq 1$, since the vectors are integral. The distance of a_1 to the sub-space generated by $a_2, \ldots, a_n$ is thus at least $1/\|b_1\| \geq 1/\sqrt{n}$ which means that totally unimodular matrices have the δ-distance property for $\delta = 1/n$. Integral matrices whose sub-determinants are bounded by Δ in absolute value have the δ-distance property with $\delta \geq 1/(\Delta^2 n)$.

In this paper, we analyze the simplex algorithm [4] with a variant of the *random edge* pivoting rule. **Our main result** is a strongly polynomial running time bound for linear programs satisfying the δ-distance property.

Our result is an extension of a randomized simplex-type algorithm of Dyer and Frieze [5] that solves linear programs (1) for totally unimodular $A \in \{0, \pm 1\}^{m \times n}$ and arbitrary b and c. Also, our algorithm is a strengthening of a recent randomized algorithm of Brunsch and Röglin [2] who compute a path between two given vertices of a polytope that is defined by matrix satisfying the δ-distance property. Their algorithm cannot be used to solve a linear program (1) since both vertices have to be given. The expected length of the path is bounded by a polynomial in n, m and $1/\delta$. Bonifas et al. [1] have shown that the diameter of a polytope defined by an integral constraint matrix A whose sub-determinants are bounded by Δ is polynomial in n and Δ and independent of the number of facets. In the setting of the δ-distance property, their proof yields a polynomial bound in $1/\delta$ and the dimension n on the diameter that is independent of m. Our result is an extension of this result in the setting of linear programming. We show that there is a variant of the simplex algorithm that uses a number of pivots that is polynomial in $1/\delta$ and the dimension n.

The number δ can be seen as a measure of how far the bases of A are from being singular. More precisely, if A_B is an $n \times n$ minor of A that is non-singular, then each column of A_B^{-1} has Euclidean length $\geq \delta$. In a way, δ can be seen as a *conditioning number* of A. Conditioning plays an increasing role in the intersection of complexity theory and optimization, see, e.g. [3].

The notion of δ-distance property is independent of the *binary encoding* of the numbers in A. The matrix A can be irrational even. Also, there are examples of combinatorial optimization problems that have exponentially large subdeterminants, but δ is a polynomial. An *edge-node* incidence matrix of an undirected graph has two ones in each row. The remaining entries are zero. Although the largest subdeterminant can be exponential (it is 2 raised to the *odd-cycle-packing number* of G [6]) one has $\delta = \Omega(1/\sqrt{|V|})$.

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A dual analysis of the shadow simplex method

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(joint work with Daniel Dadush)

The simplex method for solving linear programs $\max\{c^T x \ : \ Ax \leq b\}$ can be thought of geometrically as a family of algorithms that walk on the vertex-edge graph or 1-skeleton of the polyhedron $P = \{x \in \mathbb{R}^n \ : \ Ax \leq b\}$ of feasible points according to a *pivot rule* which determines the direction of the walk. It remains one of the most important methods for solving linear programs in practice, despite the fact that up to now, all variants of the simplex method that have been analyzed successfully have been shown to have exponential running time in the worst case.

The *shadow simplex* variant of the simplex method is given an initial vertex of P that maximizes some objective function $d^T x$ and then finds vertices maximizing the objective functions $\lambda c^T x + (1-\lambda) d^T x$ for λ increasing from 0 to 1. This procedure can be understood primally as following the outline of the projection of P onto its *shadow* on the 2-dimensional plane spanned by c and d, or dually as following the line segment from d to c in the normal fan of P. The number of expected steps of this method has been successfully bounded in various randomized settings, most of them involving a randomization of both the feasible polyhedron and the objective functions [2] [5] [6].

We propose an analysis of a variant of the shadow simplex where neither the polyhedron P nor the initial and final objective function need to be randomized. Instead, we sample a single, exponentially distributed random vector $X \in \mathbb{R}^n$ and analyze the number of pivot steps along the 3-segment path

$$d \longrightarrow d + X \longrightarrow c + X \longrightarrow c$$

We say that a normal cone of a vertex of P is τ-wide if it contains a ball of radius τ centered at a point of norm 1. We show that the diameter of P is bounded by $O(\frac{n}{\tau} \log \frac{1}{\tau})$ if all normal cones of P are τ-wide, and we show slightly weaker bounds on the expected number of steps required to find an optimal solution to a linear program or to decide its feasibility algorithmically, given similar conditions on P. Overall, we get a set of related results that significantly improves upon recent results shown in the same setting [1] [3].

To prove our results, we need to bound the expected number of facets of normal cones crossed by segments of the type $[c + X, d + X]$ and by segments of the type

$[c, c+X]$. For both types of segment, we express the probability of crossing each normal cone facet using integrals over certain shifts of facets and cones. The probability that a facet of a τ-wide normal cone C is crossed by a segment of the first type is bounded by

$$\frac{\|d-c\|}{\tau}\int_0^1\int_{(C-((1-\lambda)c+\lambda d))}\zeta(x)\mathrm{d}x\mathrm{d}\lambda,$$

where ζ is the probability density function of X. Summing over this bound for all cones bounds the expected number of pivots on such a segment by $\frac{\|d-c\|}{\tau}$.

For the second type of segment, we use a similar but more involved computation to bound the number of pivot steps on a segment $[c+\alpha X, c+X]$ by $\frac{2n}{\tau}\ln\frac{1}{\alpha}$ for $\alpha \in (0,1)$. The idea is to choose α small enough so that there is no pivot step on the initial segment $[c, c+\alpha X]$. Due to the τ-wideness of normal cones, this can be achieved easily for the segment $[d, d+\alpha X]$ by choosing the initial objective function appropriately. For bounding the diameter of P, we can also choose the final objective function vector c appropriately. For optimizing a given fixed objective function vector we improve on an idea of [4], which allows us to avoid following the segment $[c, c+\alpha X]$ entirely. Instead, we can determine at least one entry of an optimal basis for c once $c+\alpha X$ is reached. This reduces the dimension of the problem by at least 1 so that we can find the optimum by recursion.

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Topology of the Grünbaum–Hadwiger–Ramos hyperplane mass partition problem

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(joint work with Pavle V. M. Blagojević, Florian Frick, and Albert Haase)

1. History

In 1960 Branko Grünbaum [4] asked whether any convex body in $\mathbb{R}^d$ could be cut into 2^d parts of equal volume/measure by d suitably chosen affine hyperplanes. He proved this for $d = 2$ via a simple application of the intermediate value theorem.

In 1966 Hugo Hadwiger [5] proved this for $d = 3$, while answering a problem by Jaworowski posed at the 1963 Topology meeting in Oberwolfach. As an intermediate step, he considered *two* masses in $\mathbb{R}^3$ and showed that they can simultaneously be cut into four pieces of equal measure by suitable hyperplanes. This was reproved later by Yao et al. [10] using the Borsuk–Ulam theorem.

In 1984 David Avis [1] noted that the answer is "no" for $d > 4$, by considering a measure concentrated on the moment curve. This also follows from a parameter count, as the space of d affine hyperplanes has dimension d^2, while $2^d - 1$ independent conditions have to be met. See [10] for still a different argument.

In 1996 Edgar A. Ramos [9] posed the general question: For $j, k \geq 1$, determine the smallest dimension $d = \Delta(j, k)$ such that any j masses in $\mathbb{R}^d$ can be simultaneously cut into 2^k orthants of equal measure by k suitably chosen affine hyperplanes. In view of the history just sketched, we call this the "Grünbaum–Hadwiger–Ramos hyperplane mass partition problem."

As a special case, this problem contains the ham-sandwich theorem: j measures can be halved in $\mathbb{R}^d$ by a single hyperplane if and only if $j \leq d$: That is, $\Delta(j, 1) = j$. Hadwiger's results quoted above amount to $\Delta(2, 2) = \Delta(1, 3) = 3$.

2. Summary

In [2] we provide a detailed status report about the results obtained for the Grünbaum–Hadwiger–Ramos problem up to now. This in particular includes

$$\left\lceil \tfrac{2^k-1}{k} j \right\rceil \;\leq\; \Delta(j,k) \;\leq\; j + (2^{k-1}-1)2^{\lfloor \log_2 j \rfloor} \qquad \text{for all } j,k \geq 1.$$

The lower bound was derived for $j = 1$ by Avis [1] and for general j by Ramos [9] from measures concentrated on the moment curve. The upper bound was obtained by Mani-Levitska, Vrećica & Živaljević [8] from a Fadell–Husseini index [3] calculation. All the available evidence is consistent with the belief that the lower bound is tight for all j and k; we refer to this as the "Ramos conjecture."

In particular, the lower and upper bounds coincide for the cases $k = 1$ (the ham-sandwich theorem) and for $k = 2$, $j = 2^t - 1$. A number of further values and bounds have been claimed in the literature up to now; however, as documented in [2], the proofs for all of these (except for the values listed above) seem to be incomplete. The last remaining case $\Delta(4, 1)$ of Grünbaum's problem was the subject of Blagojević's lecture at this Oberwolfach workshop.

3. Ansatz

Affine halfspaces (or oriented hyperplanes) are naturally parameterized by S^d, if one includes the special cases of all of $\mathbb{R}^d$ (for the north pole) and the empty subset of $\mathbb{R}^d$ (given by the south pole of $\mathbb{R}^d$). Whence we get $(S^d)^k$ as the configuration space for k affine half-spaces on $\mathbb{R}^d$. Thus the natural configuration space/test map scheme provides the following criterion:

Proposition 1 (Product Scheme)**.** *If there is no continuous map*

$$(S^d)^k \quad \longrightarrow_{\mathfrak{S}_k^\pm} \quad S(\{X \in \mathbb{R}^{j\times 2^k} : \text{sum of columns} = 0\})$$

that is equivariant with respect to the natural actions of the hyperoctahedral group $\mathfrak{S}_k^\pm$, then there is no counterexample, i.e. "simultaneous mass partition works" for j measures and k hyperplanes in $\mathbb{R}^d$, that is, $\Delta(j,k) \leq d$.

The target space for this scheme, which records the values of measures in the orthants, is a sphere of dimension $(2^k-1)j-1$.

In the special case $k=1$ this says that if there is no map $(S^d) \to_{\mathbb{Z}/2} S(\mathbb{R}^j)$, which by Borsuk–Ulam happens when $j \leq d$, then $\Delta(j,1) \leq d$. This yields the ham-sandwich theorem, as discussed above.

We note that this scheme fails already for $k=1$ if we do not use the full configuration space S^d, e.g. by deleting north and south pole, as then the equivariant map *does exist*.

We also note that for $k>1$ the group action of $\mathfrak{S}_k^\pm$ is not free on $(S^d)^k$, which makes the treatment of the equivariant problem more difficult; however, the action of the subgroup $(\mathbb{Z}/2)^k$ is free—this is used for the upper bound quoted above.

4. A Theorem

Theorem 2. $\Delta(2^t+1,2) = \frac{3}{2}2^t + 2$ *for* $t \geq 1$.

This theorem was previously claimed by Živajević in [11], but we concluded in [2] that his proof technique, which employs a newly-designed "equivariant algebraic obstruction theory," is not valid.

As a corollary (using a reduction detailed in [2]) we obtain that $\frac{3}{2}2^t \leq \Delta(2^t,2) \leq \frac{3}{2}2^t+1$ for $t \geq 1$. It was previously claimed by Ramos in [9] that indeed the lower bound holds, but we concluded in [2] that Ramos' proofs for this are not valid.

Sketch of proof. This is known to hold for $t=1$, so we may assume $t \geq 2$.

(i) For parameters $2d = 3j+1$ we try to prove that there is no equivariant map

$$S^d \times S^d \quad \longrightarrow_{\mathfrak{S}_2^\pm} \quad S^{3j-1}$$

induced by a test map. By restricting to the equator spheres (which parameterize linear hyperplanes in $\mathbb{R}^d$), we obtain

$$\Psi : S^{d-1} \times S^{d-1} \quad \longrightarrow_{\mathfrak{S}_2^\pm} \quad S^d \times S^d \quad \longrightarrow_{\mathfrak{S}_2^\pm} \quad S^{3j-1},$$

which is a map between orientable manifolds of the same dimensions, which has to have degree 0 if $d > 2$.

(ii) Let $N \subset S^{d-1} \times S^{d-1}$ be the non-free subset (that is, all points where the 8-element dihedral group $\mathfrak{S}_2^{\pm}$ has non-trivial stabilizer).

If Ψ, Ψ' are test maps for different families of measures, then we observe that $\Psi|_N \simeq \Psi'|_N$ as on the non-free part the test maps go to an affine subspace that is not linear (does not contain the origin) and thus they can be connected by a linear equivariant homotopy. From this we get $\deg \Psi \equiv \deg \Psi' \pmod 8$ from a generalized equivariant Hopf theorem [7].

(iii) We compute $\deg \Psi'$ for some specific measures on the moment curve without support at the origin, by looking at zeros of the extended map $\hat{\Psi}' : S_+^d \times S^{d-1} \to \mathbb{R}^{3j}$, where S_+^d denotes the upper hemisphere. When d is odd these zeros come in pairs with opposite orientations, so they cancel and $\deg \Psi' = 0$. When d is even the zeros come in pairs with the same orientations, and we count solutions to get

$$\deg \Psi' = 2\binom{j}{\frac{j-1}{2}}.$$

This is nonzero mod 8 if and only if $j = 2^t + 1$ by Kummer's criterion [6] from 1852. $\square$

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The last Grünbaum hyperplane mass partition problem

Pavle V. M. Blagojević

Branko Grünbaum in his paper [2] from 1960 suggested the following innocent-looking problem.

The Grünbaum mass partition problem: Can any convex body in $\mathbb{R}^d$ be cut into 2^d pieces of equal volume by d suitably-chosen affine hyperplanes?

As Grünbaum noted, this is quite easy to prove for $d \leq 2$. In 1966 Hadwiger [3] answered Grünbaum's question (positively) for $d = 3$, while solving a problem raised by J. W. Jaworowski (Oberwolfach, 1963).
In 1984 Avis [1] answered Grünbaum's problem negatively for $d \geq 5$. Indeed, one cannot expect a positive answer there, since d hyperplanes in $\mathbb{R}^d$ can be described by d^2 parameters, while the hyperplanes one is looking for need to satisfy $2^d - 1$ independent conditions, and $2^d - 1 > d^2$ for $d > 4$.

In our lecture, we also sketched a possible proof for the remaining open case of $d = 4$; however, based on discussions in Oberwolfach we found after the lecture that our line of proof does not work out. Therefore, the last Grünbaum hyperplane mass partition problem in dimension 4 remains open.

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Eliminating Tverberg Points

Uli Wagner

(joint work with Isaac Mabillard)

Motivated by *topological Tverberg-type problems* and by classical results about *embeddings* (maps without double points), we study the question whether a finite simplicial complex K can be mapped into $\mathbb{R}^d$ without triple, quadruple, or, more generally, *r-fold points* (image points with at least r distinct preimages), for a given multiplicity $r \geq 2$. In particular, we are interested in maps $f \colon K \to \mathbb{R}^d$ that have no *r-Tverberg points*, i.e., no r-fold points with preimages in r *pairwise disjoint* simplices of K, and we seek necessary and sufficient conditions for the existence of such maps.

Generalizing classical results of Van Kampen, Shapiro and Wu [13, 11, 16] regarding embeddings of m-dimensional complexes into $\mathbb{R}^{2m}$, we show that under suitable dimension restrictions a well-known *deleted product criterion* is complete for maps without r-Tverberg points:

Theorem 1. *Suppose that $r \geq 2$, $k \geq 3$, K is a finite simplicial complex of dimension $m = (r-1)k$, and $d = rk$. Then the following statements are equivalent:*

(1) There exists a piecewise-linear map $f : K \to \mathbb{R}^d$ without r-fold Tverberg points.

(2) There exists a map $F\colon K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$ that is equivariant with respect to the natural action of the symmetric group $\mathfrak{S}_n$ on both spaces.

Here, K^r_Δ is the deleted r-fold product *of K, i.e., the subcomplex of the Cartesian product K^r whose cells are the products of r pairwise disjoint simplices of K.*

The existence of the equivariant map in (2) can be checked by equivariant obstruction theory (an appropriate r-fold generalization of the *van Kampen obstruction* has to vanish).

