

# A computational framework for determining square-maximal strings <sup>★</sup>

Antoine Deza<sup>1,2</sup>, Frantisek Franek<sup>1</sup>, and Mei Jiang

<sup>1</sup> Advanced Optimization Laboratory  
Department of Computing and Software  
McMaster University, Hamilton, Ontario, Canada  
{deza,franek,jiangm5}@mcmaster.ca  
<http://optlab.cas.mcmaster.ca/>

**Abstract.** We investigate the function  $\sigma_d(n) = \max\{ \mathbf{s}(x) \mid x \text{ is a } (d, n)\text{-string} \}$ , where  $\mathbf{s}(x)$  denotes the number of primitively rooted distinct squares in a string  $x$  and  $(d, n)$ -string denotes a string of length  $n$  with exactly  $d$  distinct symbols. New properties of  $\sigma_d(n)$  function are presented. The notion of  $s$ -cover is presented and discussed with emphasis on recursive computational determination of  $\sigma_d(n)$ . This notion is used as a key element of a computational framework for an efficient computation of the maximum number of primitively rooted distinct squares. In particular, we were able to determine all values of  $\sigma_2(n)$  for  $n \leq 53$  and  $\sigma_3(n)$  for  $n \leq 41$  and to point out that  $\sigma_2(33) < \sigma_3(33)$ ; that is, among all strings of length 33, no binary string achieves the maximum number of distinct primitively rooted squares. Noticeably, these computations reveal the unexpected existence of a  $(d, n) = (2, 33), (2, 34)$  satisfying  $\sigma_{d+1}(n+2) - \sigma_d(n) > 1$ , and of three consecutive equal values:  $\sigma_2(31) = \sigma_2(32) = \sigma_2(33)$ . In addition we show that  $\sigma_2(n) \leq 2n - 66$  for  $n \geq 53$ .

**Keywords:** *string, square, maximum number of distinct squares, parameterized approach,  $(d, n - d)$  table*

## 1 Introduction

In [2] the notion of an  $r$ -cover was introduced as a means to represent the distribution of the runs in a string and thus describe the structure of the run-maximal strings. Ignoring the number of distinct symbols  $d$ , a key assertion from [2] is that essentially any run-maximal string has an  $r$ -cover. A similar approach was adapted for run-maximal  $(d, n)$ -strings in [1] and we show in Section 2 how this approach can be adapted for square-maximal  $(d, n)$ -strings. This notion is used to speed up computations of the maximum number of primitively rooted distinct squares.

We encode a square as a triple  $(s, e, p)$  where  $s$  is the starting position of the square,  $e$  is the ending position of the square, and  $p$  is its period. Note that  $e = s + 2p - 1$ . The *join*  $x[i_1 .. i_k] \cup x[j_1 .. j_m]$  of two substrings of a string  $x = x[1 .. n]$  is defined if  $i_1 \leq j_1 \leq i_k + 1$  and then  $x[i_1 .. i_k] \cup x[j_1 .. j_m] = x[i_1 .. \max\{i_k, j_m\}]$ , or if  $j_1 \leq i_1 \leq j_m + 1$  and then  $x[i_1 .. i_k] \cup x[j_1 .. j_m] = x[j_1 .. \max\{i_k, j_m\}]$ . Simply put, the join is defined when the two substrings either are adjacent or overlap. The alphabet of  $x$  is denoted by  $\mathcal{A}(x)$ ,  $(d, n)$ -string refers to a string of length  $n$  with exactly  $d$  distinct symbols,  $\mathbf{s}(x)$  denotes the number of distinct squares in a string  $x$ , and  $\sigma_d(n)$  refers to the maximum number of distinct squares over all  $(d, n)$ -strings, i.e.  $\sigma_d(n) = \max\{ \mathbf{s}(x) \mid x \text{ is a } (d, n)\text{-string} \}$ . A singleton is a symbol which occurs

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exactly once in the string under consideration. To simplify the notation, for an empty string  $\varepsilon$  we define  $\mathbf{s}(\varepsilon) = 0$  and  $\sigma_d(0) = 0$ .

## 2 Computational approach to distinct squares

In the computational framework for determining  $\sigma_d(n)$  we will be discussing later, we first compute a lower bound of  $\sigma_d(n)$  denoted as  $\sigma_d^-(n)$ . Then only  $(d, n)$ -strings  $x$  that could achieve  $\mathbf{s}(x) > \sigma_d^-(n)$  need to be considered for determining  $\sigma_d(n)$ , thus significantly reducing the search space.

The purpose of this section is to introduce two string properties that guarantee that for such an  $x$ ,  $\mathbf{s}(x) > \sigma_d^-(n)$  for a given  $\sigma_d^-(n)$ . The two properties are *s-cover* and *density*, and we will show that for a string  $x$  to have a chance for  $\mathbf{s}(x) > \sigma_d^-(n)$ , it must have an s-cover and be dense. The s-cover is guaranteed through generation, while the density is verified incrementally during the generation in the earliest possible stages. Note that the notion of s-cover, though similar to r-cover, see [1, 2], is slightly different.

