Optimization approaches to the Solitaire Game

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History

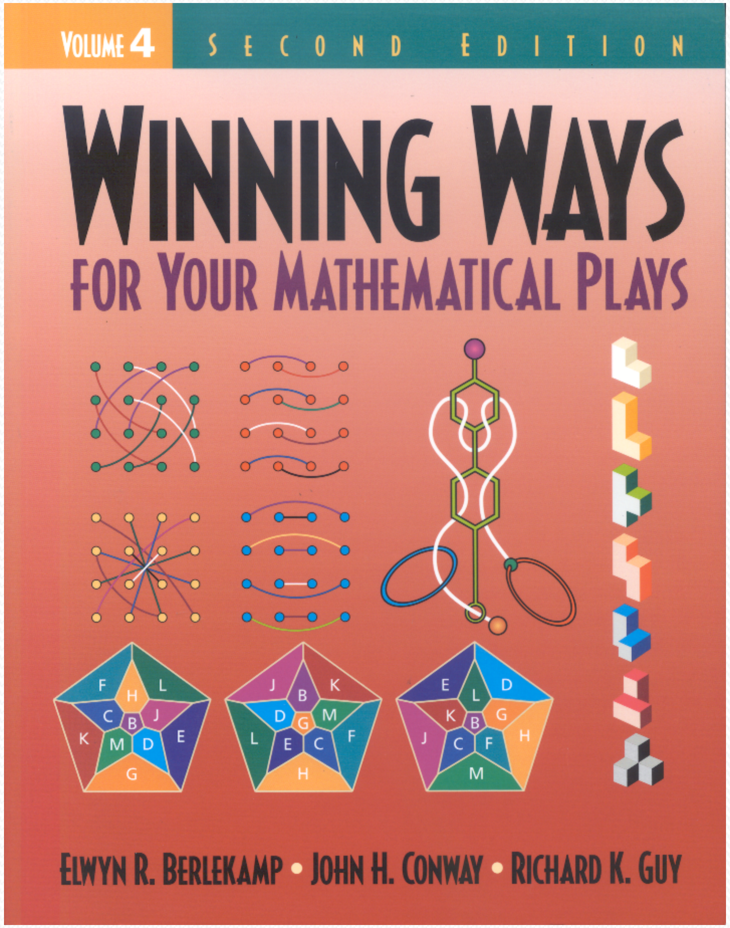
- Uncertain origins (French noblemen, American Indian, Chaldaea, China …)

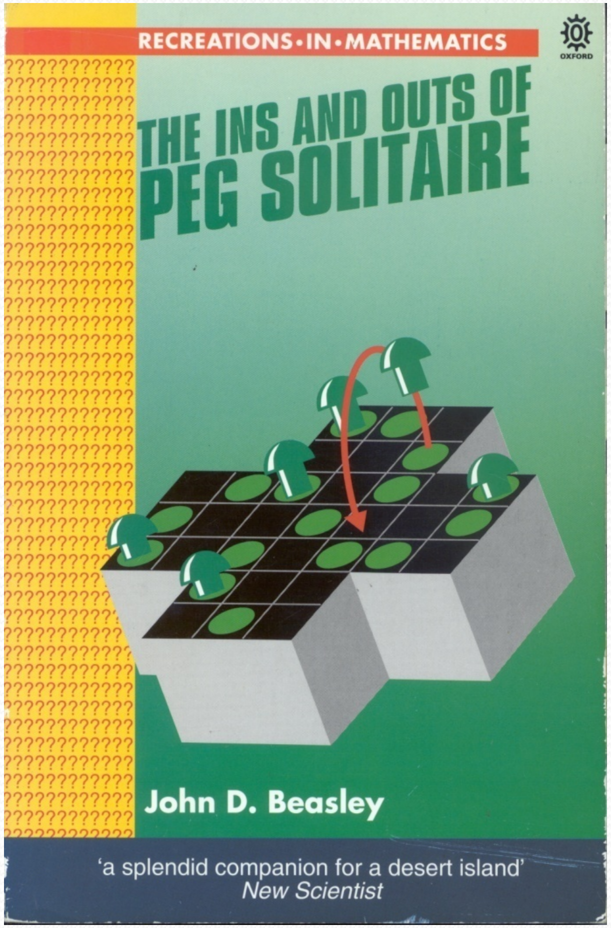
- Fashionable in the court of Louis XIV

- Engraving of Madame la Princesse de Soubize in 1697

- Described by Leibniz in 1710 (paper for the Berlin Academy)

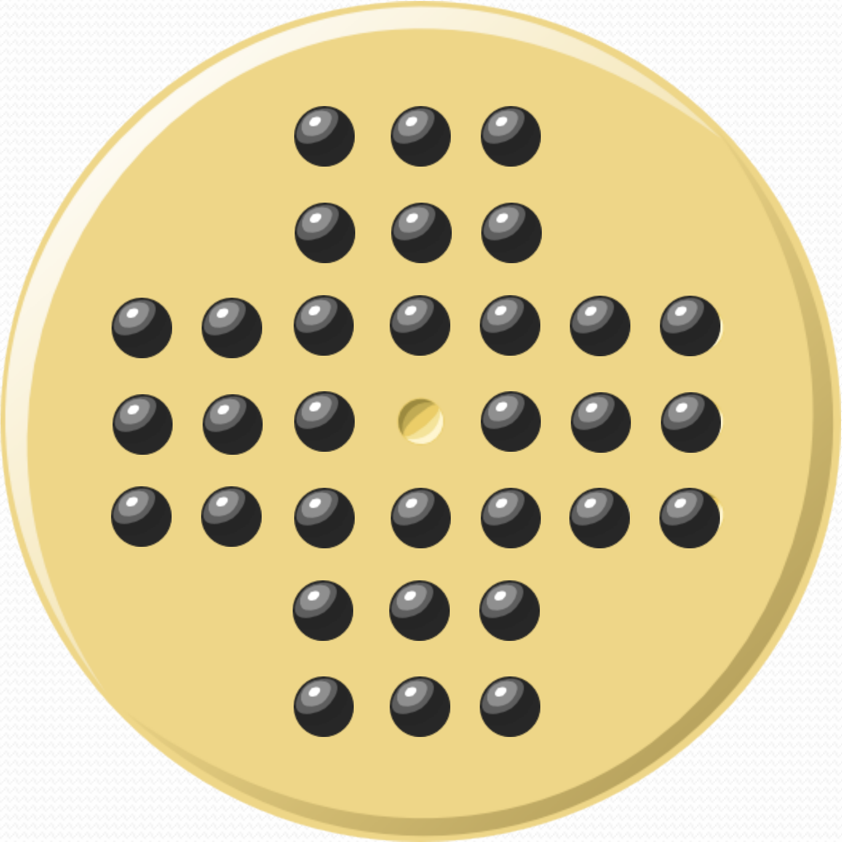
Books


VOLUME 4
SECOND EDITION
WINNING WAYS
FOR YOUR MATHEMATICAL PLAYS
ELWYN R. BERLEKAMP • JOHN H. CONWAY • RICHARD K. GUY


RECREATIONS•IN•MATHEMATICS
OXFORD
THE INS AND OUTS OF
PEG SOLITAIRE
John D. Beasley
'a splendid companion for a desert island'
New Scientist