As our main technical tool, we generalize the classical *Whitney trick* (more precisely, its piecewise-linear version, see, e.g., [15, p. 179] or [10, Lemma 5.12]) and show that in codimension 3, pairs of isolated r-Tverberg points of *opposite sign* can be eliminated by local isotopies:

Theorem 2. *Suppose that $r \geq 2$, and that $f\colon M_1 \sqcup \ldots \sqcup M_r \to B^d$ is a proper*[1] *piecewise-linear map in general position, where each M_i is a connected piecewise-linear manifold of dimension $m_i \leq d-3$, $1 \leq i \leq r$, B^d is a d-dimensional piecewise-linear ball, and $\sum_{i=1}^r (d - m_i) = d$.*

If the intersection $f(\sigma_1)\cap\ldots\cap f(\sigma_r)$ consists of pairs of r-fold points of opposite signs, then there are ambient piecewise-linear isotopies $H^i = (H^i_t)\colon B^d \times [0,1] \to B^d \times [0,1]$, $2 \leq i \leq r$, fixed on ∂B^d, such that

$$f(M_1) \cap H^2_1(f(M_2)) \cap \ldots \cap H^r_1(f(M_r)) = \emptyset.$$

The long-standing *topological Tverberg conjecture* ([4, Problem 84] and [1]) asserts that if $r \geq 2$, $d \geq 1$, and $N = (d+1)(r-1)$, then any continous map $f\colon \sigma^N \to \mathbb{R}^d$ from the N-dimensional simplex σ^N to $\mathbb{R}^d$ has an r-Tverberg point. (This is true for *affine* maps, by the classical geometric *Tverberg theorem* [12].) The conjecture was proved by Bajmoczy and Bárány [1] for $r = 2$, by Bárány, Shlosman, and Szűcs [2] for all primes r, and by Özaydin [9] for prime powers r, but the case of arbitrary r has been a long-standing open problem, considered to be one of the most challenging in the area [8, p. 154].

An important inspiration for our work was an old result of Özaydin [9, Theorem 4.2], which guarantees, among other things, that for r *not a prime power*, an equivariant map $K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$ exists whenever $\dim K^r_\Delta \leq d(r-1)$; in particular, this applies if $\dim K \leq \frac{r-1}{r}d$, or if $K = \sigma^N$, $N = (d+1)(r-1)$. One of our guiding ideas was that, together with Özaydin's result, sufficiency of the deleted product criterion might yield an approach to constructing counterexamples to the remaining open cases of the conjecture. Unfortunately, Theorem 1 cannot be applied with $K = \sigma^N$ because the crucial assumption of codimension at least 3 is not satisfied, and we did not know a way around this.

[1]That is, $f^{-1}(\partial B^d) = \partial M_1 \sqcup \ldots \sqcup \partial M_r$.

In a recent breakthrough, following the announcement of our results in the extended abstract [7], Frick [6] found an extremely elegant way to overcome this "codimension 3" difficulty and to obtain counterexamples to the conjecture for $d = 3r + 1$, r not a prime power, by reducing the problem to a lower-dimensional skeleton to which Özaydin's result and Theorem 1 can be applied. Frick's argument exemplifies the *constraints method* of Blagojević–Frick–Ziegler [3]; see also Frick's abstract in the present volume.

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Counterexamples to the topological Tverberg conjecture

Florian Frick

The "topological Tverberg conjecture" states that for given integers $r \geq 2$, $d \geq 1$, $N = (r-1)(d+1)$, and for any continuous map $f\colon \Delta_N \to \mathbb{R}^d$ from the N-simplex Δ_N into $\mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ_N such that

$f(\sigma_1)\cap\cdots\cap f(\sigma_r)\neq\emptyset$. This holds if f is an affine map: this is a reformulation of Tverberg's original theorem [8]. The conjecture for continuous f was introduced, and proven for r a prime, by Bárány, Shlosman and Szűcs [1], and later extended to the case when r is a prime power by Özaydin [6]. The conjecture is trivial for $d = 1$. All other cases have remained open. According to Matoušek [5, p. 154], the validity of the conjecture for general r is one of the most challenging problems in topological combinatorics.

Here we prove the existence of counterexamples to the topological Tverberg conjecture for any r that is not a power of a prime and dimensions $d \geq 3r+1$. Our construction builds on recent work of Mabillard and Wagner [4], from which we first obtain counterexamples to r-fold versions of the van Kampen–Flores theorem. Counterexamples to the topological Tverberg conjecture are then obtained by an additional application of the constraint method of Blagojević, Ziegler and the author [2].

In the conference proceedings version [4] Mabillard and Wagner announced the generalized van Kampen theorem together with an extended sketch of its proof; a full version of the paper is forthcoming. To state the generalized van Kampen theorem, we first need to fix some notation. We refer to Matoušek [5] for further explanations. For a simplicial complex K denote by

$$K^{\times r}_{\Delta(2)} = \{(x_1,\dots,x_r)\in\sigma_1\times\cdots\times\sigma_r \mid \sigma_i \text{ face of } K, \sigma_i\cap\sigma_j=\emptyset\ \forall i\neq j\}$$

the 2-*wise deleted product* of K and by $K^{(d)}$ the d-*skeleton* of K. The space $K^{\times r}_{\Delta(2)}$ is a polytopal cell complex in a natural way (its faces are products of simplices). Denote by W_r the vector space $\{(x_1,\dots,x_r)\in\mathbb{R}^r \mid \sum x_i = 0\}$ with the action by the symmetric group $\mathfrak{S}_r$ that permutes coordinates.

Theorem 1 (Mabillard & Wagner [4, Theorem 3])**.** *Suppose that $r\geq 2$, $k\geq 3$, and let K be a simplicial complex of dimension $(r-1)k$. Then the following statements are equivalent:*

(i) *There exists an $\mathfrak{S}_r$-equivariant map $K^{\times r}_{\Delta(2)}\to S(W_r^{\oplus rk})$.*

(ii) *There exists a continuous map $f\colon K\to\mathbb{R}^{rk}$ such that for any r pairwise disjoint faces $\sigma_1,\dots,\sigma_r$ of K we have $f(\sigma_1)\cap\cdots\cap f(\sigma_r)=\emptyset$.*

An important result on the existence of equivariant maps was shown by Özaydin.

Lemma 2 (Özaydin [6, Lemma 4.1])**.** *Let $d\geq 3$ and G be a finite group. Let X be a d-dimensional free G-CW complex and let Y be a $(d-2)$-connected G-CW complex. There is a G-map $X\to Y$ if and only if there are G_p-maps $X\to Y$ for every Sylow p-subgroup G_p, p prime.*

Özaydin uses this result to prove the existence of $\mathfrak{S}_r$-equivariant maps

$$(\Delta_{(r-1)(d+1)})^{\times r}_{\Delta(2)}\to S(W_r^{\oplus d})$$

for r not a prime power. An initial motivation for Mabillard and Wagner was to use such a map to construct counterexamples to the topological Tverberg conjecture via r-fold versions of the Whitney trick. However, for their approach to work they need codimension $k \geq 3$. Here we first derive counterexamples to r-fold versions of the van Kampen–Flores theorem, which is a Tverberg-type statement with a bound on the dimension of faces, see Corollary 3, from the result of Mabillard and Wagner and Özaydin's work, and eventually obtain counterexamples to the topological Tverberg conjecture by a combinatorial reduction.

Corollary 3. *Let $r \geq 6$ be an integer that is not a prime power and $k \geq 3$ an integer. Then for any N there exists a continuous map $f\colon \Delta_N \to \mathbb{R}^{rk}$ such that for any r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N with $\dim \sigma_i \leq (r-1)k$ we have $f(\sigma_1) \cap \dots \cap f(\sigma_r) = \emptyset$.*

Proof. Let $K = \Delta_N^{((r-1)k)}$ denote the $((r-1)k)$-dimensional skeleton of the simplex Δ_N on $N+1$ vertices. We only need to construct f on K and extend continuously to Δ_N in an arbitrary way. By Theorem 1 we need to show that there exists an $\mathfrak{S}_r$-equivariant map $K^{\times r}_{\Delta(2)} \to S(W_r^{\oplus rk})$. The reasoning is the same as in [6, Proof of Theorem 4.2]: the free $\mathfrak{S}_r$-space $K^{\times r}_{\Delta(2)}$ has dimension at most $d = r(r-1)k$, and $S(W_r^{\oplus rk}) \cong S^{(r-1)rk-1}$ is $(d-2)$-connected. By Lemma 2 the existence of an $\mathfrak{S}_r$-map $K^{\times r}_{\Delta(2)} \to S(W_r^{\oplus rk})$ reduces to the existence of equivariant maps for Sylow p-subgroups, but p-groups have fixed points in $S(W_r^{\oplus rk})$ for r not a prime power by [6, Lemma 2.1], so a constant map will do. □

The existence of the $\mathfrak{S}_r$-equivariant map $K^{\times r}_{\Delta(2)} \to S(W_r^{\oplus rk})$ also follows immediately from [6, Theorem 4.2] by observing that an n-dimensional, finite, free $\mathfrak{S}_r$-complex always admits an equivariant map into an $(n-1)$-connected $\mathfrak{S}_r$-space, see for example Matoušek [5, Lemma 6.2.2].

Any r generic affine subspaces of dimension $(r-1)k$ in $\mathbb{R}^{rk}$ intersect in a point by codimension reasons. Here we see that a continuous map $\Delta_N^{((r-1)k)} \to \mathbb{R}^{rk}$ can avoid this intersection, and indeed a map without any such intersection exists for any N, but only if r is not a prime power. Volovikov [9] proved that a map as postulated by Corollary 3 does not exist if r is a prime power and $N \geq (r-1)(d+2)$ — the case r prime was proved by Sarkaria [7]; see [2] for more general results with significantly simplified proofs.

The map f in Corollary 3 could not be constructed if the topological Tverberg conjecture were true, since the validity of the topological Tverberg conjecture would imply such an intersection result for faces of bounded dimension by the constraint method. For the sake of completeness we will present a construction that does not rely on [2].

Theorem 4 (The topological Tverberg conjecture fails)**.** *Let $r \geq 6$ be an integer that is not a prime power, and let $k \geq 3$ be an integer. Let $N = (r-1)(rk+2)$. Then there exists a continuous map $F\colon \Delta_N \to \mathbb{R}^{rk+1}$ such that for any r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N we have $F(\sigma_1) \cap \dots \cap F(\sigma_r) = \emptyset$.*

Proof. Let $f\colon \Delta_N \to \mathbb{R}^{rk}$ be a continuous map as constructed in Corollary 3, that is, such that for any r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N with $\dim \sigma_i \leq (r-1)k$ we have $f(\sigma_1) \cap \dots \cap f(\sigma_r) = \emptyset$. Define $F\colon \Delta_N \to \mathbb{R}^{rk+1}$, $x \mapsto$

$(f(x), \mathrm{dist}(x, \Delta_N^{((r-1)k)}))$. Suppose there were r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N such that there are points $x_i \in \sigma_i$ with $F(x_1) = \cdots = F(x_r)$. By restricting to subfaces if necessary we can assume that x_i is in the relative interior of σ_i. Then all the x_i have the same distance to the $(r-1)k$-skeleton of Δ_N.

Suppose all σ_i had dimension at least $(r-1)k+1$. Then these faces would involve at least $r((r-1)k+2) = (r-1)(rk+2)+2 > N+1$ vertices. Thus, one face σ_j has dimension at most $(r-1)k$ and $\mathrm{dist}(x_j, \Delta_N^{((r-1)k)}) = 0$. But then we have $\mathrm{dist}(x_i, \Delta_N^{((r-1)k)}) = 0$ for all i, so $x_i \in \Delta_N^{((r-1)k)}$ and thus $\sigma_i \subseteq \Delta_N^{((r-1)k)}$ for all i. This contradicts our assumption on f. □

If the topological Tverberg conjecture holds for r pairwise disjoint faces and dimension $d+1$, then it also holds for dimension d and the same number of faces. Thus, we are only interested in low-dimensional counterexamples. If r is not a prime power then the topological Tverberg conjecture fails for dimensions $3r+1$ and above. Hence, the smallest counterexample this construction yields is a continuous map $\Delta_{100} \to \mathbb{R}^{19}$ such that any six pairwise disjoint faces have images that do not intersect in a common point.

This and further applications of these methods will be presented in [3].

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M-vector analogue for the *cd*-index

Kalle Karu

The well-known conjecture of McMullen [2], proved by Billera, Lee [1] and Stanley [5], describes which numbers can occur as the face numbers of simplicial polytopes. The condition is that the toric g-vector is an M-vector, that is, the vector of dimensions of graded pieces of a standard graded algebra A.

A similar conjecture was made recently by Murai and Nevo [3] for the cd-index of a Gorenstein* poset P. The coefficients of the cd-index are conjectured to be the dimensions of graded pieces in a standard multi-graded algebra A. To be more precise, associate to every cd-monomial M of degree n a zero-one vector $v \in \mathbb{Z}^n$ and let $\psi_{P,v}$ be the coefficient of the monomial M in the cd-index of P. Then the conjecture is that there exists a $\mathbb{Z}^n$-graded algebra A, generated in single degree one, such that for any $v \in \mathbb{Z}^n$

$$\dim A_v = \psi_{P,v}.$$

We give numerical evidence for this conjecture by proving:

Theorem 1. *Let P be a shellable Gorenstein* poset of rank $n+1$. Then for any $v, w \in \mathbb{Z}^n$*

$$\Psi_{P,v+w} \leq \Psi_{P,v} \cdot \Psi_{P,w}.$$

The proof of this theorem follows the proofs of weaker inequalities (but for more general Gorenstein* posets) given by Murai-Nevo [3] and Murai-Yanagawa [4].

For simplicial spheres we prove the conjecture by explixitly constructing the algebra A using lattice paths:

Theorem 2. *The conjecture of Murai and Nevo is true for Gorenstein* simplicial complexes.*

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Hyperspace configurations and universal Gröbner bases

Aldo Conca
(joint work with Emanuela De Negri, Elisa Gorla)

1. Regularity and products of ideals

Let K be a field and $S = K[x_1, \dots, x_n] = \bigoplus_{i \geq 0} S_i$ be the polynomial ring. Here S_i denotes the K-vector space of the homogeneous polynomials of degree $i \in \mathbb{N}$. One of the most important homological invariant of a finitely generated graded S-module M is its Castelnuovo-Mumford regularity:

$$\operatorname{reg}(M) = \max\{j - i : \beta_{i,j}(M) \neq 0\}$$

where $\beta_{i,j}(M)$ denotes the (i, j)-th graded Betti number of M. Denote by $t_0(M)$ the largest degree of a minimal generator of M and observe that $t_0(M) \leq \operatorname{reg}(M)$. Let I, J be homogeneous ideals of S. By definition one has $t_0(IJ) \leq t_0(I) + t_0(J)$. This observation leads to the question of whether the regularity behaves similarly, that is, whether $\operatorname{reg}(IJ) \leq \operatorname{reg}(I)+\operatorname{reg}(J)$ holds in general. The answer is negative, in [10, 2] many examples are given showing that $\operatorname{reg}(I^2)$ can be larger that $2\operatorname{reg}(I)$. But there are families of ideals behaving very well in this respect. One of them is associated to hyperspace configurations.

2. Hyperspace configurations: ideals

Let

$$W = \{W_i\}_{i=1}^m$$

be a projective hyperspace configuration, i.e. a collection of linear subspaces of the projective space $\mathbb{P}_K^{n-1}$. Repetitions and inclusions among the W_i's are allowed. Consider the defining ideal I_i of W_i, that is, $I_i = \{f \in S \ : \ f(p) = 0 \ \forall \ p \in W_i\}$. Being W_i a linear subspace, I_i is generated by a subspace V_i of S_1, i.e., the linear homogeneous defining equations of W_i. Hence we may as well describe W by means of the subvector space configuration

$$V = \{V_i\}_{i=1}^m$$

of S_1. Since each I_i is generated by a regular sequence of linear forms, one has $\operatorname{reg}(I_i) = 1$. It has been proved in [4] that

$$\operatorname{reg}(I_1 I_2 \cdots I_m) = m.$$

This result is a consequence of a (surprisingly simple) primary decomposition formula for the ideal $I_1 I_2 \cdots I_m$:

$$I_1 I_2 \cdots I_m = \bigcap_{\emptyset \neq A \subseteq [m]} \left(\sum_{i \in A} I_i \right)^{\#A}$$

Furthermore it is proved in [7] that the graded Betti numbers of $I_1 I_2 \cdots I_m$ are combinatorial invariants, that is, they depend only on the intersection lattice of W. More precisely, the graded Betti numbers depend only on the function

$$f : \{A : A \subseteq [m]\} \to \mathbb{N}$$

defined by:

$$f(A) = \dim_K \left(\bigcap_{i \in A} W_i \right).$$

3. Hyperspace configurations: algebras

Another algebraic object attached to the hyperplane configuration W is the K-subalgebra $K[V]$ of S generated by the elements in the product $V_1 V_2 \cdots V_m \subset S_m$. We have:

Theorem 1. For every hyperspace configuration W the K-algebra $K[V]$ is Koszul and normal.