**Definition 1.** An s-cover of a string  $x = x[1 .. n]$  is a sequence of primitively rooted squares  $\{ S_i = (s_i, e_i, p_i) \mid 1 \leq i \leq m \}$  so that

- (1) for any  $1 \leq i < m$ ,  $s_i < s_{i+1} \leq e_i + 1$  and  $e_i < e_{i+1}$ , i.e. two consecutive squares are either adjacent or overlap;
- (2)  $\bigcup_{1 \leq i \leq m} S_i = x$ ;
- (3) For any occurrence of square  $S$  in  $x$ , there is  $1 \leq i \leq m$  so that  $S$  is a substring of  $S_i$ , denoted by  $S \subseteq S_i$ .

A string which has an s-cover is referred to as s-covered.

**Lemma 2.** If a string  $x$  is s-covered, its s-cover is unique.

*Proof.* Let us assume that we have two different s-covers of  $x$ ,  $\{ S_i \mid 1 \leq i \leq m \}$  and  $\{ S'_j \mid 1 \leq j \leq k \}$ . We shall prove by induction that they are identical. By Definition 1 (3),  $S_1 \subseteq S'_1$  and, by the same argument,  $S'_1 \subseteq S_1$ , and thus  $S_1 = S'_1$ . Let the induction hypothesis be  $S_i = S'_i$  for  $1 \leq i \leq t$ . If  $\bigcup_{1 \leq i \leq t} S_i = x$ , we have  $t = m = k$  and we are done. Otherwise consider  $S_{t+1}$ . By Definition 1 (3), there is  $S'_v$  so that  $S_{t+1} \subseteq S'_v$  and  $v > t$ . We need to show that  $v = t + 1$ . If not, then  $S_{t+1}$  would neither be a substring of  $S'_t$  nor of  $S'_{t+1}$  contradicting Definition 1 (3). Therefore  $S_{t+1} \subseteq S'_{t+1}$ . By the same argument,  $S'_{t+1} \subseteq S_{t+1}$  and so  $S_{t+1} = S'_{t+1}$ .  $\square$

**Lemma 3.** If a string  $x$  has an s-cover, then it is singleton free.

*Proof.* Let  $\{ S_j \mid 1 \leq j \leq m \}$  be the s-cover of  $x = x[1 .. n]$ . For any  $1 \leq i \leq n$ ,  $x[i] \in S_t$  for some  $t$  by Definition 1 (2). Since  $S_t$  is a square, the symbol  $x[i]$  occurs in  $x$  at least twice.  $\square$

Before we can define what a *dense string* is, we must first define the notion of a *core* of a square, similarly to the core of a run, see [1, 6]. For a square, its core is the set of indices formed by the intersection of the indices of all its occurrences in the string.

**Definition 4.** The core vector  $k(x)$  of a  $(d, n)$ -string  $x$  is defined by  $k_i(x) =$  the number of cores of squares of  $x$  containing  $i$  for  $i = 1, \dots, n$ . A singleton-free  $(d, n)$ -string  $x$  is dense, if its core vector  $k(x)$  satisfies  $k_i(x) > \sigma_d^-(n) - \mathbf{s}(x[1 \dots i-1]) - m_i$  for  $i = 1, \dots, n$ , where  $m_i = \max \{ \sigma_{d'}(n-i) : d - |\mathcal{A}(x[1 \dots i-1])| \leq d' \leq \min(n-i, d) \}$ .

**Lemma 5.** If a  $(d, n)$ -string is not dense, then  $\mathbf{s}(x) \leq \sigma_d^-(n)$ .

*Proof.* The proof follows from the basic observation that for any string  $x$ ,  $\mathbf{s}(x) \leq \mathbf{s}(x[1 \dots i-1]) + \mathbf{s}(x[i+1 \dots n]) + k_i(x)$  for any  $i$ . Note that the inequality occurs when there are the same type of squares in both  $x[1 \dots i-1]$  and  $x[i+1 \dots n]$ . If  $x$  is not dense, then for some  $i_0$ ,  $k_{i_0}(x) \leq \sigma_d^-(n) - \mathbf{s}(x[1 \dots i_0-1]) - m_{i_0}$ . Then  $\mathbf{s}(x) \leq \mathbf{s}(x[1 \dots i_0-1]) + \mathbf{s}(x[i_0+1 \dots n]) + k_{i_0}(x) \leq \mathbf{s}(x[1 \dots i_0-1]) + m_{i_0} + k_{i_0}(x) \leq \mathbf{s}(x[1 \dots i_0-1]) + m_{i_0} + \sigma_d^-(n) - \mathbf{s}(x[1 \dots i_0-1]) - m_{i_0} = \sigma_d^-(n)$ .  $\square$

**Lemma 6.** If the core vector  $k(x)$  of a  $(d, n)$ -string  $x$  satisfies  $k_i(x) > 0$  for  $i = 1, \dots, n$ , then  $x$  has an  $s$ -cover.