Rules of the game

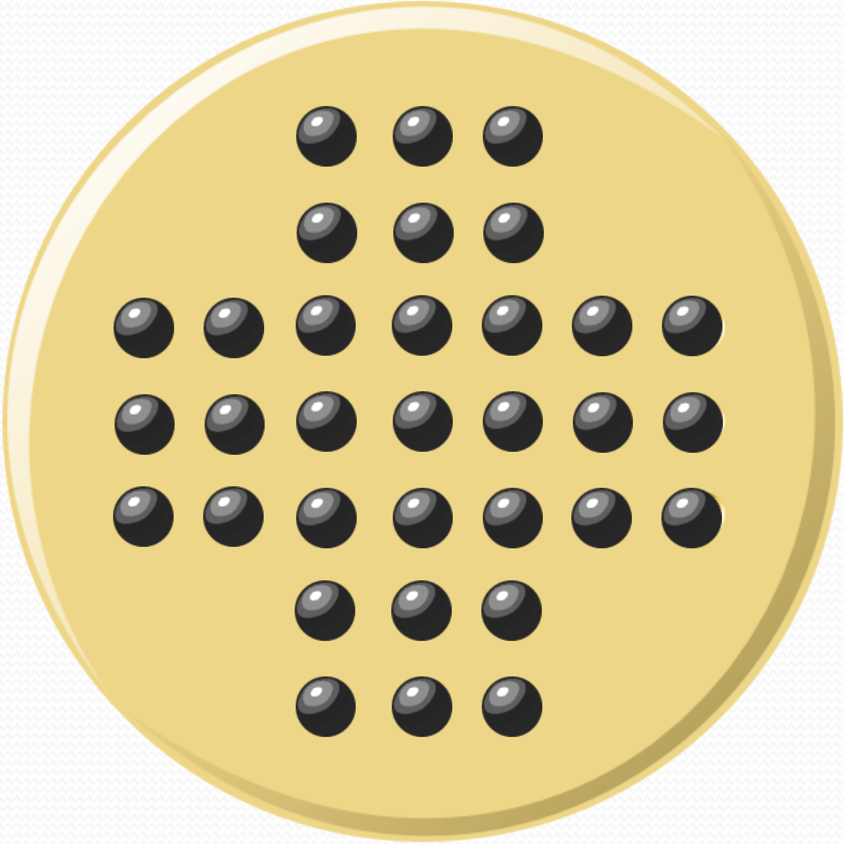
- Initially only the central hole is empty



English board

Rules of the game

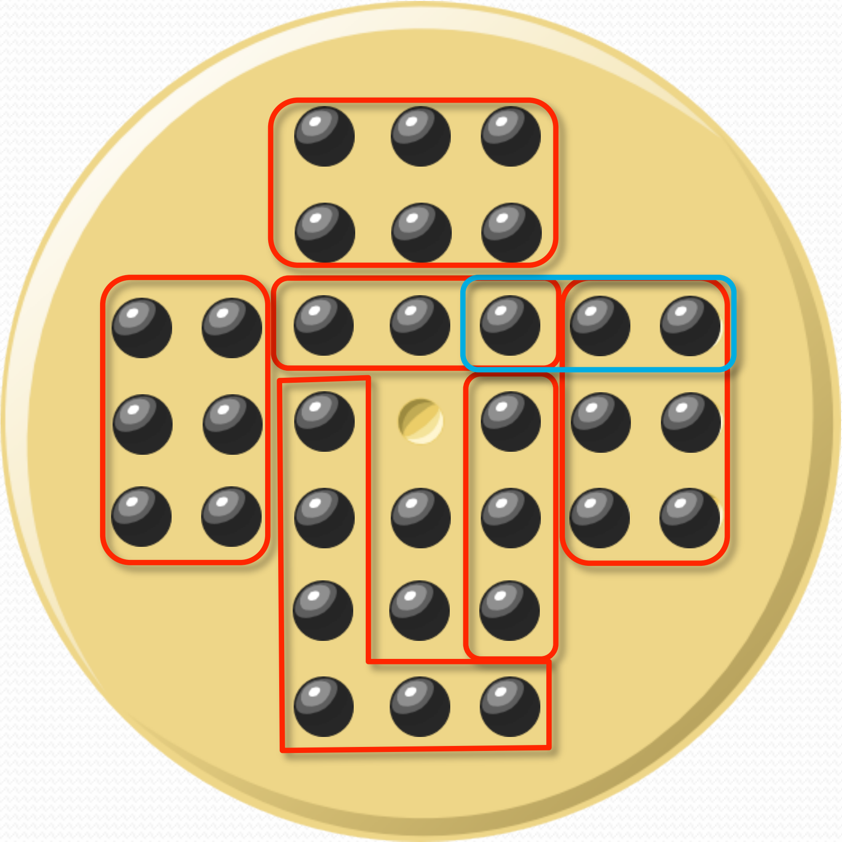
- Choose **2** consecutive pegs in a row (or column) adjacent to an empty hole in the same row (or column)
- Remove the **2** consecutive pegs and place one peg in the empty hole
- You win if only **1** peg is left in the central hole