In [3] we have proved Theorem 1 when the V_i's are generated by subsets of $\{x_1, \dots, x_n\}$. (i.e. the W_i are coordinate hyperspaces) and in the "generic" case (i.e. the V_i are vector spaces of a given dimension generated by generic linear forms).

When the V_i's are generated by variables the K-algebra $K[V]$ is the toric ring attached to a "transversal" discrete polymatroid. Discrete polymatroids are a non-square free generalization of matroids. White's conjecture, originally stated in [11] for matroids but then extended to discrete polymatroids in [8], asserts that $K[V]$ is defined by the quadrics associated to the so-called symmetric exchange property. Since Koszul algebras are defined by quadrics, Theorem 1 proves, for transversal polymatroid, the part of White conjecture that asserts that the algebra is defined by quadrics. That the generators of the defining ideal is generated by the exchange relations base has been verified in [9]. An important breakthrough in [9] is the proof of White conjecture "up to saturation".

Furthermore in [3] we have reduced the proof of the Theorem 1 (in the general case) to the following assertion:

Theorem 2. Let $S = K[x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$ with graded structure induced by $\deg(x_{ij}) = e_i \in \mathbb{Z}^m$. Let $L = (L_{ij})$ be a $m \times n$ matrix with $L_{ij} = \sum_{k=1}^{n} \alpha_{ijk} x_{ik}$ and $\alpha_{ijk} \in K$. Let I_2 be the ideal generated by the 2-minors of L. Then I_2 has a universal Gröbner basis of elements of $\mathbb{Z}^m$-degree bounded above by $(1, 1, \dots, 1) \in \mathbb{Z}^m$.

Theorem 2 has been proved firstly in [3] for "generic" L_{ij}, then in [1] for linearly independent L_{ij} and finally in [6] in general. The proof of Theorem 2 given in [6] has two main ingredients. First, a rigidity statement, proved in [5], for Borel-fixed radical ideal in the multigraded setting. Secondly, a detailed analysis the structure and of the general properties of ideals that have the multigraded Hilbert series of a radical Borel-fixed multigraded ideal. A typical family of ideals of this kind was described in [1] by Cartwright and Sturmfels and plays an important role in their studies of the multigraded Hilbert scheme associated to the Segre product of two projective spaces.

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Stanley depth and the lcm lattice

Lukas Kattän

(joint work with Bogan Ichim, Julio José Moyano Fernández)

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \ldots, x_n]$ the $\mathbb{Z}^n$-graded polynomial ring. A *Stanley decomposition* of a finitely generated $\mathbb{Z}^n$-graded S-module M is a finite vector space decomposition

$$M = \bigoplus_i m_i \mathbb{K}[Z_i] \tag{1}$$

where the $m_i \in M$ are homogeneous and $Z_i \subset \{x_1, \ldots, x_n\}$. The depth of such a decomposition is defined as the depth of the right-hand side of (1) and the *Stanley depth* of M is the maximum depth of a Stanley decomposition of M. The *Stanley conjecture* [8, Conj 5.1] asks if the Stanley depth of every finitely generated graded S-module is bounded below by its depth.

If M is the Stanley-Reisner ring of a Cohen-Macaulay simplicial complex Δ, then this conjecture specializes to the question whether Δ admits a partition

$$\Delta = \bigcup_i [C_i, D_i]$$

into intervals, such that every upper bound D_i is a facet of Δ. In fact, by combining results of Herzog, Soleyman Jahan and Zheng [2] and my coauthors and myself

[3], the Stanley conjecture for cyclic modules S/I is equivalent to the partition conjecture.

For a monomial ideal $I \subset S$ consider its *lcm-lattice* L_I, i.e. the lattice of all least common multiples of a generating set of I. It is known that the isomorphism type of L_I determines the projective dimension of S/I [1]. In order to formulate an analogous result for the Stanley depth, we define the *Stanley projective dimension* of a module M as the minimal projective dimension of a Stanley decomposition of M. We have the following result:

Theorem 1 ([4]). *The Stanley projective dimensions of S/I and I depend only on the isomorphism type of L_I.*

More general, let $I \subset S$ and $I' \subset S'$ be two monomial ideals in two polynomial rings S and S'. If there exists a surjective join-preserving map $L_I \to L_{I'}$, then $\operatorname{Stpdim} S/I \geq \operatorname{Stpdim} S'/I'$ *and* $\operatorname{Stpdim} I \geq \operatorname{Stpdim} I'$.

This result allows defines the Stanley projective dimension of a finite lattice and to define a version of the Stanley conjecture in this context. A natural question is the relation to the Stanley projective dimension and the classical invariants of finite lattices. For example, the length and the order-dimension are both known to give upper bounds for the Stanley projective dimension [7].

A consequence of Theorem 1 is the following: If $I \subset S$ and $I' \subset S'$ are two monomial ideals with the same projective dimension, such that there exists a surjective join-preserving map $L_I \to L_{I'}$, then is it enough to prove the Stanley conjecture for I. Generalizing this, we are led to study those ideals whose lcm lattice is not the image of another lcm lattice with the same number of generators and the same projective dimension. Let us call these lcm lattices *maximal*. So, to prove the Stanley conjecture for cyclic modules, one only needs to study ideals with maximal lcm lattice. The maximal lattices are characterized by the following.

Theorem 2 ([6]). *There is a bijection between the set $L(k,p)$ of maximal lattices with k atoms and projective dimension p, and the set $ST(k,p)$ of acyclic $(p-1)$-dimensional simplicial complexes on k vertices, which are acyclic.*

These complexes have already been studied by Kalai [5] and are known as spanning trees of skeletons of a simplex. The bijection maps a lattice to its Scarf complex. The inverse map is given by

$$L(\Delta) := \{V \subset [k] \,:\, \Delta|_V \text{ is acyclic}\}.$$

Using Theorem 2, we can prove the Stanley conjecture for ideals with up to six generators, cf. [6].

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Cambrian Trees

Vincent Pilaud

(joint work with Grégory Chatel, Carsten Lange)

Cambrian trees provide a natural generalization of binary search trees, where each node can have one parent and two children, or two parents and one child. Reinterpreting Nathan Reading's work [Rea06] in terms of trees, we present the Cambrian correspondence, a bijection between signed permutations and leveled Cambrian trees. This yields surjective maps ϕ from signed permutations to Cambrian trees, and ψ from Cambrian trees to binary sequences. We explore applications of Cambrian trees and of these maps in three directions:

Combinatorics. Cambrian trees provide a simple model for the type A Cambrian lattices defined by Nathan Reading in [Rea06]. The surjective maps ϕ and ψ define lattice homomorphisms from the weak order to the Cambrian lattice, and from the Cambrian lattice to the boolean lattice. See Figure 1.

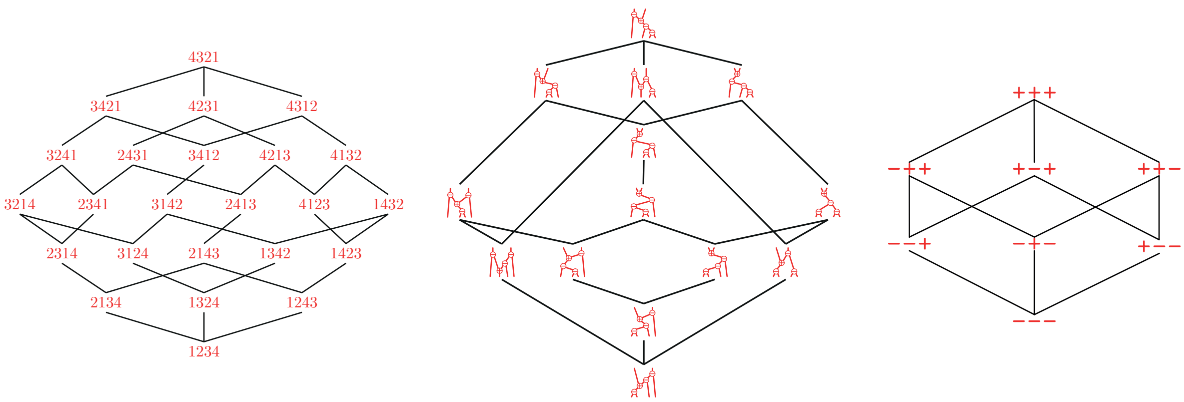

Figure 1. The weak order, a Cambrian lattice, and the boolean lattice.

Geometry. Cambrian trees naturally encode the maximal normal cones of Christophe Hohlweg and Carsten Lange's realizations of the associahedron [HL07], generalizing Jean-Louis Loday's construction [Lod04]. The maps ϕ and ψ correspond to the refinement of the normal cones of the permutahedron, these associahedra, and the cube. See Figure 2.

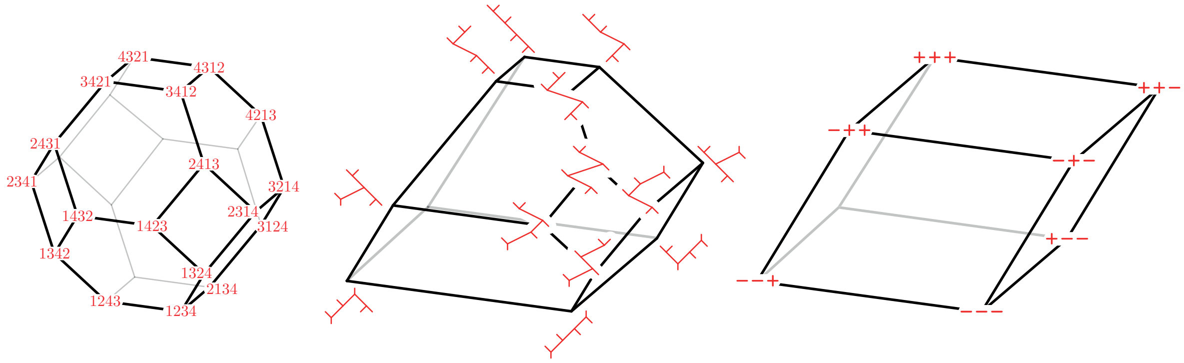

Figure 2. The permutahedron, an associahedron, and the cube.

Algebra. Cambrian trees provide a convenient tool to define a Hopf algebra structure, called the Cambrian algebra, and generalizing the binary tree algebra by Jean-Louis Loday and Maria Ronco [LR98]. The maps ϕ and ψ translate to Hopf algebra inclusions between the signed version of the algebra on permutations by Claudia Malvenuto and Christophe Reutenauer [MR95], the Cambrian algebra [CP14], and the recoils algebra by Louis Solomon [Sol76].

The presentation is based on joint works with Grégory Chatel [CP14] and Carsten Lange [LP13].

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High Dimensional Expansion

Roy Meshulam

(joint work with Alexander Lubotzky and Shahar Mozes)

Expander graphs have been a focus of intensive research in the last four decades, with numerous applications throughout mathematics and theoretical computer science (see [2, 4]). In view of the ubiquity of expander graphs, there a recent growing interest in high dimensional notions of expansion. The k-dimensional version of the graphical Cheeger constant, called "coboundary expansion", came up independently in the work of Linial, Meshulam and Wallach [3, 7] on homological connectivity of random complexes and in Gromov's remarkable work [1] on the topological overlap property.

We proceed with the formal definitions. Let X be a finite n-dimensional pure simplicial complex. For $k \geq 0$, let $X^{(k)}$ denote the k-dimensional skeleton of X and let $X(k)$ be the family of k-dimensional faces of X, $f_k(X) = |X(k)|$. Define a positive weight function $w = w_X$ on the simplices of X as follows. For $\sigma \in X(k)$, let $c(\sigma) = |\{\eta \in X(n) : \sigma \subset \eta\}|$ and let

$$w(\sigma) = \frac{c(\sigma)}{\binom{n+1}{k+1} f_n(X)}.$$

Note that $\sum_{\sigma \in X(k)} w(\sigma) = 1$. Let $C^k(X)$ denote the space of $\mathbb{F}_2$-valued k-cochains of X with the coboundary map $d_k : C^k \to C^{k+1}$. The space of k-coboundaries of X is $B_k(X) = d_{k-1}C^{k-1}(X)$. For $\phi \in C^k(X)$, let $[\phi]$ denote the image of ϕ in $C^k(X)/B^k(X)$. Let

$$\|\phi\| = \sum_{\{\sigma \in X(k) : \phi(\sigma) \neq 0\}} w(\sigma)$$

and

$$\|[\phi]\| = \min\{\|\phi + d_{k-1}\psi\| : \psi \in C^{k-1}(X)\}.$$

Definition 1. The *k-th coboundary expansion constant* of X is

$$h_k(X) = \min\left\{\frac{\|d_k\phi\|}{\|[\phi]\|} : \phi \in C^k(X) - B^k(X)\right\}.$$

Clearly, $h_k(X) > 0$ if and only if $\tilde{H}_k(X; \mathbb{F}_2) = 0$. Indeed, $h_k(X)$ can be thought of as a measure of the resiliency of the property of having trivial k-dimensional $\mathbb{F}_2$-cohomology, under deletion of $(k+1)$-simplices from X.

As in the classical 1-dimensional case, one basic question concerns the existence of bounded degree higher dimensional expanders. Let $D_k(X)$ be the maximum number of $(k+1)$-dimensional faces of X containing a common k-dimensional face. A complex X is a (k, d, ϵ)-*expander* if

$$D_{k-1}(X) \leq d \text{ and } h_{k-1}(X) \geq \epsilon.$$

In joint work with Lubotzky [5] we establish the existence of an infinite family of $(2, d, \epsilon)$-expanders for some fixed d and $\epsilon > 0$. Our proof is probabilistic and depends on the following new model, based on Latin squares, of random 2-dimensional simplicial complexes with bounded edge degrees. Let $\mathbb{S}_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$. A k-tuple $(\pi_1, \ldots, \pi_k) \in \mathbb{S}_n^k$ is *legal* if $\pi_i \pi_j^{-1}$ is fixed point free for all $1 \leq i < j \leq k$. A *Latin Square* of order n is a legal n-tuple of permutations $L = (\pi_1, \ldots, \pi_n) \in \mathbb{S}_n^n$. Let $\mathcal{L}_n$ denote the uniform probability space of all Latin squares of order n. Let $V_1 = \{a_i\}_{i=1}^n, V_2 = \{b_i\}_{i=1}^n, V_3 = \{c_i\}_{i=1}^n$ be three disjoint sets. The complete 3-partite complex $T_n = V_1 * V_2 * V_3$ consists of all $\sigma \subset V = V_1 \cup V_2 \cup V_3$ such that $|\sigma \cap V_i| \leq 1$ for $1 \leq i \leq 3$. An $L = (\pi_1, \ldots, \pi_n) \in \mathcal{L}_n$

determines a subcomplex $T_n^{(1)} \subset Y(L) \subset T_n$ whose 2-simplices are $[a_i, b_j, c_{\pi_i(j)}]$ where $1 \leq i, j \leq n$. In particular $Y(L)$ has $3n^2$ edges and every edge lies in a unique 2-simplex, i.e. $D_1(Y(L)) = 1$. Fix d and regard $\mathcal{L}_n^d$ as a uniform probability space. For $\underline{L}^d = (L_1, \ldots, L_d) \in \mathcal{L}_n^d$, let $Y(\underline{L}^d) = \cup_{i=1}^d Y(L_i)$. Note that $D_1(Y(\underline{L}^d)) \leq d$. Let $\mathcal{Y}(n,d)$ denote the probability space of all complexes $Y(\underline{L}^d)$ with measure induced from $\mathcal{L}_n^d$.

Theorem 2 ([5]). *There exist $\epsilon > 0, d < \infty$ such that*

$$\lim_{n\to\infty} Pr[Y \in \mathcal{Y}(n,d) : h_1(Y) > \epsilon] = 1.$$

Another natural problem is to provide lower bounds on the expansion of highly symmetric complexes. Let G be a subgroup of $Aut(X)$ and let S be a finite G-set. For $0 \leq k \leq n-1$, let $\mathcal{F}_k = S \times X(k)$ with a G-action given by $g(s,\tau) = (gs, g\tau)$. Let

$$\mathcal{B} = \{B_{s,\tau} : -1 \leq k < n, (s,\tau) \in \mathcal{F}_k\}$$

be a family of subcomplexes of X such that $\tau \in B_{s,\tau} \subset B_{s,\tau'}$ for all $s \in S$ and $\tau \subset \tau' \in X^{(n-1)}$.