*Proof.* We build an  $s$ -cover by induction: Since the  $k_1(x) \geq 1$ , 1 is in at least one core, hence there must be at least one square starting at position 1. Among all squares starting at position 1, set the one with the largest period to be  $S_1$ . Suppose that we have built the  $s$ -cover up to  $i \leq t$ . If  $\bigcup_{1 \leq i \leq t} S_i = x$ , we are done. Otherwise  $\bigcup_{1 \leq i \leq m} S_i = x[1 \dots v]$  for some  $v < n$ . Since  $k_{v+1}(x) \geq 1$ , there is at least one square  $(s, e, p)$  in  $x$  so that  $s \leq v+1 \leq s+2p-1$ . From all such squares choose the leftmost ones, and among them choose the one with the largest period and set it as  $S_{t+1}$ . It is straightforward to verify that all the conditions of Definition 1 are satisfied and that we have built the  $s$ -cover of  $x$ .  $\square$

Note that for a  $(d, n)$ -string, having an  $s$ -cover implies being singleton free. However it does not imply that every  $k_i(x) \geq 1$ , even though it is very close to it. Consider the  $s$ -cover  $\{S_j = (s_j, e_j, p_j) : 1 \leq j \leq m\}$  of  $x$ . If  $S_1$  has another occurrence in  $x$  and there is no other square in  $x$  starting at position 1, then 1 is not in any core and  $k_1(x) = 0$ . Similarly, if the  $s$ -cover has two consecutive adjacent squares  $S_j$  and  $S_{j+1}$ , if there is no other square occurring at position  $s_{j+1}$ , and if the square  $S_{j+1}$  has some other occurrence, then  $k_{s_{j+1}}(x) = 0$ . In this sense, the  $s$ -cover is a computationally efficient structural generalization of the property that every  $k_i(x) \geq 1$ .

**Lemma 7.** Let  $\{S_i = (s_i, e_i, p_i) \mid 1 \leq i \leq m\}$  be an  $s$ -cover of  $x$ . Let  $k = k(x)$  be the core vector of  $x$ . Then for any  $2 \leq i < m$  and the core vector  $k' = k(x[1 \dots e_i])$ ,  $(\forall 1 \leq j < s_{i+1})(k'_j \geq k_j)$ .

*Proof.* Let us assume that from some  $2 \leq i < m$  there is a  $j$  so that  $k_j > k'_j$ . Then there must exist a square  $(s, e, p)$  in  $x = x[1 \dots e_m]$  that is not a square of  $x[1 \dots e_i]$ , i.e.  $e > e_i$  and  $s < s_{i+1}$ , so it is an interim square violating the definition of  $s$ -cover.  $\square$

**Lemma 8.** If a square-maximal  $(d, n)$ -string  $x$  has an  $s$ -cover with two consecutive adjacent squares, then  $\sigma_d(n) \leq \sigma_{d_1}(n_1) + \sigma_{d_2}(n_2)$  for some  $2 \leq d_1, d_2 \leq d \leq d_1 + d_2$  and some  $n_1, n_2$ , possibly equal to zero, such that  $n_1 + n_2 = n$ .

*Proof.* Let  $\{S_i : 1 \leq i \leq m\}$  be the s-cover of  $x$  and let  $S_j \cap S_{j+1} = \emptyset$ . Then  $\mathbf{s}(x) \leq \mathbf{s}(x_1) + \mathbf{s}(x_2)$ , where  $x_1 = \bigcup_{1 \leq i \leq j} S_i$  and  $x_2 = \bigcup_{j < i \leq m} S_i$ . Therefore  $\sigma_d(n) = \mathbf{s}(x) \leq \mathbf{s}(x_1) + \mathbf{s}(x_2) \leq \sigma_{d_1}(n_1) + \sigma_{d_2}(n_2)$  where  $x_1$  and  $x_2$  are, respectively, a  $(d_1, n_1)$ - and a  $(d_2, n_2)$ -string.  $\square$

**Lemma 9.** *If a singleton-free square-maximal  $(d, n)$ -string  $x$  does not have an s-cover, then  $\sigma_d(n) = \sigma_d(n - 1)$ .*

*Proof.* Since  $x$  does not have an s-cover, there exist some  $i_0$  such that  $k_{i_0} = 0$  by Lemma 6. Remove  $x[i_0]$  to form a  $(d, n - 1)$ -string  $y$ . This will not decrease the number of distinct squares in  $x$  since there is no core of any square contains  $i_0$ . Then  $\sigma_d(n) = \mathbf{s}(x) \leq \mathbf{s}(y) \leq \sigma_d(n - 1)$ . Since  $\sigma_d(n) \geq \sigma_d(n - 1)$  (see [4]), therefore  $\sigma_d(n) = \sigma_d(n - 1)$ .  $\square$