English board

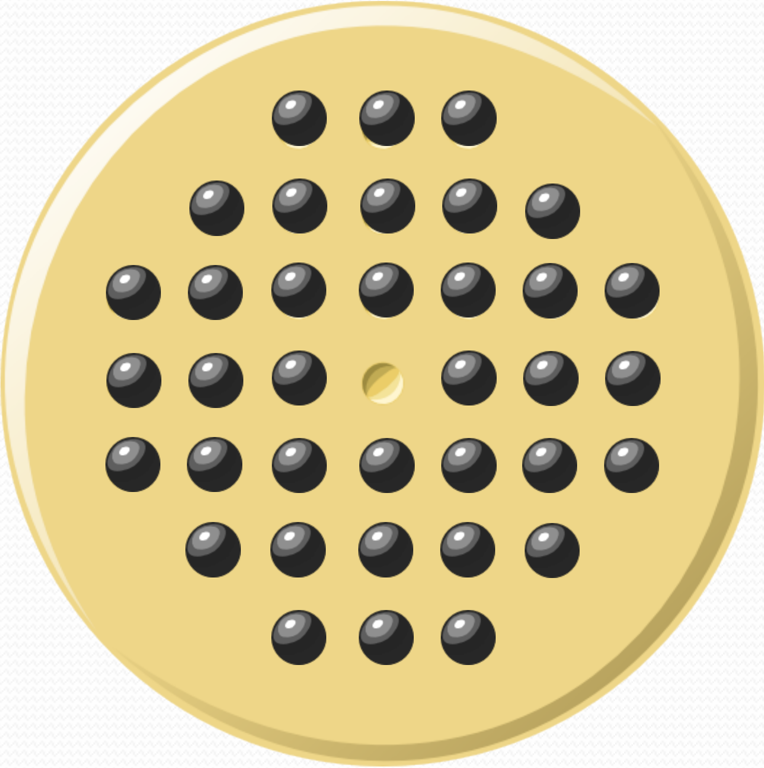
The purging strategy

- 3-purge : some triples can be removed without affecting others
- 6-purge
  includes a 3-purge
- L-purge
- *Game over*



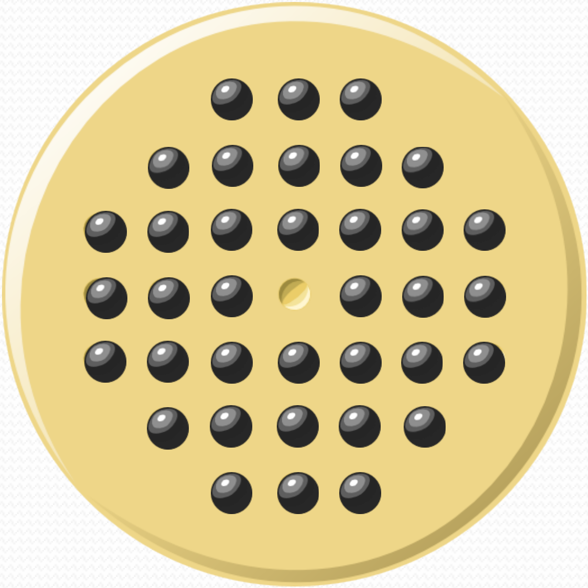
Can we solve any game?

- Toy shops allegedly promised free tickets to New York to the first person able to solve the game on a French board
- But one had to buy the game from the toy shop to enter the contest…
- But this game is infeasible

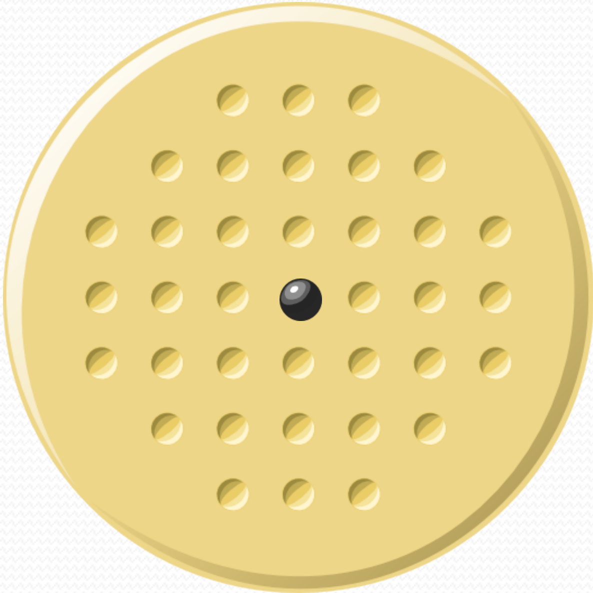


French board

Infeasibility of French solitaire game



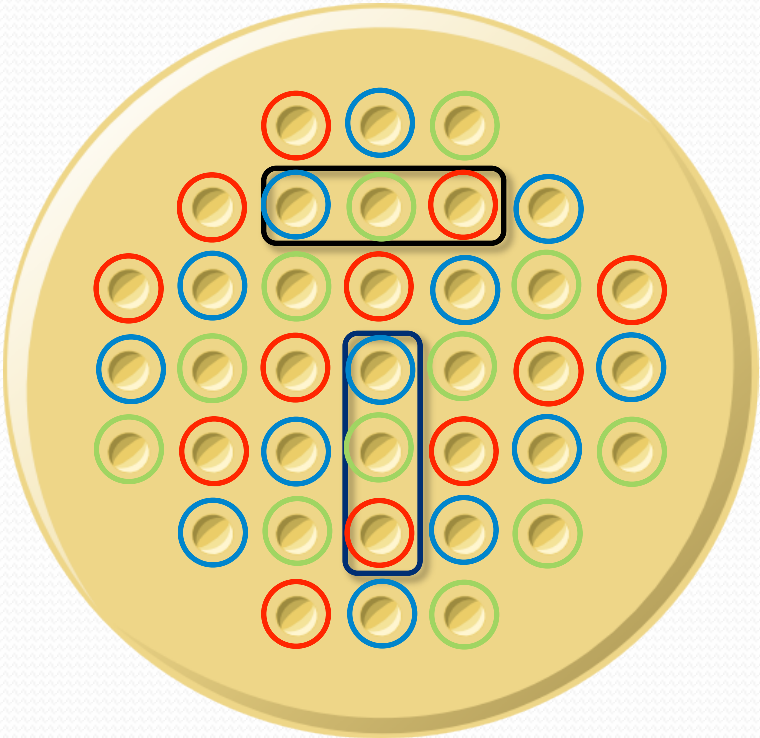
Initial configuration



Final configuration

Infeasibility of French solitaire game
Rule-of-Three [Suremain de Missery 1841]

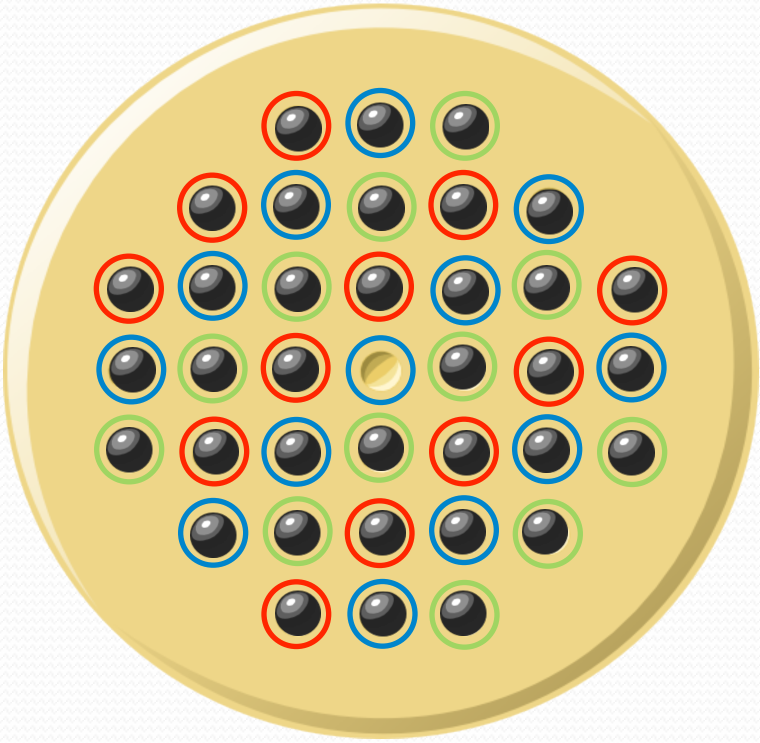
- Colour the diagonals of the board in red, blue, and green
- Any 3 adjacent positions in a row (or column) have all 3 colours