Definition 3. A *building-like complex* is a 4-tuple $(X, S, G, \mathcal{B})$ as above with the following properties:

(C1) G is transitive on $X(n)$.
(C2) $gB_{s,\tau} = B_{gs,g\tau}$ for all $g \in G$ and $(s,\tau) \in S \times X^{(n-1)}$.
(C3) $\tilde{H}_i(B_{s,\tau}) = 0$ for all $(s,\tau) \in \mathcal{F}_k$ and $-1 \leq i \leq k < n$.

Examples of building-like complexes include basis-transitive matroid complexes and spherical buildings. In joint work with Lubotzky and Mozes [6] we give a lower bound on the expansion of building-like complexes. For a simplex $\eta \in X$, let $G\eta$ denote the orbit of η under G. For $0 \leq k \leq n-1$, let

$$a_k = a_k(X, S, G, \mathcal{B}) = \max\{|G\eta \cap B_{s,\tau}(k+1)| : \eta \in X(k+1), (s,\tau) \in \mathcal{F}_k\}.$$

Theorem 4 ([6]). *Let $(X, S, G, \mathcal{B})$ be an n-dimensional building-like complex. Then for $0 \leq k \leq n-1$,*

$$h_k(X) \geq \left(\binom{n+1}{k+2} a_k \right)^{-1}. \tag{1}$$

Example: Let $G = \langle B, N \rangle$ is a finite group with BN-pair of rank $n+1$ and let $W = N/(B \cap N)$ be its Weyl group. Let $\Delta = \Delta(G; B, N)$ be the associated spherical building. Theorem 4 implies the following

Corollary 5 ([1, 6]).

$$h_{n-1}(\Delta(G; B, N)) \geq \frac{1}{|W|}.$$

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Random simplicial complexes

Nati Linial

(joint work with L. Aronshtam, T. Łuczak, R. Meshulam and Y. Peled)

Random graphs have had a huge impact on modern combinatorics and on various application domains. On the theoretical side they are used in order to show the existence of graphs with unexpected properties. It has become a standard practice to establish the existence of graphs with various desired properties using the probabilistic method. In some cases one can later come up with explicit constructions with similar features, but in numerous cases only probabilistic proofs of existence are known and the explicit construction still alludes us. In addition, random graphs play an important role in modeling natural phenomena in biology, statistical physics, social science and more. All in all, random graphs have become a great scientific success story.

Generally speaking, graphs are our method of choice when we study large systems that are defined in terms of pairwise interactions. However, we are currently much less equipped when we try to mathematically model systems whose underlying interactions involve more than two constituents. Keeping in mind that a graph is a one-dimensional simplicial complex, it is natural address this problem by developing an analogous theory for higher-dimensional simplicial complexes.

The systematic study of random graphs started in the late 50's and early 60's when Erdős and Rényi developed the theory of random $G(n,p)$ graphs. Their discoveries have inspired much of our work. About ten years ago, Linial and Meshulam [4] introduced $X_d(n,p)$ - a model of random d-dimensional simplicial complexes. Such a complex X has n vertices and a full $(d-1)$-skeleton, and each d-face is included in X independently and with probability p. Note that $X_1(n,p)$ is identical with $G(n,p)$. Linial and Meshulam sought a higher-dimensional analog of Erdős and Rényi's discovery that the threshold probability for graph connectivity is $p = \frac{\log n}{n}$. It was shown in [4] and subsequently in [6] that in $X_d(n,p)$ the threshold for the vanishing of the $(d-1)$-st homology with an arbitrary finite Abelian group of coefficients is $p = \frac{d \log n}{n}$.

A fascinating aspect of the $G(n,p)$ theory is the so-called *phase transition* that occurs at $p = \frac{1}{n}$. At this point two main things happen: (i) The graph almost surely ceases to be a forest, (ii) The giant (of $\Omega(n)$ vertices) component emerges. We have sought high-dimensional analogs of these phenomena, but before we could do that, some conceptual issues had to be addressed: There are at least two natural analogs of (i), namely, the collapsibility of all d-faces, or the vanishing of the d-th homology. With regards to (ii), note that there is not even a well defined notion of connected components, so clearly a new idea is needed.

In several recent papers [3, 1, 2, 5] we found that in $X_d(n,p)$:

- The threshold probability for the collapsibility of all d-faces is $p = (1+o(1))\frac{\log d}{n}$.
- The threshold probability for the vanishing of the d-th homology is $p = \frac{d+1-o(1)}{n}$.

Regarding point (ii) and the emergence of a giant component we first had to view the graph-theoretic ($d=1$) situation from a new perspective. Let G be a graph and e an edge not in G. We say that e is in the *shadow* of G if the addition of e to G creates a new cycle. It is not hard to see that the following two events occur in $G(n,p)$ concurrently: (a) The emergence of the giant ($\Omega(n)$ vertices) component and (b) The emergence of a giant ($\Omega(n^2)$ edges) shadow. The notion of a shadow extends naturally to higher dimensional simplicial complexes, and in [5] we show that concurrent with the vanishing of the d-homology, a giant shadow of $\Omega(n^{d+1})$ d-faces emerges. Moreover, there is a substantial difference in the way the giant shadow emerges in different dimensions. Whereas in the one-dimensional case of graphs this is a second-order phase transition, in dimensions 2 and above, this is a much more abrupt first-order phase transition.

A recurring theme in all these four papers is the work within the theory of *weak local limits*, although this point is fully developed and made explicit only in [5].

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On the topology of the poset of weighted partitions

Michelle Wachs
(joint work with Rafael González D'León)

In [7] we study a weighted version of the partition lattice, introduced by Dotsenko and Khoroshkin [3] in their study of Koszulness of certain quadratic binary operads. The maximal intervals of the weighted partition poset provide a generalization of the lattice Π_n of partitions of the set $[n] := \{1, 2, \dots, n\}$. We show that various classical topological results for Π_n generalize to the maximal intervals of the weighted partition poset. We use these results to study Lie algebras with two compatible brackets. Here we report on some of the results of [7]

Let us first recall some topological and representation theoretic properties of Π_n. The Möbius invariant of Π_n is given by

$$\mu_{\Pi_n}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!.$$

It was proved by Björner [1], using an edge labeling of Stanley [9], that Π_n is EL-shellable; consequently the order complex $\Delta(\overline{\Pi}_n)$ of the proper part $\overline{\Pi}_n$ of the partition lattice Π_n has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $n-3$.

A basis for cohomology $\tilde{H}^{n-3}(\overline{\Pi}_n)$ of the order complex $\Delta(\overline{\Pi}_n)$, obtained from the ascent-free chains of the EL-labeling of Π_n, can be described in terms of Lyndon trees with n leaves. Using Björner's [2] NBC basis construction for geometric lattices, one can describe the dual basis for homology $\tilde{H}_{n-3}(\overline{\Pi}_n)$ in terms of increasing rooted trees on node set $[n]$.

The symmetric group $\mathfrak{S}_n$ acts naturally on Π_n and this action induces isomorphic representations of $\mathfrak{S}_n$ on $\tilde{H}_{n-3}(\overline{\Pi}_n)$ and on $\tilde{H}^{n-3}(\overline{\Pi}_n)$. The symmetric group $\mathfrak{S}_n$ also acts naturally on the multilinear component $Lie(n)$ of the free Lie algebra on n generators. It is a classical result that there is an $\mathfrak{S}_n$-module isomorphism,

$$\tilde{H}_{n-3}(\overline{\Pi}_n) \simeq_{\mathfrak{S}_n} Lie(n) \otimes sgn_n, \tag{1}$$

where sgn_n is the sign representation of $\mathfrak{S}_n$.

A weighted partition of $[n]$ is a set $\{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$ where $\{B_1, B_2, ..., B_t\}$ is a partition of $[n]$ and $v_i \in \{0, 1, 2, ..., |B_i| - 1\}$ for all i. The *poset of weighted partitions* Π_n^w is the set of weighted partitions of $[n]$ with covering relation given by

$$\{A_1^{w_1}, A_2^{w_2}, ..., A_s^{w_s}\} \lessdot \{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$$

if the following conditions hold:

- $\{A_1, A_2, \dots, A_s\} \lessdot \{B_1, B_2, \dots, B_t\}$ in Π_n
- if $B_k = A_i \cup A_j$, where $i \neq j$, then $v_k - (w_i + w_j) \in \{0, 1\}$
- if $B_k = A_i$ then $v_k = w_i$.

The poset Π_n^w has a minimum element

$$\hat{0} := \{\{1\}^0, \{2\}^0, \dots, \{n\}^0\}$$

and n maximal elements

$$\{[n]^0\},\ \{[n]^1\},\ldots,\{[n]^{n-1}\}.$$

Thus there are n maximal intervals $[\hat{0},\{[n]^i\}]$, where $i=0,\ldots,n-1$. Note that the maximal intervals $[\hat{0},\{[n]^0\}]$ and $[\hat{0},\{[n]^{n-1}\}]$ are isomorphic to Π_n.

The $\mathfrak{S}_n$-module isomorphism (1) can be generalized. Let $Lie_2(n)$ be the multilinear component of the free Lie algebra on n generators with two compatible brackets and let $Lie_2(n,i)$ be the component of $Lie_2(n)$ generated by bracketed permutations with i brackets of one type and $n-1-i$ brackets of the other type. The symmetric group acts naturally on each $Lie_2(n,i)$ and on each open interval $(\hat{0},\{[n]^i\})$. It follows from operad theoretic results of Vallette [10] and Dotsenko-Khoroshkin [4] that the following $\mathfrak{S}_n$-module isomorphism holds:

$$\tilde{H}_{n-3}((\hat{0},\{[n]^i\}))\simeq_{\mathfrak{S}_n} Lie_2(n,i)\otimes sgn_n. \tag{2}$$

We give an explicit $\mathfrak{S}_n$-module isomorphism that generalizes an isomorphism for (1) given by Wachs [11].

EL-shellability of the maximal intervals can be obtained by generalizing the Björner-Stanley EL-labeling of Π_n mentioned above. The ascent-free chains of this EL-labeling yield a basis for cohomology of each maximal interval that can be described in terms of a colored version of the Lyndon trees. The explicit isomorphism of (2) enables one to transfer the colored Lyndon basis to $Lie_2(n,i)$. We show that the Möbius invariant of the ith maximal interval is given up to sign by the number of rooted trees on node set $[n]$ having i descents. The formula

$$\dim Lie_2(n)=n^{n-1},$$

which was first obtained by Dotsenko and Khoroshkin [3] and by Liu [8], is a consquence of this result and (2).

We construct an elegant basis for $\tilde{H}_{n-3}((\hat{0},\{[n]^i\}))$ that generalizes Björner's NBC basis for $\tilde{H}_{n-3}(\Pi_n)$. This basis is constructed by "splitting" rooted trees with i descents. It is not dual to the colored Lyndon basis for cohomology obtained from the ascent-free chains of the EL-labeling, however. Thus the proof that our construction yields a basis is considerably more difficult than that for Π_n.

A more general weighted partition poset is introduced by González D'Leon [5, 6] in order to study Lie algebras with an arbitrary number of compatible brackets. In a forthcoming paper we will study another general weighted partition poset, which is obtained by assigning weights to the bonds of an arbitrary graph on n-vertices.

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Quasitoric manifolds, colored simple polytopes and the Lebesgue, KKM, and Hex theorems

Rade Živaljević
(joint work with Djordje Baralić)

1. Introduction

Following and developing ideas of R. Karasev [2] we extend the Lebesgue theorem (on covers of cubes) and the Knaster-Kuratowski-Mazurkiewicz theorem (on covers of simplices) to different classes of convex polytopes (colored in the sense of M. Joswig). We also show that the n-dimensional Hex theorem admits a generalization where the n-dimensional cube is replaced by a n-colorable simple polytope. The use of specially designed *quasitoric manifolds*, with easily computable cohomology rings and the cohomological cup-length, offers a great flexibility and versatility in applying the general method.

2. Colorful Lebesgue theorem

Theorem 2.1. (Lebesgue) *If the unit cube $[0,1]^n$ is covered by a finite family $\{X_i\}_{i\in I}$ of closed sets so that no point is included in more than n sets, then one of them must intersect two opposite facets of the cube.*

Theorem 2.2. (Colorful Lebesgue theorem) *Suppose that an n-colorable simple polytope P^n is covered by a family of closed sets $P^n = \cup_{i=1}^N X_i$ such that each point $x \in P^n$ is covered by no more than n of the sets X_j. Then for some i, a connected component of X_i intersects at least two distinct facets of P^n of the same color.*

Theorem 2.3. *Suppose that P^n is an n-colorable simple polytope, M^{2n} its canonical quasitoric manifold, and $\pi : M^{2n} \to P^n$ the associated projection map. Let $\omega = v_1 + \ldots + v_n$ be the 2-dimensional 'vertex class' associated to a vertex $V \in P^n$. Suppose that $\mathcal{F} = \{X\}_{i=1}^N$ is a finite family of closed subsets of P^n such that each X_i intersects at most one of the facets in each of the color classes. If the covering multiplicity of $\mathcal{F}$ is at most $k \leq n$ then there exists a connected component Z of* *the set $P^n \setminus \cup_{i=1}^N X_i$ which is ω^{n-k}-essential in the sense that the restriction of the class ω^{n-k} on $\pi^{-1}(Z)$ is non-trivial. Moreover, if $\mathcal{K}$ is the collection of all k-faces K of P^n such that $Z \cap K \neq \emptyset$ then $\mathcal{K}$ contains a collection of k-faces of size at least 2^{n-k} which are all in the same I-color class for some $I = \{i_1, \dots, i_{n-k}\} \subset [n]$.*

3. Colorful KKM-theorem

Theorem 3.1. (KKM) *If a non-degenerate simplex $\Delta^n \subset \mathbb{R}^n$ is covered by a finite family $\{F_i\}_{i\in I}$ of closed sets so that no point is covered more than n times then one of the sets F_i intersects all the facets of $\mathbb{R}^n$.*

Definition 3.2. A simple polytope P^n is called *specially $(n+1)$-colorable* if the associated coloring function $h : \{F_1, \dots, F_m\} \to [n+1]$ has the property that all facets in $h^{-1}(n+1)$ are n-simplices.

Theorem 3.3. (Colorful KKM theorem) *Let P^n be a specially $(n+1)$-colorable polytope in the sense of Definition 3.2. Suppose that P^n is covered by a family of closed sets $P^n = \cup_{i=1}^N X_i$ with the covering multiplicity $\leq n$ (i.e. each point $x \in P^n$ is covered by no more than n of the sets X_j). Then there exists $i \in [N]$ and a connected component Y_i of X_i such that among the faces of P^n intersected by Y_i are facets of all $n+1$ colors.*

Theorem 3.4. *Let P^n be a specially $(n+1)$-colorable polytope (Definition 3.2). Suppose that P^n is covered by a family of closed sets $P^n = \cup_{i=1}^N X_i$ with the covering multiplicity $k \leq n$ and there is no X_i intersecting some $n+1$ distinct colored facets. Then there exists a connected component W of $P^n \setminus \cup_{i=1}^N X_i$ which is t^{n-k}-essential in the sense that the restriction of the class t^{n-k} on $\pi^{-1}(W) \subset M^{2n}$ is non-trivial. Moreover, the collection of all k-faces K of P^n such that $W \cap K \neq \emptyset$ contains the k-skeleton of some simplicial face T_i and at least $\binom{n}{k}$ k-faces of P^n not contained in T_i.*

4. Colorful Hex Theorem

Theorem 4.1. (Colorful Hex theorem) *Suppose that P^n is an n-colorable simple polytope and let $h : [m] \to [n]$ be a selected coloring function which associates to each facet F_j the corresponding color $h(F_j) = h(j)$. Let V be a vertex of P^n and let $\{F_{\nu_i}\}_{i=1}^n$ be the collection of all facets of P^n which contain V such that $h(\nu_i) = i$. Suppose that the polytope P^n is covered by a family of n closed sets $P^n = \cup_{i=1}^n X_i$. Then for some i, a connected component of X_i intersects both F_{ν_i} and some other facet F_j of P^n colored by the color i.*

4.1. **A generalized Game of Hex.** Let $S \subset \mathbb{R}^n$ be a finite set of points and let $\{V_x\}_{x\in S}$ the associated Voronoi partition of $\mathbb{R}^n$. Let P^n be a simple n-colorable polytope with m facets and an associated coloring function $h : [m] \to [n]$. Choose a vertex V of P^n and let $\{F_{\nu_i}\}_{i=1}^n$ be the collection of all facets containing the vertex V such that $h(\nu_i) = i$.