**Lemma 10.** *If a square-maximal  $(d, n)$ -string has a singleton, then  $\sigma_d(n) = \sigma_{d-1}(n - 1)$ .*

*Proof.* Remove the singleton to form a  $(d - 1, n - 1)$ -string  $y$  with  $\sigma_d(n) = \mathbf{s}(x) \leq \mathbf{s}(y) \leq \sigma_{d-1}(n - 1)$ . Since  $\sigma_d(n) \geq \sigma_{d-1}(n - 1)$  (see [4]), therefore  $\sigma_d(n) = \sigma_{d-1}(n - 1)$ .  $\square$

### 3 Heuristics for a lower bound $\sigma_d^-(n)$

Recall that  $\sigma_d^-(n)$  denotes the best available lower bound for  $\sigma_d(n)$ . The higher the value of  $\sigma_d^-(n)$ , the less computational effort must be spent on determining  $\sigma_d(n)$ . For  $d = 2$ , generate  $\mathcal{L}_2(n)$ , the set of  $(2, n)$ -strings which are: s-covered, balanced over every prefix (the frequencies of  $a$ 's and  $b$ 's differ by at most a predefined constant), have a maximum period bounded by at most a predefined constant, and contain no triples ( $aaa$  or  $bbb$ ). Then

$$\sigma_2^-(n) = \max \{ \sigma_2(n - 1), \max_{x \in \mathcal{L}_2(n)} \mathbf{s}(x) \}.$$

For  $d \geq 3$ , we simply set  $\sigma_d^-(n) = \max \{ \sigma_{d-1}(n - 1), \sigma_{d-1}(n - 2) + 1, \sigma_d(n - 1) \}$ . This heuristic was found to be very good.

### 4 Generating special s-covered $(d, n)$ -strings

Rather than generating strings, we generate their s-covers. By *special* we mean only s-covers that have no consecutive adjacent squares. The generation proceeds by extending the partially built s-cover in all possible ways. Every time a potential square of the s-cover is to be extended by one position, all previously used symbols and the first unused symbol are tried. For each symbol, the frequency counter is checked that the symbol does not exceed  $n + 2 - 2d$ . Once a symbol is used, the frequency counter is updated. When the whole s-cover is generated, the counter is checked whether all  $d$  symbols occurred in the resulting string; if not, the string is rejected. A typical implementation of the generation of the s-cover would be through recursion as backtracking is needed. For computational efficiency reasons we opted instead for a

user-stack controlled backtracking implemented as a co-routine `Next()` allowing us to call the co-routine repeatedly to produce the next string. Note that the strings are generated in a lexicographic order. The generation of the s-cover follows these principles: The generator for the first square is created by iterative calls to `Next()` producing all the possible generators. Each generator is checked for the additional properties (must be primitive, did not create an intermediate square in the partial string, etc.) before it is accepted. For each subsequent square, its generator may be partially or fully determined. If it is partially determined, iterative calls to `Next()` are used to generate all possible completions of the generator. The complete generator is checked and accepted or rejected. In addition, if the density of the string being generated is to be checked, we use Lemma 7 and the core vector of the partially generated string to reject the string or allow it to be extended further.

## 5 Recursive computation of $\sigma_d(n)$

First,  $\sigma_d^-(n)$  is computed by the heuristic of Section 3. Then it is verified that  $\sigma_{d_1}(n_1) + \sigma_{d_2}(n_2) \leq \sigma_d^-(n)$  for any  $2 \leq d_1, d_2 \leq d \leq d_1 + d_2$  and any  $n_1 + n_2 = n$ . Then  $\mathcal{U}_d(n)$ , the set of all dense special s-covered  $(d, n)$ -strings is generated as described in Section 4. It follows that

$$\sigma_d(n) = \max \left\{ \sigma_d^-(n), \max_{x \in \mathcal{U}_d(n)} \mathbf{s}(x) \right\}.$$

To see that, first consider the existence of a square-maximal  $(d, n)$ -string with singletons: by Lemma 10,  $\sigma_d(n) = \sigma_{d-1}(n-1)$ . Then consider the existence of a singleton-free square-maximal string  $x$  not in  $\mathcal{U}_d(n)$ :

- (a) Either  $x$  does not have an s-cover, in which case by Lemma 9,  $\sigma_d(n) = \sigma_d(n-1)$ ;
- (b) or  $x$  has an s-cover with two consecutive adjacent squares and by Lemma 8,  $\sigma_d(n) \leq \sigma_{d_1}(n_1) + \sigma_{d_2}(n_2)$  for some  $2 \leq d_1, d_2 \leq d$  and some  $n_1 + n_2 = n$ , and so  $\sigma_d(n) \leq \sigma_d^-(n)$ ;
- (c) or  $x$  has a special s-cover, but is not dense, in which case by Lemma 5,  $\sigma_d(n) \leq \sigma_d^-(n)$ .