Infeasibility of French solitaire game
Rule-of-Three [Suremain de Missery 1841]

- Any move removes 2 pegs from 2 colours, and adds 1 peg to the other colour
- 3 cases:

#pegs in red	#pegs in blue	#pegs in green
-1	-1	+1
-1	+1	-1
+1	-1	-1



Infeasibility of French solitaire game Rule-of-Three [Suremain de Missery 1841]

We have 3 ***invariants*** under ***any move***:

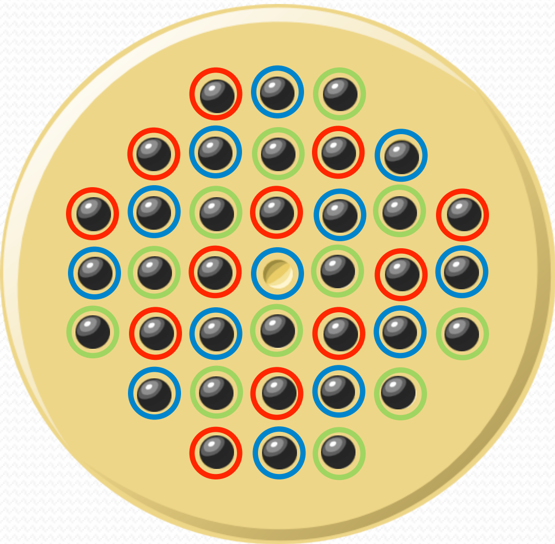
#(occupied red holes) – #(occupied green holes) (mod 2)

#(occupied green holes) – #(occupied blue holes) (mod 2)

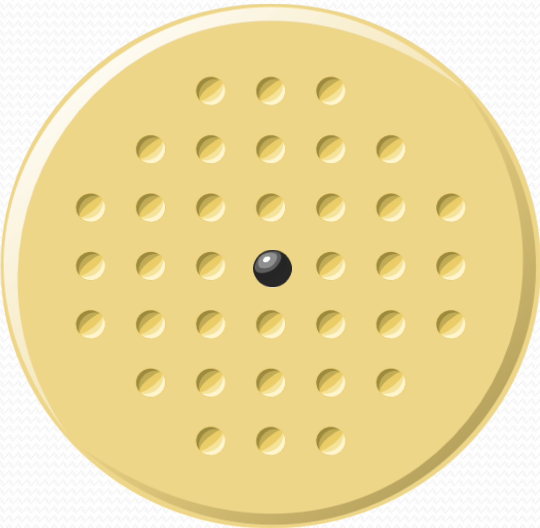
#(occupied blue holes) – #(occupied red holes) (mod 2)

#pegs in red	**#pegs in blue**	**#pegs in green**
-1	*-1*	*+1*
-1	*+1*	*-1*
+1	*-1*	*-1*

Infeasibility of French solitaire game
Rule-of-Three [Suremain de Missery 1841]



Initial configuration

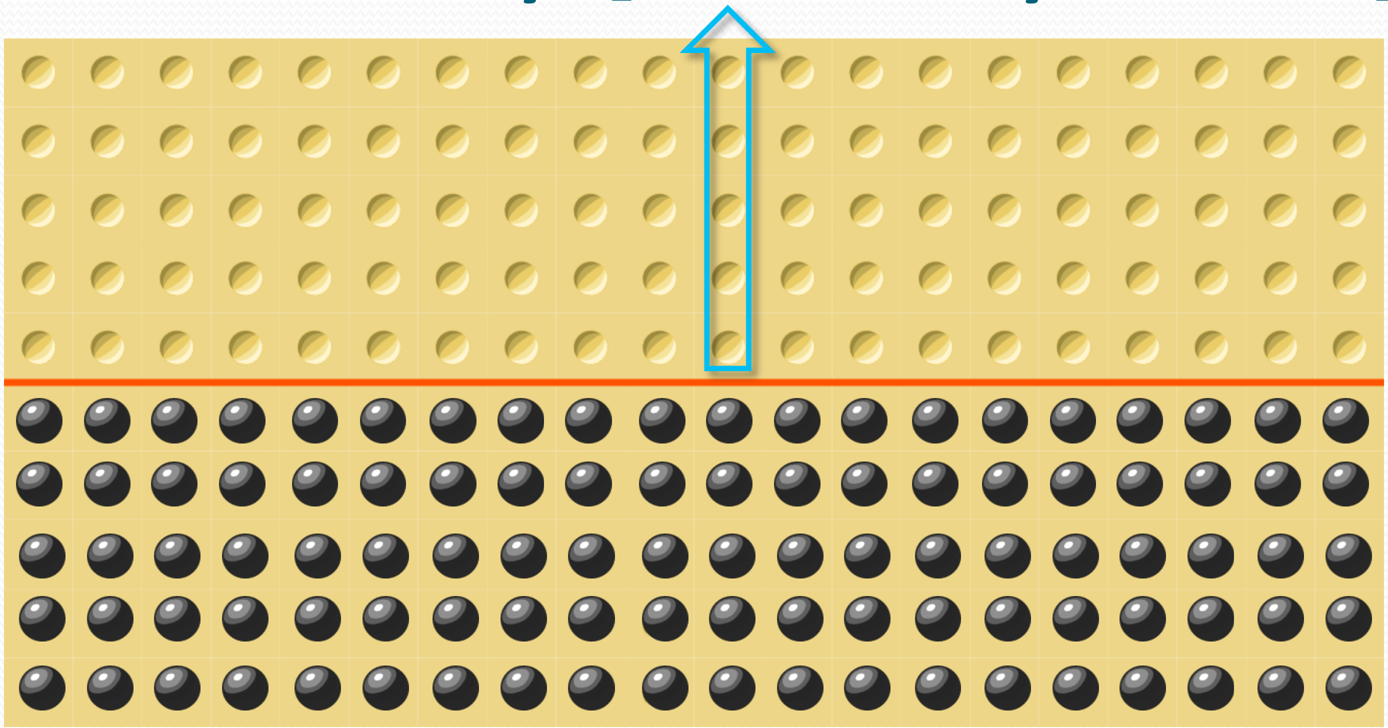


Final configuration

#Peg - #Peg = ***0*** (mod **2**)
#Peg - #Peg = ***0*** (mod **2**)
#Peg - #Peg = ***0*** (mod **2**)