There are n players $J_1, \dots, J_n$. The first player chooses a point $x_1 \in S$ and colors the corresponding Voronoi cell V_{x_1} by the color 1. The second player chooses a point $x_2 \in S \setminus \{x_1\}$ and colors the Voronoi cell V_{x_2} by color 2, etc. After the first round of the game the first player chooses a point $x_{n+1} \in S \setminus \{x_1, \dots, x_n\}$, etc. The game continues until one of the players (say the player J_i) creates a connected monochromatic set of cells (all of color i) which connect the facet F_{ν_i} with one of the facets F_j such that $h(j) = i$. Alternatively the game ends if there are no more points in S to distribute among players. An easy application of Theorem 4.1 shows that the game will always be decided i.e. sooner or later one of the players will win the game.

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“Once upon a time, there were k lattice points inside a rational polytope...”

Jesús Antonio De Loera

(joint work with Iskander Aliev, Quentin Louveaux)

The topic of this report is to consider the properties of rational convex polytopes that contain k lattices points within. We are particularly interested in the computational aspects of the problem. This report presents two themes discussed in my lecture which summarized three new papers [1, 2, 3].

1. A Quantitative Integral Version of Helly's theorem

In a *Helly-type theorem*, there is a family of objects F, a property P and a constant μ such that if every subfamily of F with μ elements has property P, then the entire family has property P (this topic expands a large literature, we recommend [4, 7, 10, 14] and the references there for a glimpse of this fertile subject). One of the most famous Helly-type theorems, due to its many applications in the theory of integer programming and computational geometry of numbers is the 1973 theorem of Doignon [9], later reproved by Bell and Scarf [5, 13].

Theorem. Let A be a $m \times n$ integer matrix and b an vector in $\mathbb{Z}^m$. If the set $\{x \in \mathbb{Z}^n : Ax \leq b\}$ is empty, then there is a subset S of the rows of A, of cardinality no more than 2^n, with the property that the set $\{x \in \mathbb{Z}^n : A_S x \leq b_S\}$ is also empty.

Our key contribution is to a quantitive generalization of Doignon-Bell-Scarf's theorem, one that is close in spirit to the quantitative versions of Helly theorem of Bárány Katchalski and Pach [4]. Note that the original Doignon theorem is simply the case of $k = 0$:

Theorem 1.1. *Given n, k two non-negative integers there exists a universal constant $c(n,k)$, depending only on k and n, such that for any $m \times n$ integer matrix A, and m-vector b if the polyhedron $\{x \in \mathbb{R}^n : Ax \leq b\}$ has exactly k integer solutions, then there is a subset S of the rows of A, of cardinality no more than $c(n,k)$, with the property that the polyhedron $\{x \in \mathbb{R}^n : A_S x \leq b_S\}$ has exactly the same k integer solutions as $\{x \in \mathbb{R}^n : Ax \leq b\}$.*

The upper bound we prove in this paper is $c(n,k) = \lceil 2(k+1)/3 \rceil 2^n - 2\lceil 2(k+1)/3 \rceil + 2$.

We were also able to obtain lower bounds for the case $k = 1$ proving that our upper bound is in fact the exact value of $c(n,1)$. Given a polyhedron $P \subset \mathbb{R}^n$, we say that an inequality $f^T x \leq g$ in the description of P is *k-necessary* if the removal of $f^T x \leq g$ from P results in the inclusion of at least one additional integer point in the interior of P.

Theorem 1.2. *There exists a polyhedron P in $\mathbb{R}^n$ that has exactly one interior integer point, $2(2^n - 1)$ facets and one integer point in the relative interior of each facet. Thus all inequalities in P are k-necessary. As a consequence, the upper bound of Theorem 1.1 is tight for $k = 1$ and thus $c(n,1) = 2(2^n - 1)$.*

We discuss several consequences and conjectures.

2. Parametric Polytopes with k lattice points and the k-Frobenius number

Here we consider parametric polytopes of the form $\{x : Ax = b, x \geq 0\}$ with an integer matrix $A \in \mathbb{Z}^{d \times n}$ and a vector $b \in \mathbb{Z}^d$. This is a polyhedra semigroup $\mathrm{Sg}(A) = \{b : b = Ax,\ x \in \mathbb{Z}^n, x \geq 0\}$, and one has a "stratification" by the number of lattice points inside the polytope of fixed element b. One may have *at least* k lattice solutions, *exactly* k, or *less than* k. The classification of right-hand side vectors b in a semigroup according with cardinality is of great interest. We have several contributions to the study of k-feasibility for the semigroup $\mathrm{Sg}(A)$, and the associated polyhedra.

(1) First, we prove a structural result that implies that the set $\mathrm{Sg}_{\geq k}(A)$ of b's inside the semigroup $\mathrm{Sg}(A)$ that provide $\geq k$-feasible fibers $IP_A(b)$ is finitely generated.

Let $\mathrm{Sg}_{\geq k}(A)$ (respectively $\mathrm{Sg}_{=k}(A)$ and $\mathrm{Sg}_{<k}(A)$) be the set of right-hand side vectors $b \in \mathrm{cone}(A) \cap \mathbb{Z}^d$ that make $IP_A(b) \geq k$-feasible (respectively $= k$-feasible, $< k$-feasible). Note that $\mathrm{Sg}_{\geq 1}(A)$ is equal to $\mathrm{Sg}(A)$, the semigroup generated by the column vectors of the matrix A.

We give an algebraic description of the sets $\mathrm{Sg}_{\geq k}(A)$ and $\mathrm{Sg}_{<k}(A)$. Let $e_1, \ldots, e_n$ be the standard basis vectors in $\mathbb{Z}^n_{\geq 0}$. We define the *coordinate subspace of* $\mathbb{Z}^n_{\geq 0}$ of dimension $r \geq 1$ determined by $e_{i_1}, \ldots, e_{i_r}$ with $i_1 < \cdots < i_r$ as the set $\{e_{i_1} z_1 + \cdots + e_{i_r} z_r : z_j \in \mathbb{Z}_{\geq 0} \text{ for } 1 \leq j \leq r\}$. By the 0-dimensional coordinate subspace of $\mathbb{Z}^n_{\geq 0}$ we understand the origin $0 \in \mathbb{Z}^n_{\geq 0}$.

Theorem 2.1. $\mathrm{Sg}_{\geq k}(A)$ *is a finite union of translated copies of the semigroup* $A\mathbb{Z}^n_{\geq 0}$*, more precisely*

(i) There exists a monomial ideal $I^k(A) \subset \mathbb{Q}[x_1, \ldots, x_n]$ *such that*

$$\mathrm{Sg}_{\geq k}(A) = \{A\lambda : \lambda \in E^k(A)\}\,, \tag{1}$$

where $E^k(A)$ *is the set of exponents of monomials in* $I^k(A)$*.*

(ii) The set $\mathrm{Sg}_{<k}(A)$ *can be written as a finite union of translates of the sets* $\{A\lambda : \lambda \in S\}$*, where* S *is a coordinate subspace of* $\mathbb{Z}^n_{\geq 0}$*.*

(2) We propose several ways to compute the k-holes, i.e., $\mathrm{Sg}_{<k}(A)$, of the semigroup $\mathrm{Sg}(A)$. First using Hilbert bases, but then we are able to use generating functions for an efficient algorithm to detect all the $\geq k$-feasible vectors b's, not explicitly one by one, but rather the entire set of k-feasible vectors is encoded as a single multivariate *generating function*, $\sum_{\geq k-\text{feasible}} t^b$.

Let a be a positive integral n-dimensional primitive vector, i.e., $a = (a_1, \ldots, a_n)^T \in \mathbb{Z}^n_{>0}$ with $\gcd(a_1, \ldots, a_n) = 1$. For a positive integer k the *k-Frobenius number* $\mathrm{F}_k(a)$ is the largest number which cannot be represented in *at least* k different ways as a non-negative integral combination of the a_i's.

A key contribution of this paper is a generalization of the celebrated theorem of R. Kannan [11] for $k = 1$.

Corollary 2.2. *Consider the parametric knapsack problem* $a^T x = b$*,* $x \geq 0$ *associated with the vector* $a = (a_1, \ldots, a_n)^T \in \mathbb{Z}^n_{>0}$ *with* $\gcd(a_1, \ldots, a_n) = 1$*. For fixed positive integers* k *and* n*, the* k*-Frobenius number can be computed in polynomial time.*

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Generalized Tchebyshev triangulations

Eran Nevo
(joint work with Gábor Hetyei)

After fixing a triangulation L of a k-dimensional simplex that has no new vertices on the boundary, we introduce a triangulation operation on all simplicial complexes that replaces every k-face with a copy of L, via a sequence of induced subdivisions. The operation may be performed in many ways, see Example and Figure below, but we show that the face numbers of the subdivided complex depend only on the face numbers of the original complex, in a linear fashion. We use this linear map to define a sequence of polynomials generalizing the Tchebyshev polynomials of the first kind; the latter correspond to the case where L is the path with two edges. Our results generalize results from [1] already for this case. We show that in many cases, but not all, the resulting polynomials have only real roots, located in the interval $(-1, 1)$. Some analogous results are shown also for generalized Tchebyshev polynomials of the higher kind, defined using summing over links of all original faces of a given dimension in our generalized Tchebyshev triangulations. Generalized Tchebyshev triangulations of the boundary complex of a cross-polytope play a central role in our calculations, and for some of these we verify the validity of a generalized lower bound conjecture in [2].

Recall that the Tchebyshev polynomials $T_n(x)$ of the first kind satisfy:
(1) their degree is n and their coefficients are integer,
(2) symmetry: $(-1)^n T_n(-x) = T_n(x)$, and
(3) all their roots are real, and
(4) all the (real) roots are simple and belong to the interval $(-1, 1)$.

Let the *F-polynomial* of a simplicial complex K be

$$F(K, x) = \sum_{j \geq 0} f_{j-1}(K)(\frac{x-1}{2})^j,$$

where $f_{j-1}(K)$ is the number of faces of K of size j. Then,

Theorem 1. *Fix L as above. Then there is a unique linear map $T^L : \mathbb{R}[x] \to \mathbb{R}[x]$ that takes $F(K,x)$ to $F(K',x)$, for any complex K and any subdivision K', induced by L by the above procedure.*

Example 2. Let L be the path with 2 edges (triangulating the 1-simplex), K be the union of the two triangles $\{v_1, v_2, v_3\}$ and $\{v_1, v_2, v_4\}$ sharing the edge $\{v_1, v_2\}$. Let K' be the generalized Tchebyshev triangulation of K defined by the ordering of edges $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_2, v_3\}$, $\{v_1, v_4\}$, $\{v_2, v_4\}$ and let K'' be the generalized Tchebyshev triangulation of K defined by the ordering $\{v_1, v_3\}$, $\{v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_1, v_4\}$, $\{v_1, v_2\}$. Then K' is a cone (over an 8-cycle) and K'' is not, see Fig. 1. However, K' and K'' have the same face numbers.

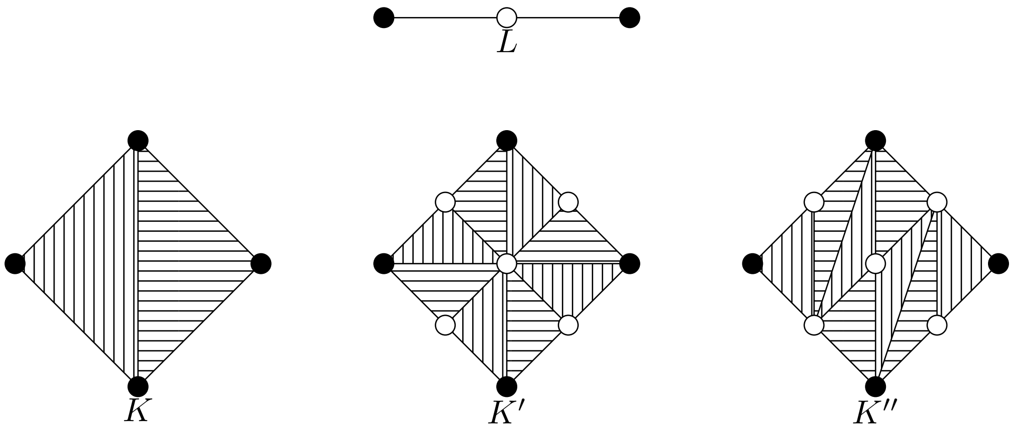

Figure 1. Illustration to Example 2

We define

$$T_n^L(x) := T^L(x^n),$$

the n^{th} *generalized Tchebyshev polynomial of the first kind* associated with L. The following combinatorial interpretation holds: let $\diamond^n$ denote the boundary complex of an n-dimensional cross-polytope, and $\diamond'$ (any of) its triangulation induced by L.

Corollary 3. *$T_n^L(x) = F(\diamond', x)$.*

One can then show, using the h-vector of $\diamond'$, that

$$T_n^L(x) = \frac{1}{2^n} \sum_{i=0}^{n} h_i(\diamond')(x-1)^i(x+1)^{n-i}.$$

This interpretation together with the Dehn-Sommerville relations and the nonnegativity of the h-vector of $\diamond'$ imply that (2) and (4) above hold for $T_n^L(x)$ as well. While (3) does not hold for L in general (there are examples for any dimension > 2), we show it does hold if $\dim L \leq 2$. As for (1) above, it is easily proved using the following recurrence relation for $T_n^L(x)$: define the two-variable polynomial

$$r_L(u,v) = \sum_{\sigma \in L \setminus \partial L} u^{|\sigma \cap V(\partial L)|} v^{|\sigma \cap V(\mathrm{int}(L))|} - u^{k+1},$$

which counts for any i, j how many faces L has with i vertices on the boundary and j vertices in the interior. Then

Theorem 4. *The polynomials $T_n^L(x)$ satisfy $T_n^L(x) = x^n$ for $n \leq k$, and for all $n \geq k+1$ the following recurrence holds:*

$$T_n^L(x) = \sum_{j=1}^{k+1} p_j^L(x) T_{n-j}^L(x),$$

where each $p_j^L(x)$ is the polynomial of x equals to the coefficient of t^j in $(-2t)^{k+1} r_L\left(-\frac{1+t}{2t}, -\frac{1+x}{2}\right)$.

We close by an open question:
For which L are all roots of $T_n^L(x)$ real for any n?

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Balanced generalized lower bound inequality for simplicial polytopes

Satoshi Murai
(joint work with Martina Juhnke-Kubitzke)

The study of face numbers of convex polytopes is one of the central problems in combinatorics. In particular, face numbers of simplicial polytopes have been of great interest and studied by many researchers (see [St2]). We prove a new necessary condition for face numbers of balanced simplicial polytopes.

For a simplicial d-polytope P, let $f_i(P)$ be the number of i-dimensional faces of P and let $h(P) = (h_0(P), h_1(P), \ldots, h_d(P))$ be the *h-vector* of P, where $h_i(P)$ is the number defined by

$$h_i(P) = \sum_{j=0}^{i} (-1)^{j-i} \binom{d-j}{i-j} f_{j-1}(P)$$

for $i = 0, 1, \ldots, d$. A simplicial d-polytope P with the vertex set V is said to be *balanced* if there is a map $\kappa : V \to [d] = \{1, 2, \ldots, d\}$ satisfying $\kappa(x) \neq \kappa(y)$ for all vertices x and y that forms an edge of P. The following result is known as the Generalized Lower Bound Theorem (GLBT for short).

Theorem 1 (Generalized Lower Bound Theorem)**.** *Let P be a simplicial d-polytope. Then*

$$h_0(P) \leq h_1(P) \leq \cdots \leq h_{\lfloor \frac{d}{2} \rfloor}(P).$$

Moreover, $h_{i-1}(P) = h_i(P)$ for some $i \leq \frac{d}{2}$ if and only if P is $(i-1)$-stacked, that is, P can be triangulated without introducing faces of dimension $< d - i$.

The GLBT was conjectured by McMullen and Walkup in 1971, and was settled recently: The inequalities were proved by Stanley [St1], the if part of the equality case was proved by McMullen and Walkup [MW], and the only if part of the equality case was proved by Nevo and the author [MN]. Inspired by the GLBT, Klee and Novik [KN, Conjecture 5.5] proposed the following conjecture.