## 6 Recursive computation of $\sigma_d(2d)$

To compute the values on the main diagonal we can use s-covers satisfying additional conditions. The s-cover  $\{S_i = (s_i, e_i, p_i) : 1 \leq i \leq m\}$  of  $x = x[1 .. n]$  satisfies the *parity condition* if for any  $1 \leq i < m$ ,  $\mathcal{A}(x[1 .. e_i]) \cap \mathcal{A}(x[s_{i+1} .. n]) \subseteq \mathcal{A}(x[s_{i+1} .. e_i])$ .

**Lemma 11.** *The singleton-free part of a square-maximal  $(d, 2d)$ -string  $x$  with all its singletons at the end has an s-cover satisfying the parity condition.*

*Proof.* We can assume that  $x$  has  $0 \leq v \leq d-2$  singletons, all at the end. Let  $k(x)$  be the core vector of  $x$ . Suppose the singleton-free part  $x[1 .. 2d-v]$  does not have an s-cover, then there exist some  $1 \leq i_0 \leq 2d-v$  such that  $k_{i_0}(x) = 0$ . Remove  $x[i_0]$  to form a  $(d, 2d-1)$ -string  $y$ . Therefore,  $\sigma_d(2d) = \mathbf{s}(x) \leq \mathbf{s}(y) \leq \sigma_d(2d-1) = \sigma_{d-1}(2d-2)$ , a contradiction.

So  $x[1 .. 2d-v]$  has an s-cover  $\{S_i : 1 \leq i \leq m\}$ . Let us assume that the s-cover does not satisfy the parity condition. Then either

(a)  $\bigcup_{1 \leq i \leq t} S_i$  and  $\bigcup_{t < i \leq m} S_i$  for some  $1 \leq t \leq m$  are adjacent and their respective alphabets have at least one symbol in common, say  $c$ . If we replace  $c$  in  $\bigcup_{1 \leq i \leq t} S_i$  by a new symbol  $\hat{c} \notin \mathcal{A}(x)$ , we get a new  $(d+1, 2d)$ -string  $y$  so that  $\mathbf{s}(y) \geq \mathbf{s}(x)$ . Thus  $\sigma_d(2d) = \mathbf{s}(x) \leq \mathbf{s}(y) \leq \sigma_{d+1}(2d) = \sigma_d(2d-1) = \sigma_{d-1}(2d-2)$ , a contradiction, or  
(b)  $\bigcup_{1 \leq i \leq t} S_i$  and  $\bigcup_{t < i \leq m} S_i$  for some  $1 \leq t \leq m$  are overlapping, and there is a symbol  $c$  occurring in  $\bigcup_{1 \leq i \leq t} S_i$  and in  $\bigcup_{t < i \leq m} S_i$ , but not in the overlap  $S_t \cap S_{t+1}$ . If we replace  $c$  in  $\bigcup_{1 \leq i \leq t} S_i$  by a new symbol  $\hat{c} \notin \mathcal{A}(x)$ , we get a new  $(d+1, 2d)$ -string  $y$  so that  $\mathbf{s}(y) \geq \mathbf{s}(x)$ . Thus  $\sigma_d(2d) = \mathbf{s}(x) \leq \mathbf{s}(y) \leq \sigma_{d+1}(2d) = \sigma_d(2d-1) = \sigma_{d-1}(2d-2)$ , a contradiction.  $\square$

With additional assumptions, Lemma 11 can be strengthened to exclude consecutive adjacent squares from the s-cover.

**Lemma 12.** *Let  $\sigma_{d'}(2d') = d'$  for any  $d' < d$ . Either  $\sigma_d(2d) = d$  or for every square-maximal  $(d, 2d)$ -string  $x$  with  $v$  singletons all at the end,  $0 \leq v \leq d-2$ , its singleton-free part  $x[1 \dots 2d-v]$  has an s-cover satisfying the parity condition and which has no consecutive adjacent squares.*

*Proof.* The existence of the s-cover  $\{ S_i \mid 1 \leq i \leq m \}$  of  $x[1 \dots 2d-v]$  satisfying the parity condition follow from Lemma 11. We need to prove that either  $\sigma_d(2d) = d$  or there are no adjacent squares in the s-cover. Since  $\sigma_{d'}(2d') = d'$  for any  $d' < d$ ,  $\sigma_{d'}(n') \leq n' - d'$  for any  $n' - d' < d$ . Let us assume that the s-cover of  $x$  has two adjacent squares  $S_t$  and  $S_{t+1}$ . Let  $x_1 = \bigcup_{1 \leq i \leq t} S_i$  and let  $x_2 = \bigcup_{t < i \leq m} S_i$ . Then  $\mathbf{s}(x) \leq \mathbf{s}(x_1) + \mathbf{s}(x_2)$  where  $x_1$  and  $x_2$  are, respectively, a  $(d_1, n_1)$ - and a  $(d_2, n_2)$ -string with  $n_1 + n_2 = 2d - v$  and  $d_1 + d_2 \geq d - v$ . Since the s-cover satisfies the parity condition,  $\mathcal{A}(x_1)$  and  $\mathcal{A}(x_2)$  are disjoint and hence  $d_1 + d_2 = d - v$ . Therefore  $(n_1 - d_1) + (n_2 - d_2) = d$ . Since both  $x_1$  and  $x_2$  are singleton-free,  $n_1 - d_1 > 0$  and  $n_2 - d_2 > 0$ . Hence  $n_1 - d_1 < d$  and  $n_2 - d_2 < d$ , and therefore  $\sigma_d(2d) = \mathbf{s}(x) \leq \mathbf{s}(x_1) + \mathbf{s}(x_2) \leq \sigma_{d_1}(n_1) + \sigma_{d_2}(n_2) \leq (n_1 - d_1) + (n_2 - d_2) = d$ .  $\square$