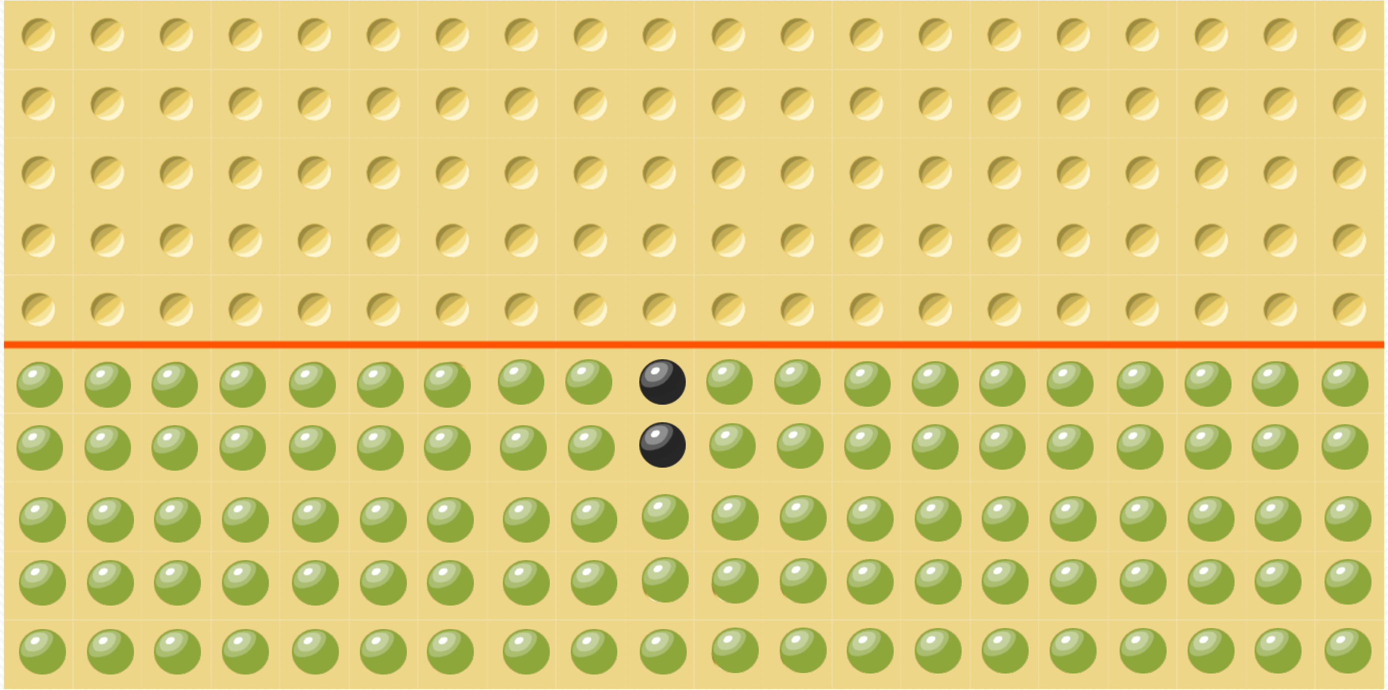
#Peg - #Peg = ***1*** (mod **2**)
#Peg - #Peg = ***1*** (mod **2**)
#Peg - #Peg = ***0*** (mod **2**)

Solitaire army [Conway 1961]



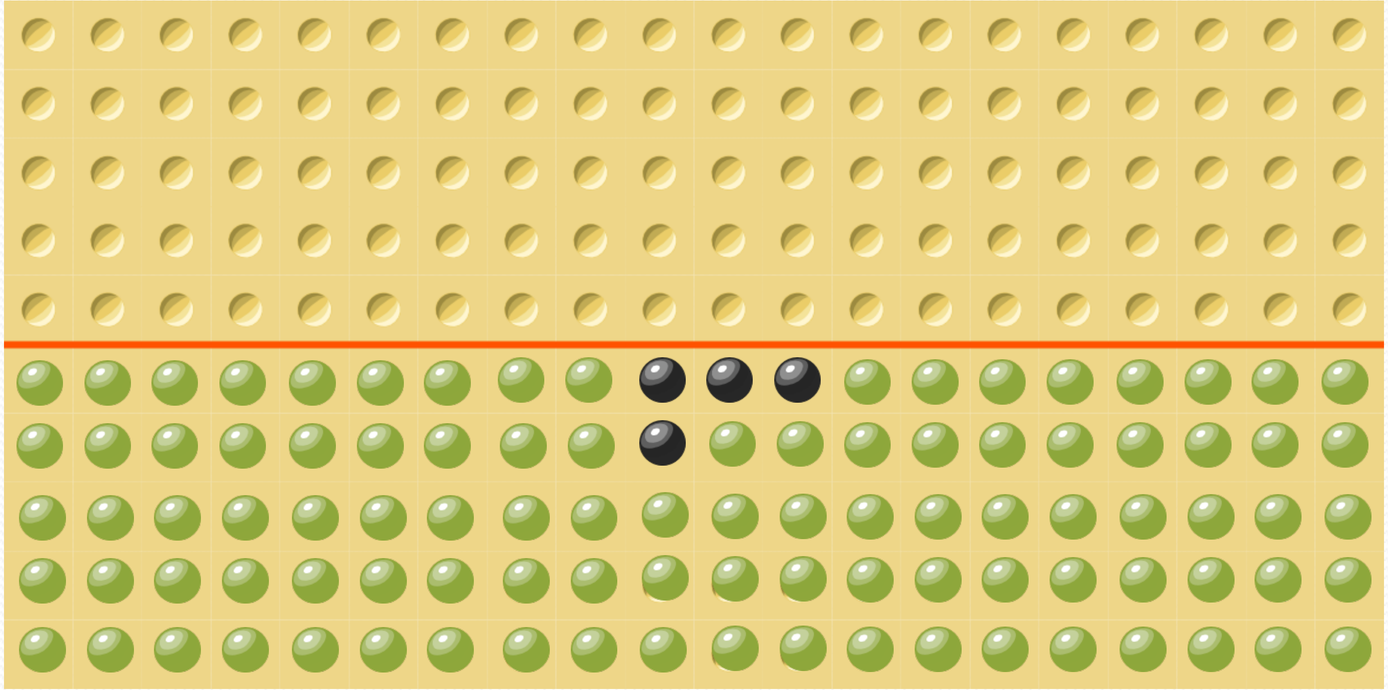
- Infinite board, as many pegs as needed
- Goal: advance one peg as far north as possible