Conjecture 2 (Balanced Generalized Lower Bound Conjecture)**.** *Let P be a balanced simplicial d-polytope. Then*

$$\frac{h_0(P)}{\binom{d}{0}} \leq \frac{h_1(P)}{\binom{d}{1}} \leq \cdots \leq \frac{h_{\lfloor \frac{d}{2} \rfloor}(P)}{\binom{d}{\lfloor \frac{d}{2} \rfloor}}.$$

Moreover, one has $\frac{h_{i-1}(P)}{\binom{d}{i-1}} = \frac{h_i(P)}{\binom{d}{i}}$ for some $i \leq \frac{d}{2}$ if and only if P has the balanced $(i-1)$-stacked property.

Note that a simplicial d-polytope is said to have the *balanced $(i-1)$-stacked property* if there is a regular CW-complex χ satisfying that (a) any maximal cell of χ is combinatorially isomorphic the boundary of a cross d-polytope, (b) χ is a homology d-ball whose boundary is equal to that of P, (c) χ has no interior faces of dimension $< d-i$.

The first two inequalities of the conjecture and the equality case of the conjecture for $i \leq 2$ are known to be true (see [KN]). We give an affirmative answer to the first part of the Balanced Generalized Lower Bound Conjecture.

Theorem 3. *If P is a balanced simplicial d-polytope, then we have*

$$\frac{h_0(P)}{\binom{d}{0}} \leq \frac{h_1(P)}{\binom{d}{1}} \leq \cdots \leq \frac{h_{\lfloor \frac{d}{2} \rfloor}(P)}{\binom{d}{\lfloor \frac{d}{2} \rfloor}}.$$

To study the Balanced Generalized Lower Bound Conjecture, it is important to consider rank-selected subcomplexes. We regard a simplicial d-polytope on the vertex set V as an abstract simplicial complex on V. For a balanced simplicial d-polytope P and a subset $S \subset [d]$, the simplicial complex

$$P_S = \{F \in P : \kappa(F) = S\}$$

is called a *rank-selected subcomplex* of P. It was proved in [KN] that

$$(1) \quad \binom{d}{i-1} h_i(P) - \binom{d}{i} h_{i-1}(P) = C \cdot \left(\sum_{S \subset [d],\ \#S = 2i-1} h_i(P_S) - h_{i-1}(P_S) \right),$$

where $C = \binom{d}{i} / \binom{d-i+1}{i}$. To prove Theorem 3, we show the following algebraic statement for the Stanley–Reisner rings of rank-selected subcomplexes. We refer the readers to [St2] for basics on Stanley–Reisner rings.

Theorem 4. *Let P be a balanced simplicial d-polytope, $S \subset [d]$ and let R be the Stanley–Reisner ring of P_S over $\mathbb{Q}$. There are a linear system of parameters Θ of R and a linear form ω such that the multiplication map*

$$\omega^{\#S-2i} : (R/\Theta R)_i \to (R/\Theta R)_{\#S-i}$$

is injective for $i \leq \frac{\#S}{2}$.

The above theorem proves Theorem 3 since it implies $h_i(P_S) \geq h_{i-1}(P_S)$ in (1). It also gives a partial affirmative answer to the question posed by Björner and Swartz [Sw, Problem 4.2] who asked if the Stanley–Reisner ring of every doubly Cohen–Macaulay simplicial complex satisfies the conclusion of Theorem 4.

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On the Topology of Steel

Frank H. Lutz

(joint work with Emanuel A. Lazar, Robert D. MacPherson, Jeremy K. Mason)

Polycrystalline materials, such as metals, are composed of crystal grains of varying size and shape. Typically, the occurring grain cells have the combinatorial types of 3-dimensional simple polytopes, and together they tile 3-dimensional space.

Kelvin's famous equal-volume foam [8] provides a periodic tiling of 3-space with truncated octahedra, i.e., a tessellation with a single cell type that has 14 faces — and Kelvin asked in 1887 whether his structure is best possible in the sense that it has smallest surface area per equal-volume cell. It took until 1993 for a negative answer to Kelvin's problem, when Weaire and Phelan [9] presented a more efficient periodic equal-volume foam consisting of two cell types, two dodecahedra and six 14-sided cells per fundamental domain, thus yielding an average number 13.5 of faces per cell; cf. [1].

Polycrystalline metal foams evolve from initial random Voronoi decompositions of 3-space that have grains with on average 15.535 faces [7]. For a recent grain growth model and its implementation see [2, 5]. Evolved steady state decompositions of 3-space tend to have a face average of about 13.7; cf. [2] and references contained therein. In contrast to Kelvin's foam and the Weaire–Phelan foam, polycrystalline metal foams are composed of a large number of different cell types, where some of the occurring combinatorial types are substantially more frequent than others [2, 3, 6].

Based on data from grain growth simulations [2], we see that frequent grains are *combinatorially round* in the following sense. The simplicial duals of the 2-dimensional simple boundaries of the grains are flag simplicial 2-spheres or are obtained from flag simplicial 2-spheres or the boundary of the simplex by only few stacking operations [4]. To be precise, in evolved simulated data, 99.72% of the grain duals are flag simplicial 2-spheres or are obtained by at most 5 stackings. The infrequent simplicial grain duals are *severely constricted*, i.e., they have no vertices of degree 3, but have a separating cycle in their 1-skeleton of length 3 (an empty triangle).

For any flag simplicial 2-sphere with n faces, we define its *penalty value* by

$$p = n \cdot \sum_i x_i \cdot i \cdot (n - i - 4),$$

with x_i the number of separating 4-cycles in the 1-skeleton that give a split of type $i : 4 : (n - i - 4)$. The penalty value p then allows to predict how frequent the corresponding grain may occur in a simulated grain microstructure: grains with high penalty value are infrequent.

An analysis over time further shows that the grains in initial random Voronoi decompositions are less round than in evolved data sets, with an optimum obtained during the grain growth process at the point in time when an average of faces per grain close to 14, the Kelvin value, is reached. This value can possibly be of interest for optimizing steel microstructures by quenching to freeze the microstructures at the optimum.

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Representation stability in the homology of set partitions

Victor Reiner
(joint work with Patricia Hersh)

We study representation stability in the sense of Church and Farb [2] for representations of the symmetric group S_n on the cohomology of the configuration space of n ordered points in $\mathbb{R}^d$.

Church [1] considered connected oriented d-manifolds X with $d \geq 2$, whose cohomology (with coefficients in $\mathbb{Q}$) is finite-dimensional. For such manifolds X, he showed that the i^{th} cohomology of the configuration space of n ordered points in X will vanish unless $d-1$ divides i, and that its S_n-representations eventually *stabilizes* in a precise sense. In particular, he showed that this stability will occur at least for $n \geq 2i$ when $d \geq 3$, and for $n \geq 4i$ if $d = 2$.

Using results on Whitney homology for the lattice of partitions of $\{1, 2, \ldots, n\}$ due to Sundaram [4], and to Sundaram-Welker [3], we show that for $X = \mathbb{R}^d$, the stabilization occurs significantly earlier. Specifically it begins sharply at $n = \frac{3}{d-1}i$ if d is odd, and sharply at $n = \frac{3}{d-1} + 1$ if d is even. We speculate that the stabilization can never occur earlier for (connected, orientable) d-manifolds X.

The methods come from bounding the lengths of parts among the partitions that occur in the Schur function expansions of the Whitney homology, as these lengths control the onset of stabilization.

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Geometric complexes for subsets of the circle and beyond

Michal Adamaszek
(joint work with Henry Adams, Florian Frick, Francis Motta, Chris Peterson and Corrine Previte-Johnson)

This extended abstract is a survey of the works [1, 2, 3].

Suppose that M is a metric space and $X \subseteq M$ is an arbitrary subset. For any distance parameter $r \geq 0$ the Čech complex and the Vietoris–Rips complex are two geometric constructions which capture the "proximity at scale r" between the points of X. They are simplicial complexes with vertex set X and with faces defined as follows:

- $\sigma \in \check{\mathrm{C}}\mathrm{ech}(X, M; r)$ iff $\bigcap_{x\in\sigma} B(x,r) \neq \emptyset$,
- $\tau \in \mathrm{VR}(X; r)$ iff $\mathrm{dist}(x,y) \leq r$ for all $x, y \in \tau$,

where $B(x,r)$ denotes the ball of radius r around the point x in M.

Alternatively, $\check{\mathrm{C}}\mathrm{ech}(X, M; r)$ is the nerve of the family of balls $\{B(x,r)\}_{x\in X}$. The complex $\mathrm{VR}(X;r)$ is the clique complex of the underlying Vietoris–Rips graph on X with edges between pairs of points in distance at most r. In a geodesic space the latter graph is the intersection graph of the family of balls $\{B(x,r/2)\}_{x\in X}$.

These constructions have been used from the early days of homology theory as a way of creating simplicial models for metric spaces [7]. Nowadays they have fundamental applications in topological data analysis. In this setting X is typically a finite sample of some subset M of $\mathbb{R}^d$. The persistent homology of $\check{\mathrm{C}}\mathrm{ech}(X, M; r)$ or $\mathrm{VR}(X;r)$, obtained by varying the parameter r, is a tool to recover the topological features of M. See for instance [4]. The motivation for using Vietoris–Rips complexes comes from the work of Hausmann and Latschev [5, 6], who show that if M is a closed Riemannian manifold, if r is sufficiently small compared to the injectivity radius of M, and if X is sufficiently dense or $X = M$, then $\mathrm{VR}(X;r)$ recovers the homotopy type of M. As the main idea of persistent homology is to allow r to vary, we would like to understand what happens when r is not sufficiently small, but nothing has been known about the behaviour of $\check{\mathrm{C}}\mathrm{ech}(M, M; r)$ or $\mathrm{VR}(M;r)$ for any manifold M when r is arbitrary.

Our results so far concentrate on the case when $M = S^1$ is the circle. For simplicity we assume that S^1 is given the arc-length metric scaled so as to have circumference 1. In this case the balls $B(x,r)$ are closed arcs and our first result is a statement about arbitrary collections of arcs.

Theorem 1. ([1]) *Let C be an arbitrary finite collection of arcs in S^1. Then the nerve complex of C as well as the clique complex of the intersection graph of C have the homotopy type of a point, an odd-dimensional sphere, or a wedge of even-dimensional spheres of the same dimension.*

The intersection graphs of arc configurations C form the class known as circular-arc graphs. Specializing to all arcs of the same length yields the classification of homotopy types of $\check{\mathrm{C}}\mathrm{ech}(X, S^1; r)$ and $\mathrm{VR}(X;r)$ for arbitrary finite subsets $X \subseteq S^1$. In [1] we also study the geometry of $\check{\mathrm{C}}\mathrm{ech}(X, S^1; r)$ for evenly spaced subsets $X \subseteq S^1$, in particular relating them to the classical cyclic polytopes.

For a given subset $X \subseteq S^1$ and any r the homotopy type of $\mathrm{VR}(X;r)$ can be determined efficiently by means of a combinatorial invariant of the pair (X, r) which we call the *winding fraction* [2, 3]. Suppose, for simplicity, that $r < \frac{1}{2}$ and $|X| < \infty$. For $x \in X$ let $f_r(x)$ denote the furthest point of X located in clockwise distance at most r from x. By iterating f_r we obtain a discrete dynamical system on X. One can show that the quantity

$$\mathrm{wf}(X;r) := \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathrm{dist}(f_r^{(i)}(x), f_r^{(i+1)}(x))$$

does not depend on the choice of $x \in X$. Then we obtain the following results.

Theorem 2. ([2]) *For $0 < r < \frac{1}{2}$ and $X \subseteq S^1$ with $|X| < \infty$ we have homotopy equivalences*

$$\mathrm{VR}(X;r) \simeq \begin{cases} S^{2l+1} & \text{if } \frac{l}{2l+1} < \mathrm{wf}(X;r) < \frac{l+1}{2l+3} \text{ for some } l \geq 0, \\ \bigvee^{c-1} S^{2l} & \text{if } \mathrm{wf}(X;r) = \frac{l}{2l+1}, \end{cases}$$

where c is the number of orbits of f_r.

A similar statement is also available when $|X| = \infty$. With a suitable limiting procedure we obtain the next result.

Theorem 3. ([2]) For $0 < r < \frac{1}{2}$ we have homotopy equivalences

$$\mathrm{VR}(S^1;r) \simeq \begin{cases} S^{2l+1} & \text{if } \frac{l}{2l+1} < r < \frac{l+1}{2l+3} \text{ for some } l \geq 0, \\ \bigvee^{\mathfrak{c}} S^{2l} & \text{if } r = \frac{l}{2l+1}. \end{cases}$$

To the best of our knowledge this is the first full calculation of $\mathrm{VR}(M;r)$ when M is a manifold. It confirms for $M = S^1$ some conjectures of [5], for instance that the connectivity $\mathrm{conn}(\mathrm{VR}(M;r))$ is a non-decreasing function of r.

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Solving S-unit and Mordell equations via Shimura–Taniyama conjecuture

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(joint work with Rafael von Känel)

In this talk certain geometry of numbers (discrete geometry) aspects of the project [5] have been presented. The plan of the present abstract is as follows. We first summarize the content of [5]. Then we briefly discuss S-unit and Mordell equations and we state parts of the results of [5]. Finally we mention two problems related to non-convex polytopes, which are motivated by [5].

Summary. Mordell and S-unit equations are classical Diophantine equations. In [5] we construct two types of practical algorithms that solve S-unit and Mordell equations. The first type builds on Cremona's algorithm using modular symbols. The second type combines explicit height bounds with sieving and enumeration algorithms. Here we conduct some effort to work out optimized height bounds and to construct refined enumeration algorithms (e.g. we develop a refined de Weger sieve and we obtain a global elliptic logarithm sieve). To illustrate the utility of our algorithm we solved large classes of S-unit and Mordell equations, and we used the resulting data to motivate various questions (e.g. Baker's explicit *abc*-conjecture) related to these fundamental Diophantine equations. Furthermore we establish new results for Mordell equations, which for example directly imply improved versions of two old theorems of Coates on the difference of coprime squares and cubes. Our results and algorithms all crucially rely on the Shimura–Taniyama conjecture [8, 6, 2] combined with the method of Faltings [3] (Arakelov, Paršin, Szpiro) and they do not use the theory of logarithmic forms.

Mordell and S-unit equations. Let S be a finite set of rational primes and let $N_S = \prod_{p \in S} p$. Denote by $\mathcal{O} = \mathbb{Z}[1/N_S]$ the ring of S-integers and by $\mathcal{O}^\times$ their units, the S-units. We consider the classical S-unit equation

$$x + y = 1, \quad x, y \in \mathcal{O}^\times. \tag{1}$$

Many important Diophantine problems can be reduced to the study of S-unit equations. For example, the *abc*-conjecture of Masser–Oesterlé is equivalent to a certain height bound for the solutions of S-unit equations. On using Diophantine approximations à la Thue–Siegel, Mahler (1933) showed that (1) has only finitely many solutions. Furthermore there already exists a practical algorithm of de Weger [7] which solves the S-unit equation (1) using the theory of logarithmic forms [1]. Next we take a non-zero $a \in \mathcal{O}$ and we consider the Mordell equation

$$y^2 = x^3 + a, \quad x, y \in \mathcal{O}. \tag{2}$$

This equation is a priori more difficult than (1). The simplest case $\mathcal{O} = \mathbb{Z}$ of equation (2) goes back at least to Bachet (1621). For this case, using a Diophantine approximation result of Thue, Mordell (1923) showed that (2) has only finitely many solutions. Furthermore there already exist practical algorithms which resolve Mordell equations (2) by using the theory of logarithmic forms [1].

Algorithm for S-unit equation. We first describe the main ingredients for our algorithm solving the S-unit equation (1) by using height bounds. The Shimura–Taniyama conjecture together with Faltings' method leads to explicit height bounds for the S-unit equation (1). For practical purposes, these bounds are the actual best ones. However they are still too large to check all candidates with height below this bound. A method of de Weger [7], using Diophantine approximation and the LLL lattice reduction algorithm, can considerably reduce these bounds in practice. In [5] we refined the reduction method, we worked out a refined sieve which is very efficient for sets S of cardinality at least 6, and we developed an improved enumeration of solutions with very small height. Here we used ideas and methods from geometry of numbers (discrete geometry).

We now discuss some results which we obtained by using the above described algorithm. To solve the S-unit equation (1), it is natural to consider the set $\Sigma(S)$ of solutions of (1) modulo symmetry. Here two solutions (x, y) and (x', y') are called symmetric if x' or y' lies in $\{x, \frac{1}{x}, \frac{1}{1-x}\}$. Previously, de Weger [7] computed the set $\Sigma(S)$ in the case $S = \{2, 3, 5, 7, 11, 13\}$. We obtained the following theorem.