Since the number of distinct squares in a singleton-free  $(d, 2d)$ -string is at most  $d$ , we do not need to consider the singleton-free strings. Moving a singleton to the end of a string does not decrease the number of distinct squares, therefore we shall only consider  $(d, 2d)$ -strings that have singletons at the end. Since  $\sigma_d(2d) > \sigma_{d-1}(2d-2)$ , we can set  $\sigma_d^-(2d) = \sigma_{d-1}(2d-2) + 1$  and thus consider only the strings that have the non-singleton part dense. By Lemma 12 we need only to consider strings whose s-covers of the non-singleton part satisfy the parity condition with no consecutive adjacent squares. Moreover, the number of singletons must be at least  $\lceil \frac{2d}{3} \rceil$ , see [4]. Let  $\mathcal{T}_v$  denote the set of all singleton-free  $\sigma_d^-(2d)$ -dense s-covered  $(d - \lceil \frac{2d}{3} \rceil, 2d - \lceil \frac{2d}{3} \rceil)$ -strings whose s-covers satisfy the parity condition and have no consecutive adjacent squares. Then

$$\sigma_d(2d) = \max \left\{ d, \max_{x \in \mathcal{T}_v} \mathbf{s}(x) \right\}.$$

## 7 Additional properties of $\sigma_d(n)$

Though the fundamental properties of  $\sigma_d(n)$  were presented previously in [4], here we present some additional properties concerning the gaps between consecutive values in

the  $(d, n-d)$  table where the value of  $\sigma_d(n)$  is the entry on the  $d$ -th row and the  $(n-d)$ -th column. Lemma 13, respectively Lemma 14, shows that the difference between any two consecutive entries along a row, respectively between any two consecutive entries on the main diagonal, in the  $(d, n-d)$  table is bounded by 2.

**Lemma 13.** *For any  $2 \leq d \leq n$ ,  $\sigma_d(n+1) - \sigma_d(n) \leq 2$ .*

*Proof.* Let  $(d, n+1)$ -string  $x = x[1 .. n+1]$  be square-maximal, then  $\mathbf{s}(x) = \sigma_d(n+1)$ . Without a loss of generality we can assume that the first symbol of  $x$  is not a singleton – otherwise we can move all singletons from the beginning of  $x$  to the end of  $x$  without destroying any square type. Let  $y = x[2 .. n+1]$ . Then  $y$  is a  $(d, n)$ -string and  $\mathbf{s}(y) \leq \sigma_d(n)$ . By Fraenkel-Simpson [5], there are at most two rightmost occurrences of squares starting at the same position in a string. In other words, the removal of  $x[1]$  destroyed at most two square types. That is,  $\mathbf{s}(x) - 2 \leq \mathbf{s}(y)$ . Therefore,  $\sigma_d(n+1) - 2 \leq \mathbf{s}(y) \leq \sigma_d(n)$ , implying  $\sigma_d(n+1) - \sigma_d(n) \leq 2$ .  $\square$

**Lemma 14.** *For any  $2 \leq d$ ,  $\sigma_{d+1}(2d+2) - \sigma_d(2d) \leq 2$ .*

*Proof.* By Lemma 13,  $\sigma_{d+1}(2d+2) - \sigma_{d+1}(2d+1) \leq 2$ . By the results from [4], the entries under and on the main diagonal along a column are constant; that is,  $\sigma_{d+1}(2d+1) = \sigma_d(2d)$ . Therefore,  $\sigma_{d+1}(2d+2) - \sigma_d(2d) \leq 2$ .  $\square$

**Lemma 15.** *For any  $d \geq 2$ , if there is a square-maximal singleton free  $(d, 2d+1)$ -string  $x$ , then there exists a square-maximal  $(d, 2d+1)$ -string  $y$  of the form  $y = aaabccdd\dots$*

*Proof.* Since  $x$  contains no singletons, then  $x$  contains exactly  $d-1$  pairs and 1 triple. To prove there exists a square-maximal string in the form that all pairs consist of adjacent symbols and the triple also consists of adjacent symbols, we need to show the non-adjacent symbols can be moved together without reducing the number of distinct squares. Let us suppose that there is a non-adjacent pair of  $\mathbf{c}$ 's in  $x$ .

(i) If the  $\mathbf{c}$ 's did not occur in any square, then we could move both  $\mathbf{c}$ 's to the end of the string without destroying an square type. Moreover, we would gain a new square  $\mathbf{cc}$ , contradicting the square-maximality of  $x$ .