Solitaire army [Conway 1961]



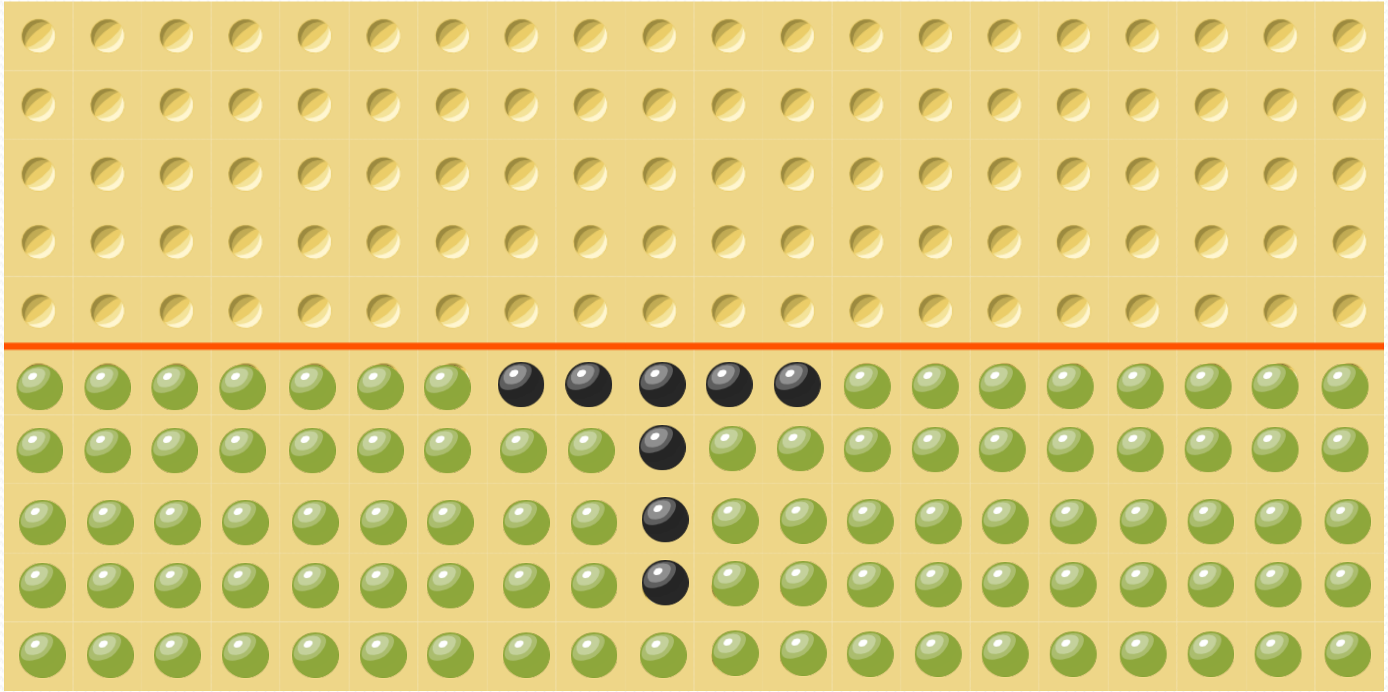
# of advances	1	2	3	4	5
min # of pegs needed	2				

Solitaire army [Conway 1961]



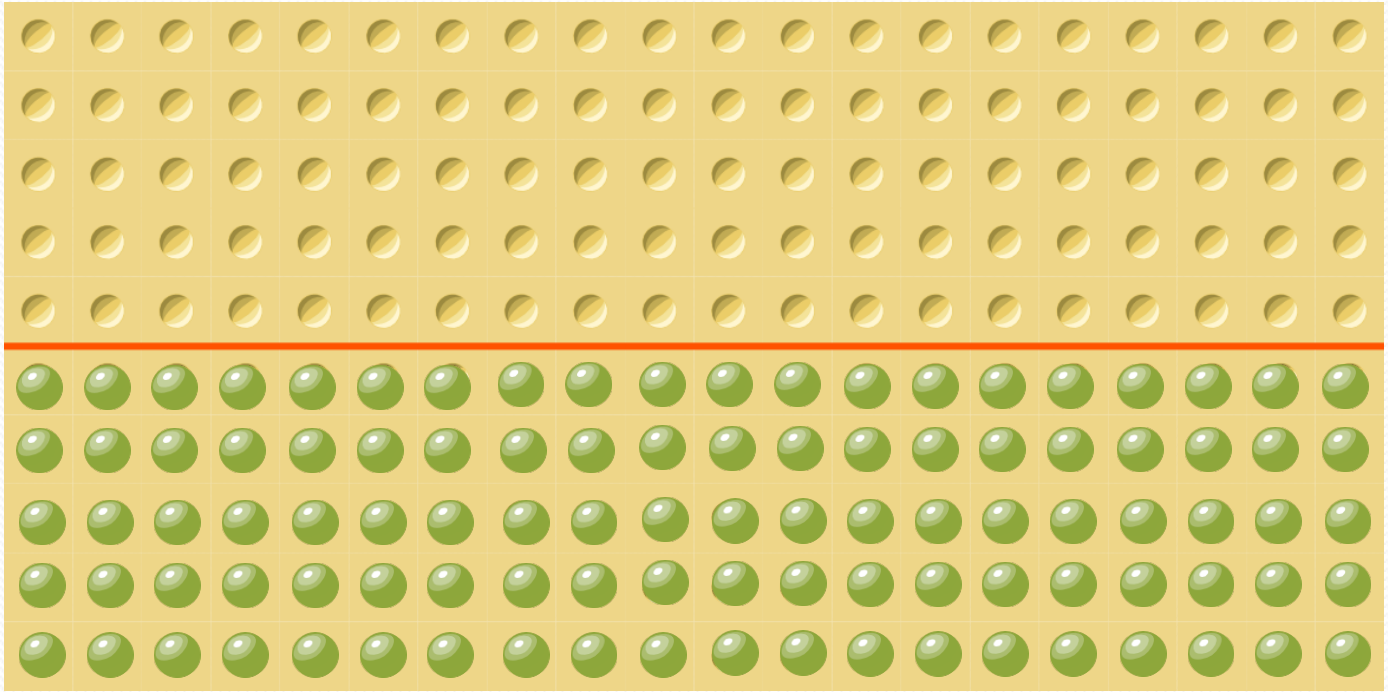
# of advances	1	2	3	4	5
min # of pegs needed	2	4			

Solitaire army [Conway 1961]