Theorem 1. *Let $n \in \{1, 2, \ldots, 16\}$ and let $S(n)$ be the set of the n smallest rational primes. The cardinality # of $\Sigma(S(n))$ is given in the following table.*

n	1	2	3	4	5	6	7	8
#	1	4	17	63	190	545	1433	3649

n	9	10	11	12	13	14	15	16
#	8828	20015	44641	95358	199081	412791	839638	1663449

Let $N \in \{1, 10, \ldots, 10^7\}$. If $\Sigma(N) = \cup_S \Sigma(S)$ with the union taken over all sets S with $N_S \le N$, then the cardinality # of $\Sigma(N)$ is given in the following table.

N	1	10	10^2	10^3	10^4	10^5	10^6	10^7
#	0	5	42	354	2362	13902	79125	432408

In fact our algorithm completely determined the sets $\Sigma(S(n))$ and $\Sigma(N)$ appearing in the above theorem. Further we remark that given the set $\Sigma(S)$, one can directly write down all solutions of the S-unit equation (1).

Mordell equations and primitive solutions. To discuss some results for Mordell equations (2), we need to introduce more notation. Following Bombieri–Gubler, we say that $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ is primitive if ± 1 are the only $n \in \mathbb{Z}$ with n^6 dividing $\gcd(x^3, y^2)$. In particular $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ is primitive if x, y are coprime. To measure the finite set S and $a \in \mathcal{O}$, we take

$$a_S = 1728 N_S^2 \prod p^{\min(2, \mathrm{ord}_p(a))}$$

with the product taken over all rational primes $p \notin S$. Let h be the usual logarithmic Weil height with $h(n) = \log |n|$ for $n \in \mathbb{Z} - \{0\}$. Building on the arguments of [4, Cor 7.4], we establish the following result.

Theorem 2. *Let $a \in \mathbb{Z}$ be nonzero. Assume that $y^2 = x^3 + a$ has a solution in $\mathbb{Z} \times \mathbb{Z}$ which is primitive. Then any $(x, y) \in \mathcal{O} \times \mathcal{O}$ with $y^2 = x^3 + a$ satisfies*

$$\max\bigl(h(x), \tfrac{2}{3}h(y)\bigr) \le a_S \log a_S.$$

We now discuss several aspects of this result. A useful feature of Theorem 2 is that it does not involve $|a|$. To illustrate this we take $n \in \mathbb{Z}_{\ge 1}$, we let $\mathcal{F}_n$ be the infinite family of integers a with radical $\mathrm{rad}(a)$ at most n, and we put $a_* = a_S$ for S empty. Then it holds $a_* \le 1728 \mathrm{rad}(a)^2$ and we obtain the following corollary.

Corollary. *For any $n \in \mathbb{Z}_{\geq 1}$, the set of primitive $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ with $y^2 - x^3 \in \mathcal{F}_n$ is finite and can in principle be determined. Furthermore if $a \in \mathbb{Z}$ satisfies $\log|a| \geq a_* \log a_*$, then there are no primitive $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ with $y^2 - x^3 = a$.*

Two problems in discrete geometry. The following problem is motivated by the global elliptic logarithm sieve constructed in [5]. This sieve is applied in our algorithm which solves the Mordell equation (2) by using height bounds.

Problem. *Let $A := \mathbb{R}^n_{\geq 0}$. Given $k \geq n$, how does one choose $x_1, \ldots, x_k \in A$ with $||x_1||_1 = \ldots = ||x_k||_1 = 1$ such that $\sup\{||a||_1 \,:\, a \in A \setminus \bigcup_i (x_i + A)\}$ is minimal?*

In [5] we chose some reasonable x_i, but they are probably not yet optimal. In practice, approximately optimal solutions are good enough. Our refined sieve for the S-unit equation (1) motivates a similar but a bit more technical problem.

Problem. *Let $[n] = \{1, \ldots, n\}$. For any $X = (J, x)$ with $J \subseteq [n]$ and $x \in \mathbb{R}^J_{\geq 0}$ with $||x||_1 = 1$, define $A(X) = \{a \in \mathbb{R}^{[n]} \,:\, a_j \leq x_j \text{ for some } j \in J\}$ and $B(X) = A(X) \cap (-A(X))$. Given $k \geq n$, how does one choose k such pairs $X_1, \ldots, X_k$ such that $\sup\{||a||_1 \,:\, a \in \bigcap_i B(X_i)\}$ is minimal?*

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Combinatorial aspects of translation-invariant valuations

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(joint work with Katharina Jochemko)

Ehrhart [3] showed that for every r-dimensional lattice polytope $P \subset \mathbb{R}^d$ there are integers $h_0^*, \ldots, h_r^*$ such that

$$E_P(n) \;:=\; |nP \cap \mathbb{Z}^d| \;=\; h_0^* \binom{n+r}{r} + h_1^* \binom{n+r-1}{r} + \cdots + h_r^* \binom{n}{r}$$

for all $n \geq 0$. In particular, $E_P(n)$ is a polynomial. The *h^*-vector* of P is defined as $h^*(P) = (h_0^*, \ldots, h_d^*)$ with the convention $h_i^* = 0$ for $r < i \leq d$. A fundamental contribution to a combinatorial understanding of the h^*-vector was made by Stanley [7, 8]:

- $h_i^*(P) \geq 0$ for every i and lattice polytope P; (*h^*-nonnegativity*)
- $h_i^*(P) \leq h_i^*(Q)$ for every i and lattice polytopes $Q \subseteq P$. (*h^*-monotonicity*)

McMullen generalized Ehrhart's result as follows: For simplicity, let $\Lambda \in \{\mathbb{R}^d, \mathbb{Z}^d\}$. A polytope with all vertices in Λ is called a *Λ-polytope* and $\mathcal{P}(\Lambda)$ is the class of all Λ-polytopes in $\mathbb{R}^d$. A *valuation* into an abelian group G is a map $\varphi : \mathcal{P}(\Lambda) \to G$ such that $\varphi(\varnothing) = 0$ and the *valuation property* holds:

$$\varphi(P \cup Q) \;=\; \varphi(P) + \varphi(Q) - \varphi(P \cap Q),$$

whenever $P, Q, P \cup Q, P \cap Q \in \mathcal{P}(\Lambda)$; cf. [6] for the case $\Lambda = \mathbb{Z}^d$. A valuation φ is called a *Λ-valuation* if it is translation-invariant with respect to Λ, that is, $\varphi(P + t) = \varphi(P)$ for all $P \in \mathcal{P}(\Lambda)$ and $t \in \Lambda$. McMullen [5] showed that for every Λ-valuation φ and every r-dimensional Λ-polytope P there are elements $h_0^\varphi, \ldots, h_r^\varphi \in G$ such that

$$\varphi_P(n) \;:=\; \varphi(nP) \;=\; h_0^\varphi \binom{n+r}{r} + h_1^\varphi \binom{n+r-1}{r} + \cdots + h_r^\varphi \binom{n}{r}$$

for all $n \geq 0$. This prompts the definition of an h^φ-vector for every Λ-valuation. The natural question is to ask for an extension of the h^*-nonnegativity and h^*-monotonicity to general Λ-valuations. The Euler characteristic χ shows that h^*-nonnegativity/-monotonicity does not hold for general (nonnegative) valuations. To state our main result, we define

$$\varphi(\operatorname{relint}(P)) \;:=\; \sum_F (-1)^{\dim P - \dim F} \varphi(F),$$

where the sum is over all faces F of P. For the discrete volume $E(P) = |P \cap \mathbb{Z}^d|$, this definition yields $E(\operatorname{relint}(P)) = |\operatorname{relint}(P) \cap \mathbb{Z}^d|$.

Theorem. *Let G be an abelian group with a partial order compatible with the group structure. For a Λ-valuation $\varphi : \mathcal{P}(\Lambda) \to G$, the following are equivalent*

(i) *φ is h^*-monotone.*
(ii) *φ is h^*-nonnegative.*
(iii) $\varphi(\operatorname{relint}(\Delta)) \geq 0$ *for all Λ-simplices Δ.*

This trivially implies Stanley's result and reproves a theorem of Beck, Robins, and Sam [1] on the h^*-vectors of solid-angle polynomials.

For $G = \mathbb{R}$, the set $\mathcal{V}$ of real-valued Λ-valuations is naturally an infinite dimensional vector space. Condition (iii) implies that

$$\mathcal{V}_+ \;:=\; \{\varphi \in \mathcal{V} : \varphi \ h^*\text{-nonnegative}\}$$

is a convex cone. If $\Lambda = \mathbb{R}^d$, we get the following new characterization of the volume.

Theorem. *Let $\varphi : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ be a $\mathbb{R}^d$-valuation. Then φ is h^*-nonnegative if and only if $\varphi = \lambda \mathrm{vol}_d$ for some $\lambda \geq 0$.*

Said differently, the cone $\mathcal{V}_+$ is 1-dimensional with generator given by the volume. If $\Lambda = \mathbb{Z}^d$, the situation is different. Let us write $\bar{\mathcal{V}}$ for the vector subspace of $\mathbb{Z}^d$-valuations that are invariant under lattice transformations, that is, $\varphi(T(P)) = \varphi(P)$ for any linear map T that satisfies $T(\mathbb{Z}^d) = \mathbb{Z}^d$. We further define $\bar{\mathcal{V}}_+ = \mathcal{V}_+ \cap \bar{\mathcal{V}}$. It was shown by Betke and Kneser [2] that $\bar{\mathcal{V}}$ is a vector space of dimension $d+1$.

Proposition. *$\bar{\mathcal{V}}_+$ is a cone of full-dimension $d+1$.*

We conjecture the following.

Conjecture. *$\bar{\mathcal{V}}_+$ is a polyhedral cone. Even stronger, $\bar{\mathcal{V}}_+ \cong \mathbb{R}^{d+1}_{\geq 0}$.*

In dimensions $d \leq 3$, we can verify the strong version of the conjecture.

If $G = V$ is a real vector space and the partial order is given by $x \preceq_C y$ if and only if $y - x \in C$ for some fixed pointed convex cone $C \subseteq V$, then we prove the following strengthenings.

Theorem. *For $G = (\mathbb{R}^m, \preceq_C)$ the following holds.*
(1) If $\Lambda = \mathbb{R}^d$, then $\mathcal{V}_+ \cong C$.
(2) If $\Lambda = \mathbb{Z}^d$ and the strong version of conjecture is true, then $\mathcal{V}_+ \cong C^{d+1}$.

The two (interesting) cases currently under consideration are
- $V \subset \mathbb{R}^{m \times m}$ symmetric matrices, C positive semidefinite matrices.
- V the character ring of a finite group, C spanned by the irreducible characters.

Most results can be found in [4].

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Matroids vs. Shifted simplicial complexes

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(joint work with Steven Klee)

The families of matroid independence complexes and pure shifted simplicial complexes are two remarkable classes of simplicial complexes that share several structural properties, but for which theorems are typically proved in different ways. In a groundbreaking paper Kook, Reiner and Stanton [4] proved that matroid complexes are Laplacian integral. Afterwards Duval and Reiner [1] proved the analogue of this result for shifted complexes using completely different techniques. Finally, Duval [2] proved that the Laplacians of both families of complexes satisfy the same (relative) deletion contraction recurrence. It is then reasonable to ask whether there is a natural class of simplicial complexes that bridges these two theories.

We propose an approach that aims to construct such a class of simplicial complexes. The idea is to relax the classical cryptomorphic matroid axioms to obtain various classes of ordered complexes that contain both the pure shifted complexes and matroids. Each of these classes inherits some nice properties of matroids and, surprisingly, the complexes in the intersection of the mentioned classes enjoy even more analogue properties of classical matroid theory. We describe two such classes here: the first one is obtained by relaxing the exchange axiom, while the second one is obtained by relaxing the circuit axiom. We then proceed to study properties in the intersection of these two classes: they admit natural shelling orders, a notion of internal and external activities, and a well behaved theory of Tutte-Grothendieck invariants with a corresponding Tutte polynomial.

In what follows, Δ is a pure finite $(d-1)$-simplicial complex. We distinguish between the ground set of Δ, which we call E, and the set of vertices which we call V. It is always true that $V \subseteq E$, but it is possible to have some $e \in E$ for which $\{e\} \notin \Delta$. We also assume that E comes endowed with a total order.

Definition 1 (Weak Exchange Axiom)**.** We say that Δ satisfies the *weak exchange axiom*, or WEA for short, if the following condition is satisfied. For every pair of facets F, G of Δ, if $f \in F \backslash G$ is bigger than every element in $G \backslash F$, then there exists $g \in G \backslash F$ such that $(F - \{f\}) \cup \{g\}$ is a facet of Δ.

It is clear that matroids and pure shifted complexes satisfy the WEA. Let A be an initial segment of A that contains a facet. For a face F disjoint from A we define Γ_F^A to be the simplicial complex on the ground set A whose faces are the faces of $\mathrm{link}(F, \Delta)$ that are contained in A.

Theorem 2. *Assume that Δ satisfies the WEA, then:*

i. *Δ is vertex decomposable. The largest vertex v which is not a cone point is a shedding vertex.*
ii. *The lexicographic order of the facets is a shelling order of Δ.*

iii. For every initial segment A we get an h-polynomial decomposition:

$$h(\Delta, t) = \sum_{F \in \Delta, F \cap A = \emptyset} t^{|F|} h(\Gamma_F^A, t)$$

Property i. was originally proved by Billera and Provan for matroid complexes. Property ii. leads to a characterisation of matroid independence complexes given by Björner: a pure simplicial complex is a matroid complex if and only if for every order of the vertices, the induced lexicographic order on the facets is a shelling order. The identity in iii. was obtained in [3] for ordered matroids whose smallest lex basis is an initial segment. This suggests a new approach to Stanley's matroid h-vector conjecture. We make a new conjecture that implies Stanley's conjecture and has the following feature: in order to prove our conjecture for rank d matroids it is enough to check that it holds for matroids with ground set of size at most $2d$. This allows us to check Stanley's conjecture for matroids of rank up to four. In general, this identity suggests a way to construct a not necessarily pure multicomplex realising the h-vector of any complex that satisfies the WEA. This construction can be done for shifted complexes and gives a new proof of Stanley's conjecture for shifted matroids.

A further interesting property is that, using the restrictions sets of the shelling, we get an internal activity theory parallel to that of matroid theory. An element v of a facet F is said to be *internally active* if it does not belong to the restriction set of F with respect to the shelling. The set of internally active elements of F is denoted by $IA(F)$. The basic bond of a face and an element is defined as in matroid theory, that is, for a facet F and $f \in F$, the basic bond $bo(F, f)$ of f with respect to F consists of f and elements e of $E - F$ such that $F - \{f\} \cup \{e\}$ is a facet. It turns out that $IA(F)$ corresponds to elements$f$ of F such that f is the smallest element in $bo(F, f)$.

As important as the matroid exchange axiom is the circuit axiom. From now on we will call the minimal missing faces of Δ circuits. We will define a weak circuit axiom similarly to how we defined the WEA.

Definition 3 (Weak Circuit Axiom)**.** We say that a complex Δ satisfies the *weak circuit axiom*, denoted by WCA for short, if the following property holds. For every pair of circuits C_1, C_2 and $c \in C_1 \cap C_2$ such that c is smaller than the maximum of $C_1 \triangle C_2$, there exists a circuit C_3 that is contained in $(C_1 \cup C_2) - \{c\}$.

Again, it is easy to check that matroids and shifted complexes satisfy the weak circuit axiom.

Theorem 4. *Assume that Δ satisfies the WCA, F is a facet of Δ and v is an element in the ground set that does not belong to F. If there are two circuits C_1 and C_2 contained in $F \cup \{v\}$, then v is not the smallest element in either of them.*

The previous theorem gives us a consistent way of defining the notion of external activity. An element v that does not belong to a facet F is externally active of F if v is the minimal element in a circuit $C \subseteq F \cup \{v\}$. The set of externally active elements of F is denoted by $EA(F)$.

We define deletion and contraction for ordered complexes just as in matroid theory. In order for the theory to work, we have to give up some flexibility. We will only allow to contract the largest element of the ground set. Deletion is denoted by $\Delta\backslash v$ while contraction is denoted by Δ/v.

Lemma 5. *Both WEA and WCA are deletion-contraction closed properties. They are also closed under (ordered) joins.*

We now consider the class of complexes that satisfy both WEA and WCA. This means that our complexes admit both internal and external activity theories, which in turn makes it feasible to define a meaningful Tutte polynomial. The Tutte polynomial turns out to be a universal deletion contraction invariant and several of the classical evaluations still hold true.