(ii) If the  $\mathbf{c}$ 's occur in exactly one square  $ucvucv$ , where  $u$  and  $v$  are some strings, we can move both  $\mathbf{c}$ 's to the end of  $x$  to form a new string  $y$ . The new squares created by this move are  $uvuv$  and  $\mathbf{cc}$  while the old square  $ucvucv$  was destroyed. If  $uvuv$  did not exist in any other part of  $x$ , then  $\mathbf{s}(y) > \mathbf{s}(x)$  which contradicts the square-maximality of  $x$ ; thus  $uvuv$  already existed in some other part of  $x$ , so we lost the square  $ucvucv$ , but gained  $\mathbf{cc}$ , so  $\mathbf{s}(y) = \mathbf{s}(x)$ .

(iii) If the  $\mathbf{c}$ 's occur in more than one square, these squares must form a non-trivial run, i.e. a run with a non-empty tail. Since there is only one symbol  $\mathbf{t}$  occurring in  $x$  3 times, the only form of such a non-trivial run can be  $tucvtucvt$ . If  $u = v = \varepsilon$ , then the run is  $\mathbf{tctct}$  containing two distinct squares  $\mathbf{tctc}$  and  $\mathbf{ctct}$ . We can change it to  $\mathbf{tttcc}$ , destroying the two squares  $\mathbf{tctc}$  and  $\mathbf{ctct}$ , but gaining two new squares  $\mathbf{tt}$  and  $\mathbf{cc}$ . If either  $u \neq \varepsilon$  or  $v \neq \varepsilon$ , then by moving both  $\mathbf{c}$ 's to the end of  $x$ , we destroy the two distinct squares  $\mathbf{tucvtucv}$  and  $\mathbf{ucvtucvt}$ , but gain three new squares  $\mathbf{tvtuv}$ ,  $\mathbf{uvtuvt}$ , and  $\mathbf{cc}$ . Note that neither  $\mathbf{tvtuv}$  nor  $\mathbf{uvtuvt}$  can exist anywhere else in  $x$

for the lack of  $\mathbf{t}$ 's. Thus we have more distinct squares than  $x$ , which contradicts the maximality of  $x$ .

Since we can move safely all pairs together to the end of  $x$ , the symbols of the triple will end up also adjacent at the beginning of the string.  $\square$

Lemma 16 shows that the two entries of the  $(d, n - d)$  table in the same column just above the main diagonal must be identical.

**Lemma 16.** *For any  $3 \leq d$ ,  $\sigma_d(2d + 1) = \sigma_{d-1}(2d)$ .*

*Proof.* We prove it by induction. Let  $(H_d)$  be the statement that  $\sigma_d(2d+1) = \sigma_{d-1}(2d)$ .  $(H_d)$  for  $2 \leq d \leq 10$  is true from the values in the  $(d, n - d)$  table computed so far, see [4]. This takes care of the base case of the induction. Thus let us assume that  $H_{d-1}$  is true, and let us prove that  $(H_d)$  is true. Let  $(d, 2d + 1)$ -string  $x$  be square-maximal. If  $x$  contains a singleton, remove it to form a new  $(d - 1, 2d)$ -string  $y$ . Then  $\sigma_d(2d + 1) = \mathbf{s}(x) \leq \mathbf{s}(y) \leq \sigma_{d-1}(2d)$  and since  $\sigma_d(2d + 1) \geq \sigma_{d-1}(2d)$ , see [4], thus  $\sigma_d(2d + 1) = \sigma_{d-1}(2d)$ . If  $x$  contains no singletons, by Lemma 15 we can assume that it has the form  $aaabbccdd\dots$ . Remove a pair from  $z$  forming a new  $(d - 1, 2d - 1)$ -string  $y$ . Then  $\sigma_d(2d + 1) - 1 = \mathbf{s}(x) - 1 = \mathbf{s}(y) \leq \sigma_{d-1}(2d - 1)$  and since  $\sigma_d(2d + 1) - 1 \geq \sigma_{d-1}(2d - 1)$  by [4], therefore  $\sigma_d(2d + 1) = \sigma_{d-1}(2d - 1) + 1$ . Since  $H_{d-1}$ ,  $\sigma_{d-1}(2d) \geq \sigma_{d-2}(2d - 2) + 1$  and  $\sigma_d(2d + 1) \geq \sigma_{d-1}(2d)$  according to [4], hence  $\sigma_d(2d + 1) = \sigma_{d-1}(2d)$ .  $\square$

Corollary 17 demonstrates the fact that the difference between any two consecutive entries on the two diagonals immediately above the main diagonal is also bounded by 2.