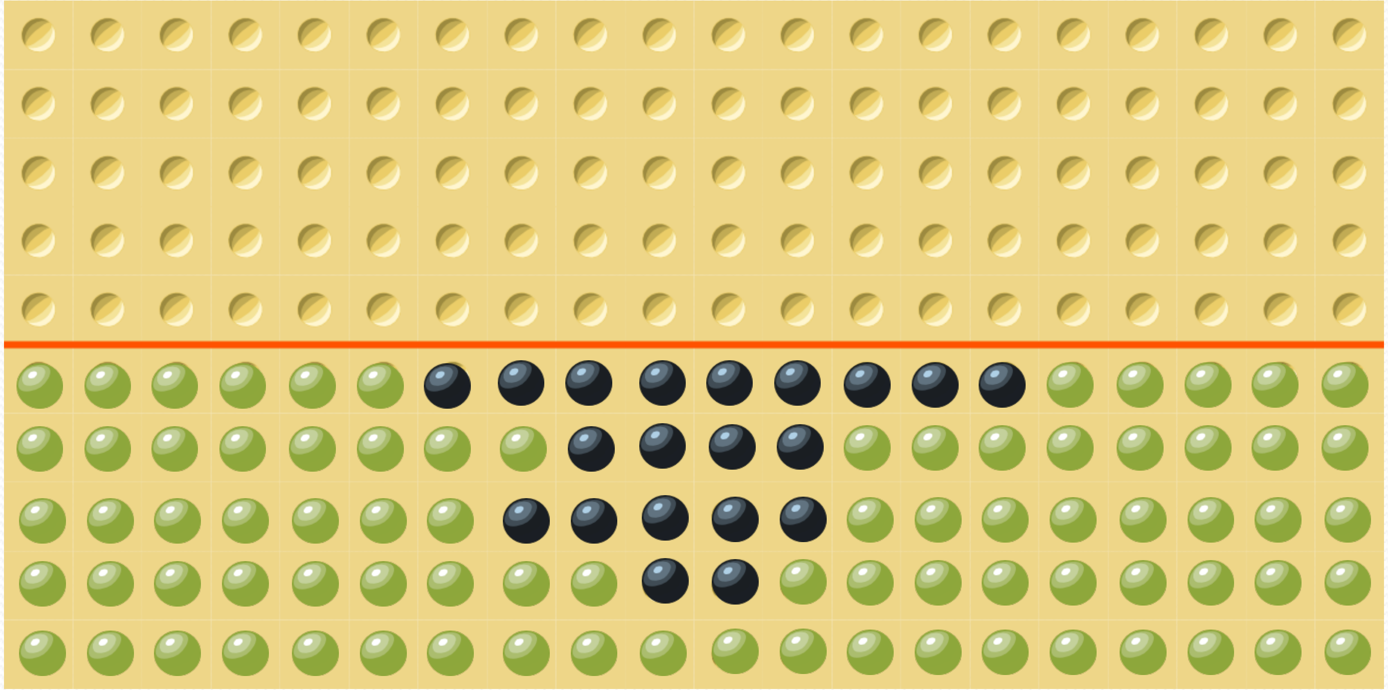
# of advances	1	2	3	4	5
min # of pegs needed	2	4	8		

Solitaire army [Conway 1961]



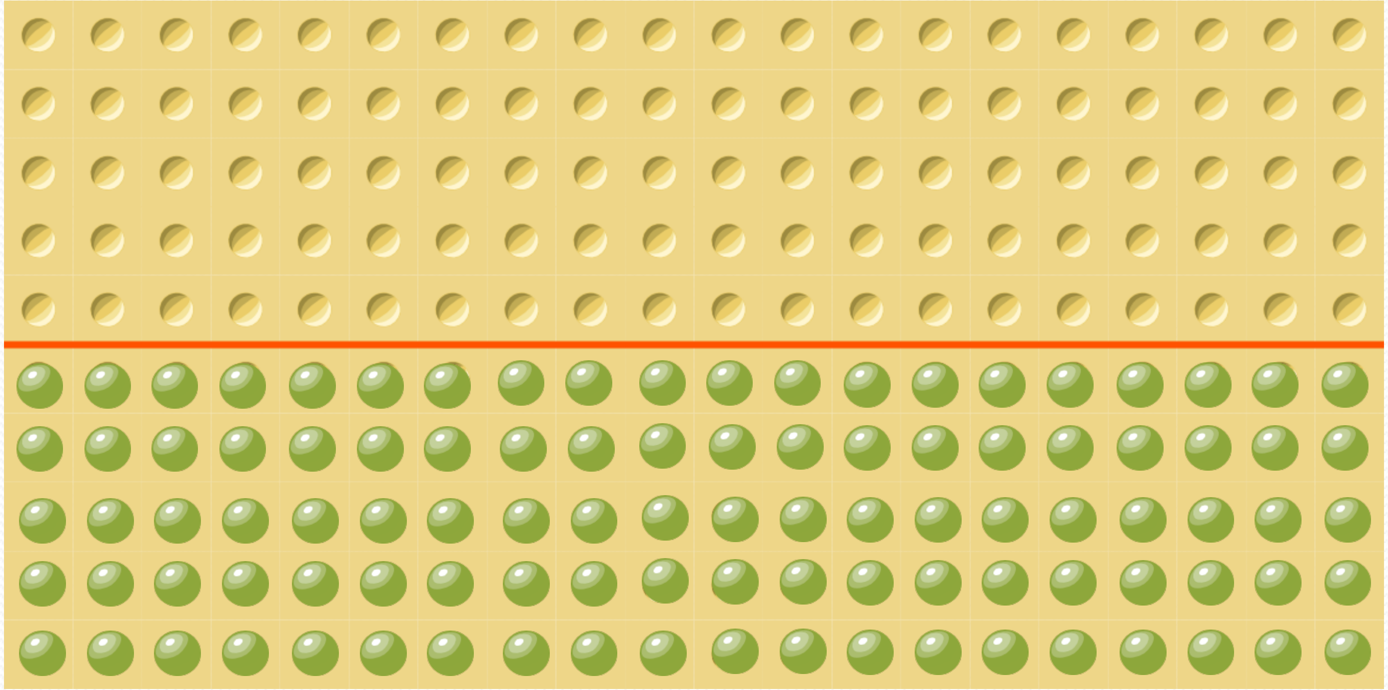
# of advances	1	2	3	4	5
min # of pegs needed	2	4	8	?	