Definition 6 (The Tutte polynomial)**.** Assume that Δ satisfies the WEA and WCA. The *Tutte polynomial* of Δ is defined as:

$$T(\Delta, x, y) = \sum_{F \text{ facet}} x^{|IA(F)|} y^{|EA(F)|}$$

Define the loop L and coloop C to be the (-1)- and 0-dimensional complexes on a one element ground set. A Tutte-Grothendieck invariant over a ring R assigns to each WEA+WCA complex Δ an element $f(\Delta) \in R$ that satisfies the following condition:

If v is the largest element in the ground set of Δ, then

$$f(\Delta) = \begin{cases} f(\Delta/v) + f(\Delta\backslash v) & \text{if } v \text{ is not a loop or a coloop.} \\ f(C)f(\Delta/v) & \text{if } v \text{ is a coloop.} \\ f(L)f(\Delta\backslash v) & \text{if } v \text{ is a loop.} \end{cases}$$

Theorem 7. *If Δ satisfies the WEA and the WCA, then:*

i. The Tutte polynomial is a Tutte-Grothendieck invariant.
ii. If f is a WEA+WCA deletion-contraction invariant, then $f(\Delta) = T(\Delta, f(C), f(L))$.

Most classical Tutte-Grothendieck invariants are also Tutte-Grothendieck invariants in this case, meaning evaluations of the Tutte polynomial give combinatorial data such as the number of facets and faces, the reversed f- and h-polynomials, and the number of spanning sets. Furthermore we define a broken circuit to be the result of removing the smallest element of a circuit, just as in classical matroid theory. An *nbc-facet* is a facet of Δ that contains no broken circuit. Equivalently, no element is externally active with respect to F. If the ground set and the vertex set coincide, then the *nbc-complex* of Δ is the simplicial complex $nbc(\Delta)$ whose facets are the nbc-facets of Δ. We obtain the following theorem:

Theorem 8. *The lexicographic order of the nbc-facets is a shelling order. The complements of the restriction sets in this shelling order are the internally active elements. Thus the reverse h-polynomial of $nbc(\Delta)$ is given by $T(\Delta, x, 0)$.*

We conclude by mentioning that we are currently investigating other relaxations of axioms and properties of matroids, such as the semi-modularity of rank functions, the lattice of flats and duality. We hope to obtain enough analogues of theorems from matroid theory to extend the integrality of the spectra of the Laplacian operators to this framework.

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What's next?

Ed Swartz

In the last ten years there has been a dramatic increase in our understanding the f-vectors of manifolds. Today we examine two recent developments and ask, "What's next?" While most of the results hold in the more general category of homology manifolds, we stick to the PL-category as it is the only one where all of the results are known to hold. Throughout we assume that Δ is a $(d-1)$-dimensional finite simplicial complex which is a PL-manifold with or without boundary.

1. Lower bounds for manifolds with boundary

Throughout this section, unless specifically stated otherwise, $\partial\Delta \neq \emptyset$.

Define

$$h''_i(\Delta) = h_i(\Delta) + \binom{d}{i}\sum_{j=0}^{i-1}(-1)^{i-j}\tilde{\beta}_j(\Delta).$$

As usual, $\tilde{\beta}_j$ is the reduced j^{th} Betti-number of Δ with respect to some fixed field. We suppress the dependence of h'' on the field. In joint work with Novik on Buchsbaum complexes we proved the following lower bound.

Theorem 1.1. [7] $h''(\Delta) \geq 0$.

While some examples of triangulations with h'' were given in [7], there was no systemic study of complexes for which the lower bound was sharp. As a follow up to their proof of the generalized lower bound conjecture for polytopes [5], Murai and Nevo introduced i-stacked manifolds and proved that these were precisely the complexes which satisfied $h''=0$ [6]. If $\partial\Delta \neq \emptyset$, then Δ is i-**stacked** if there are no interior faces of codimension $i+1$. So 0-stacked complexes are disjoint union of simplices and 1-stacked spheres are stacked polytopes. If $\partial\Delta = \emptyset$, then Δ is i-stacked if it is the boundary of an i-stacked manifold.

Theorem 1.2. [6] If $\partial\Delta \neq \emptyset$, then Δ is i-stacked if and only if $h''_{i+1}(\Delta) = 0$.

What does the combinatorial condition of being i-stacked imply for the topology of Δ?

Theorem 1.3. (S., 2014) If $h''_{i+1}(\Delta) = 0$, then Δ has a handle decomposition in which every handle has index less than or equal to i.

The obvious question is,

Problem 1.1. If X is a PL-manifold with boundary and handle decomposition using handles of index i or less does there exist a triangulation Δ of X with $h''_{i+1}(\Delta) = 0$?

When $i = 0, 1$ or $d-2$ the answer is yes. For all other values of i the question is open.

The nonnegativity of h'' is not the only known lower bound for manifolds with boundary. Kalai proved that $h_2 - \#\text{interior points} \geq 0$ [4]. It is not hard to show that the number of interior points of Δ is always $d\ h_d + h_{d-1}$. In joint work with Novik this was generalized to include the Betti numbers of the boundary.

Theorem 1.4. [8] Suppose Δ is connected with orientable boundary. If $d \geq 5$, then $h_2 - d\ h_d - h_{d-1} \geq \binom{d}{2}\beta_1(\partial\Delta) + d\ \tilde{\beta}_0(\partial\Delta)$. If $d = 4$, then $h_2 - d\ h_d - h_{d-1} \geq 3\ \beta_1(\partial\Delta) + 4\ \tilde{\beta}_0(\partial\Delta)$.

Problem 1.2. What are the higher dimensional analogs of this result?

As of now I do not even know of any conjectural answers to this question.

2. Manifolds without boundary and the g-conjecture

Throughout this section Δ is a manifold without boundary. It has become apparent that progress in understanding f-vectors of manifolds in dimensions five and above will be very difficult until there is some resolution of the g-conjecture. Depending on one's degree of optimism there are many possible g-conjectures whose main point is to generalize the g-theorem to spheres other than simplicial polytopes. For our purposes the g-conjecture is the statement that face rings of PL-spheres have weak-Lefschetz elements.

As an example of the influence of the g-conjecture, consider the following result of Murai and Nevo. Define

$$\tilde{g}_i = g_i - \binom{d+1}{i} \sum_{j=1}^{i} (-1)^{i-j} \tilde{\beta}_{j-1}.$$

Theorem 2.1. [6] If the g-conjecture holds and Δ is orientable, then $(\tilde{g}_0, \tilde{g}_1, \ldots, \tilde{g}_{\lfloor d/2 \rfloor})$ is an M-vector.

While progress on verifying (or falsifying!) the g-conjecture has been difficult, Bagchi and Datta have presented a new approach and view point to the g-vector using the PL-Morse theory used by Brehm and Kühnel in [3].

Let $\sigma = v_1 < v_2 < v_3 < \cdots < v_n$ be any linear ordering of the vertices of Δ. The PL-Morse theory used by Brehm and Kühnel defines Morse indices $c_i^K(\Delta)$ in a way that preserves many of the usual properties of Morse theory. The one we focus on here is that for any σ the usual Morse inequality holds:

$$\sum_{j=0}^{i}(-1)^{i-j}c_j^K \geq \sum_{j=0}^{i}(-1)^{i-j}\beta_j.$$

Instead of looking at one ordering of the vertices, consider the average of $c_i^K(\sigma)$ over all possible orderings of the vertices. Define $\mu_i^K(\Delta) = \frac{1}{n!}\sum_{\sigma} c_i^K(\sigma)$. The μ_i were introduced by Bagchi and Datta in [2] in the context of two-neighborly triangulations. Bagchi extended their definition to arbitrary complexes and proposed the following lower bound conjecture.

Conjecture 2.1. [1]

$$g_i \geq \binom{d+1}{i}[(-1)^i + \sum_{j=1}^{i}(-1)^{i-j}\mu_{j-1}].$$

The Morse inequalities above show that this conjecture is stronger than the conjecture that the $\tilde{g}_i$ are nonnegative. In addition, Bagchi was able to prove his conjecture when $d = 4$.

Theorem 2.2. [1] If Δ is a three-dimensional homology manifold, then

$$g_2 \geq 10[\mu_1 - \mu_0 + 1].$$

While the corresponding Betti number statement, $g_2 \geq 10(\beta_1 - \tilde{\beta}_0)$ follows easily from the proofs in [7], this theorem is obviously stronger.

Problem 2.1. What is the relationship between Bagchi's conjecture and the g-conjecture?

Problem 2.2. What is the relationship between the μ_i and discrete Morse theory? Is μ_1 a lower bound for the number of generators of the fundamental group?

Kalai has a long-standing conjecture that

$$g_2 \geq \binom{d+1}{2} \cdot \text{ minimum number of generators of the fundamental group.}$$

As a possible entry of the μ_i to manifolds with boundary we suggest the following.

Problem 2.3. If Δ is three-dimensional is $h_2 - 4\ h_4 - h_3 \geq 6\mu_1 + 4\mu_2$?

The motivation behind this problem is that it follows easily from the results of [7] that $h_2 - 4\ h_4 - h_3 \geq 6\beta_1 + 4\beta_2$.

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Open Problems

Collected by Arnau Padrol

1. The Hasse diagram property
(by Patricia Hersh with Karola Mészáros)

Given a polytope P and a generic cost vector c, we obtain a directed graph $G(P, c)$ from the 1-skeleton of P by orienting each edge from u to v for $c(u) < c(v)$. When this directed graph is the Hasse diagram of a poset $L(P, c)$, we say that the pair (P, c) has the *Hasse diagram property.* This leads to several questions:

Problem 1: If a polytope P is simple and the pair (P, c) has the Hasse diagram property, does this imply that each open interval in the poset $L(P, c)$ has order complex that is homotopy equivalent to a ball or a sphere? If so, are the intervals giving the homotopy type of a sphere exactly those given by faces of the polytope P?

Problem 2: Given (P, c), the *pseudo-join* of a collection of atoms is the unique sink in $G(P, c)$ of the smallest face containing all of these atoms as well as containing the unique source in P. Is the proper part of P homotopy equivalent to the proper part of the subposet of pseudo-joins of atoms?

Problem 3: Assuming that our polytope P is simple and that (P, c) has the Hasse diagram property, does this imply that the subposet of pseudo-joins of collections of atoms is always a Boolean algebra?

Problem 4: We say that (P, c) is *non-revisiting* if each directed path in $G(P, c)$ that leaves a face F cannot revisit it. If P is a simple polytope with (P, c) having the Hasse diagram property, does this imply that (P, c) also has the non-revisiting property?

2. Translating trapezoids (by Friedrich Eisenbrand)

Given a trapezoid with two of its edges parallel to the x-axis, compute the translation in the direction of the x-axis such that the resulting trapezoid contains the maximum number of integer points. For convex polygons with an arbitrary number of vertices, this is NP-hard [1].

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3. Tight simplicial complexes (by Bhaskar Bagchi)

Let X be a finite simplicial complex and $\mathbb{F}$ be a field. Recall that X is said to be $\mathbb{F}$-tight if (a) X is connected, and (b) for every induced subcomplex Y of X the $\mathbb{F}$-linear map $H_*(Y) \to H_*(X)$, induced by the inclusion of Y in X, is injective. Prove or disprove: if X is $\mathbb{F}$-tight and if the link of some vertex in X is an $\mathbb{F}$-homology $(d-1)$-sphere (for some d), then X triangulates an $\mathbb{F}$-homology closed d-manifold. (This is trivial for $d = 1$. In the case $d = 2$, this is Lemma 3.4 in B. Bagchi, B. Datta and J. Spreer, Tight triangulations of closed 3-manifolds, Preprint, 2015. Open for $d \geq 3$.)

4. Alternating sign matrix combinatorics from suspending independence complexes of Graphs (by Alexander Engström)

We consider cohomology of independence complexes of graphs with real coefficients. The harmonic representation is the intersection between the kernel and the cokernel of the chain maps. This setup is common in topological combinatorics, see for example [1].

For a path on five vertices the cohomology representative is

$$-1[1,3]+3[1,4]-3[2,4]-2[1,5]+3[2,5]-[3,5],$$

where $a[x,y]$ should be interpreted as that the independent set $\{x,y\}$ gets weight a. With the weights normalized to all be integers of greatest common divisor one, the largest absolute value of a weight is three for the the path on five vertices.

In this problem we discuss the largest weight in cohomology representations of graphs. The independence complexes of cycles on $3k+1$ vertices has one cohomology class. For $k = 1, 2, \ldots$ its largest weight is

$$1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, \ldots$$

which is the number of alternating sign matrices. This can either be proved directly or using a correspondence in physics between supersymmetric lattice models, which essentially are independence complexes of cycles, and $O(1)/XXZ$ spin chain models together with results from [2]. By a direct argument the largest weights of paths on $3k-1$ vertices are

$$1, 1, 3, 26, 646, 45885, 9304650, 5382618660, \ldots,$$

which is the number of alternating sign matrices that are symmetric about the vertical axes. For both these cases and several similar there are nice formulas and generating functions, see [3].

In the two previous examples we got a series of graphs by replacing an edge by paths with $3k+1$ edges. This replacement suspends the independence complex k times [4].

Problem 1: Given a graph G whose independence complex has exactly one cohomology generator and an edge e of G, consider the sequence of graphs G_k given by replacing e by a path on $3k+1$ edges. Show that there is a closed formula and a "nice" generating function for the largest weight of the cohomology representatives.

Problem 2: Kuperberg defined several versions of alternating sign matrices in [3]. Find the independence complexes that realize them.

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5. Problems regarding the topological Tverberg conjecture (TTC) (by Gil Kalai)

Now that the TTC is false we can still ask:

Problem 1: Let f be a continuous map from $\Delta^m \to R^d$ with the additional property that the images of faces of Δ^m form a "good cover". Does the conclusion of TTC hold? (Perhaps it is even enough just to assume that the images of faces are contractible.)

Problem 2: Let f be a continuous map from $\Delta^m \to R^d$ is there a point common to all images of faces of dimension $m-r$? (Pavle Blagojević said that the answer is yes, cf. [1, Theorem 1.1.])

Problem 3: Tverberg proposed in the mid 70s: Let f be a continuous map from an m-dimensional polytope to R^d; then there are r disjoint faces whose images have a point in common ($\star$). Now we know that this is false even when P is a simplex but Tverberg's Theorem asserts that this is true for affine maps from the simplex. The question is if ($\star$) is always true for affine maps. (More generally you can ask if ($\star$) is true if the images of faces form a good cover or even just contractible.)

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6. On a triangulation of a manifold, is there always a unique optimal discrete Morse vector? (by Bruno Benedetti)

Is there a finite simplicial complex C and a positive integer k such that:

(1) C is homeomorphic to some manifold;
(2) C admits no discrete Morse function with fewer than k critical faces;
(3) C admits two discrete Morse functions f and g such that

$$\sum_{i=0}^{\dim C} c_i(f) = \sum_{i=0}^{\dim C} c_i(g) = k,$$

and $c_i(f) \neq c_i(g)$ for some i?

Above, $c_i(f)$ denotes the number of critical faces of dimension i of f; similarly for $c_i(g)$.

7. Combinatorics of 3D floor diagram (by Grigory Mikhalkin)

Floor diagrams are a combinatorial tool that enables us to compute the number of real and complex curves through given constraints (e.g. points) in projective spaces, see [1]. Their combinatorics in dimension 2 and 3 (some of the most geometrically interesting cases) are quite different. The 2-dimensional case is relatively well-studied. It was shown in [2] that labeled floor diagrams in genus 0 correspond to labeled trees. In particular the number of labeled floor diagrams is d^{d-2}. What can be said about the 3D case, in particular for the number of labeled floor diagrams?

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8. Polytopes with few vertices and few facets (by Arnau Padrol)

There exists a constant $D(\alpha)$ such that if $d > D(\alpha)$ and P is a d-polytope with no more than $d + 1 + \alpha$ vertices and no more than $d + 1 + \alpha$ facets, then P is a pyramid. The problem is to find the smallest possible $D(\alpha)$. The current best upper bound is quadratic in α, while the lower bound is linear.

A corollary of the statement above is that there is a constant $K(\alpha)$ such that, for every d, the number of combinatorial types of d-polytopes with no more than $d + 1 + \alpha$ vertices and no more than $d + 1 + \alpha$ facets is at most $K(\alpha)$. Which is the smallest possible $K(\alpha)$?

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