**Corollary 17.** *For any  $3 \leq d$ ,  $\sigma_d(2d + 1) - \sigma_{d-1}(2d - 1) \leq 2$  and  $\sigma_d(2d + 2) - \sigma_{d-1}(2d) \leq 2$ .*

*Proof.* By Lemma 13,  $\sigma_{d-1}(2d) - \sigma_{d-1}(2d - 1) \leq 2$ , and by Lemma 16,  $\sigma_{d-1}(2d) = \sigma_d(2d+1)$ . Therefore,  $\sigma_d(2d+1) - \sigma_{d-1}(2d-1) \leq 2$ . Similarly,  $\sigma_d(2d+2) - \sigma_d(2d+1) \leq 2$  by Lemma 13, and  $\sigma_d(2d + 1) = \sigma_{d-1}(2d)$  by Lemma 16. Therefore,  $\sigma_d(2d + 2) - \sigma_{d-1}(2d) \leq 2$ .  $\square$

*Remark 18.* Fraenkel-Simpson [5] gave the upper bound of  $2n - 8$  for  $n \geq 5$  and any  $d$ , and  $\sigma_2(n) \leq 2n - 29$  for  $n \geq 22$ . Ilie [8] provided an asymptomatic bound of  $2n - \Theta(\log n)$ . We slightly improve Fraenkel-Simpson's bounds with: for any  $2 \leq d \leq n$  and  $n \geq d_0 + 2$ ,  $\sigma_d(n) \leq 2n - d_0 - 2d$ , where  $d_0$  is the maximum  $d$  such that  $\sigma_d(2d) = d$  is known. Currently,  $d_0 = 15$ . In addition, since  $\sigma_2(53) = 40$  we get  $\sigma_2(n) \leq 2n - 66$  for  $n \geq 53$ . Similarly, since  $\sigma_3(41) = 30$ ,  $\sigma_4(31) = 22$ ,  $\sigma_5(33) = 23$ ,  $\sigma_6(28) = 17$ ,  $\sigma_7(27) = 17$ ,  $\sigma_8(23) = 13$ ,  $\sigma_9(23) = 12$  and  $\sigma_{10}(23) = 11$ , we get  $\sigma_3(n) \leq 2n - 52$  for  $n \geq 41$ ,  $\sigma_4(n) \leq 2n - 40$  for  $n \geq 31$ ,  $\sigma_5(n) \leq 2n - 43$  for  $n \geq 33$ ,  $\sigma_6(n) \leq 2n - 39$  for  $n \geq 28$ ,  $\sigma_7(n) \leq 2n - 37$  for  $n \geq 27$ ,  $\sigma_8(n) \leq 2n - 33$  for  $n \geq 23$ ,  $\sigma_9(n) \leq 2n - 34$  for  $n \geq 23$  and  $\sigma_{10}(n) \leq 2n - 35$  for  $n \geq 23$ .

*Proof.* By Lemma 14,  $\sigma_d(n) \leq d_0 + 2k$ , where  $n - d = d_0 + k$  and  $k \geq 1$ . Thus  $\sigma_d(d_0 + k + d) \leq d_0 + 2k = 2(d_0 + k + d) - d_0 - 2d$ . Therefore,  $\sigma_d(n) \leq 2n - d_0 - 2d$  for  $n \geq d_0 + 2$ .

## 8 Computational Results

We implemented the described algorithms in C++, and ran the programs in parallel on the SHARCNET computer cluster. We were able to compute all  $\sigma_2(n)$  values for  $n \leq 53$  in a matter of hours. The new largest 10 values are:  $\sigma_2(44) = 33$ ,  $\sigma_2(45) = 34$ ,  $\sigma_2(46) = 35$ ,  $\sigma_2(47) = 36$ ,  $\sigma_2(48) = 36$ ,  $\sigma_2(49) = 37$ ,  $\sigma_2(50) = 37$ ,  $\sigma_2(51) = 38$ ,  $\sigma_2(52) = 39$  and  $\sigma_2(53) = 40$ . The results and sample square-maximal strings may be found at [3]. Whenever the computation required determining the number of primitively rooted distinct squares in a concrete string, a C++ implementation of the Franek, Jiang, and Weng's algorithm [7] was used. Interesting values include: three consecutive equal values:  $\sigma_2(31) = \sigma_2(32) = \sigma_2(33)$ , the unexpected existence of a  $(d, n) = (2, 33), (2, 34)$  satisfying  $\sigma_{d+1}(n+2) - \sigma_d(n) > 1$ , and  $\sigma_2(33) < \sigma_3(33)$ ; that is, among all strings of length 33, no binary string achieves the maximum number of distinct primitively rooted squares.

## 9 Conclusion

We presented the notion of s-covers as a structural generalization of a uniform distribution of squares in a string. Then we showed that it is enough to consider special s-covered strings in order to recursively determine the maximum number of primitively rooted distinct squares  $\sigma_d(n)$ . Based on these observations, we presented a fast and efficient computational framework with significantly reduced search space for computations of  $\sigma_d(n)$  based on the notion of density and exploiting the tightness of the available lower bound. We used an implementation of this algorithm to obtain the previously unknown values of  $\sigma_d(n)$ , and in particular  $\sigma_2(n)$  up to  $n = 53$ .

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