Solitaire army [Conway 1961]



# of advances	1	2	3	4	5
min # of pegs needed	2	4	8	20	?

Solitaire army [Conway 1961]



# of advances	1	2	3	4	5
min # of pegs needed	2	4	8	20	impossible

Solitaire army : golden pagoda

…	p^5	p^4	p^3	p^2	p	p^2	p^3	p^4	p^5	…
…	p^4	p^3	p^2	p	**1**	p	p^2	p^3	p^4	…
…	p^5	p^4	p^3	p^2	p	p^2	p^3	p^4	p^5	…
…	p^6	p^5	p^4	p^3	p^2	p^3	p^4	p^5	p^6	…
…	p^7	p^6	p^5	p^4	p^3	p^4	p^5	p^6	p^7	…
…	p^8	p^7	p^6	p^5	p^4	p^5	p^6	p^7	p^8	…
…	p^9	p^8	p^7	p^6	p^5	p^6	p^7	p^8	p^9	…
…	p^{10}	p^9	p^8	p^7	p^6	p^7	p^8	p^9	p^{10}	…
…	p^{11}	p^{10}	p^9	p^8	p^7	p^8	p^9	p^{10}	p^{11}	…
…	p^{12}	p^{11}	p^{10}	p^9	p^8	p^9	p^{10}	p^{11}	p^{12}	…
…	p^{13}	p^{12}	p^{11}	p^{10}	p^9	p^{10}	p^{11}	p^{12}	p^{13}	…

p: golden ratio ($p^2 + p = 1$)

Assign a value to each hole

Solitaire army : golden pagoda

…	p^5	p^4	p^3	p^2	p	p^2	p^3	p^4	p^5	…
…	p^4	p^3	p^2	p	1	p	p^2	p^3	p^4	…
…	p^5	p^4	p^3	p^2	p	p^2	p^3	p^4	p^5	…
…	p^6	p^5	p^4	p^3	p^2	p^3	p^4	p^5	p^6	…
…	p^7	p^6	p^5	p^4	p^3	p^4	p^5	p^6	p^7	…
…	p^8	p^7	p^6	p^5	p^4	p^5	p^6	p^7	p^8	…
…	p^9	p^8	p^7	p^6	p^5	p^6	p^7	p^8	p^9	…
…	p^{10}	p^9	p^8	p^7	p^6	p^7	p^8	p^9	p^{10}	…
…	p^{11}	p^{10}	p^9	p^8	p^7	p^8	p^9	p^{10}	p^{11}	…
…	p^{12}	p^{11}	p^{10}	p^9	p^8	p^9	p^{10}	p^{11}	p^{12}	…
…	p^{13}	p^{12}	p^{11}	p^{10}	p^9	p^{10}	p^{11}	p^{12}	p^{13}	…

$= p^2$

$= p^3$

$= p^4$

$= p^5$

$= p^6$

p: golden ratio ($p^2 + p = 1$)

Initial total value = 1

$$p^6 + p^7 + p^8 + \dots = p^6/(1 - p) = p^4$$

$$p^4 + p^5 + p^4 = p^3 + p^4 = p^2$$

$$p^2 + p^3 + p^4 + \dots = p^2/(1 - p) = 1$$

Assuming we have an *infinite* number of pegs initially

Solitaire army : golden pagoda

…	p^5	p^4	p^3	p^2	p	p^2	p^3	p^4	p^5	…
…	p^4	p^3	p^2	p	**1**	p	p^2	p^3	p^4	…
…	p^5	p^4	p^3	p^2	p	p^2	p^3	p^4	p^5	…
…	p^6	p^5	p^4	p^3	p^2	p^3	p^4	p^5	p^6	…
…	p^7	p^6	p^5	p^4	p^3	p^4	p^5	p^6	p^7	…
…	p^8	p^7	p^6	p^5	p^4	p^5	p^6	p^7	p^8	…
…	p^9	p^8	p^7	p^6	p^5	p^6	p^7	p^8	p^9	…
…	p^{10}	p^9	p^8	p^7	p^6	p^7	p^8	p^9	p^{10}	…
…	p^{11}	p^{10}	p^9	p^8	p^7	p^8	p^9	p^{10}	p^{11}	…
…	p^{12}	p^{11}	p^{10}	p^9	p^8	p^9	p^{10}	p^{11}	p^{12}	…
…	p^{13}	p^{12}	p^{11}	p^{10}	p^9	p^{10}	p^{11}	p^{12}	p^{13}	…

p: golden ratio ($p^2 + p = 1$)

Initial total value = 1

Final total value $\geq$ 1

Solitaire army : pagoda function

…	p^5	p^4	p^3	p^2	p	p^2	p^3	p^4	p^5	…
…	p^4	p^3	p^2	p	1	p	p^2	p^3	p^4	…
…	p^5	p^4	p^3	p^2	p	p^2	p^3	p^4	p^5	…
…	p^6	p^5	p^4	p^3	p^2	p^3	p^4	p^5	p^6	…
…	p^7	p^6	p^5	p^4	p^3	p^4	p^5	p^6	p^7	…
…	p^8	p^7	p^6	p^5	p^4	p^5	p^6	p^7	p^8	…
…	p^9	p^8	p^7	p^6	p^5	p^6	p^7	p^8	p^9	…
…	p^{10}	p^9	p^8	p^7	p^6	p^7	p^8	p^9	p^{10}	…
…	p^{11}	p^{10}	p^9	p^8	p^7	p^8	p^9	p^{10}	p^{11}	…
…	p^{12}	p^{11}	p^{10}	p^9	p^8	p^9	p^{10}	p^{11}	p^{12}	…
…	p^{13}	p^{12}	p^{11}	p^{10}	p^9	p^{10}	p^{11}	p^{12}	p^{13}	…

p: golden ratio ($p^2 + p = 1$)

Initial total value = 1

Final total value $\geq$ 1

$$p^6 + p^7 \geq p^8$$

$$p^9 + p^{10} = p^8$$

After ***any move***, the total value can only ***decrease*** or stay the same

Solitaire army : golden pagoda

…	p^5	p^4	p^3	p^2	p	p^2	p^3	p^4	p^5	…
…	p^4	p^3	p^2	p	1	p	p^2	p^3	p^4	…
…	p^5	p^4	p^3	p^2	p	p^2	p^3	p^4	p^5	…
…	p^6	p^5	p^4	p^3	p^2	p^3	p^4	p^5	p^6	…
…	p^7	p^6	p^5	p^4	p^3	p^4	p^5	p^6	p^7	…
…	p^8	p^7	p^6	p^5	p^4	p^5	p^6	p^7	p^8	…
…	p^9	p^8	p^7	p^6	p^5	p^6	p^7	p^8	p^9	…
…	p^{10}	p^9	p^8	p^7	p^6	p^7	p^8	p^9	p^{10}	…
…	p^{11}	p^{10}	p^9	p^8	p^7	p^8	p^9	p^{10}	p^{11}	…
…	p^{12}	p^{11}	p^{10}	p^9	p^8	p^9	p^{10}	p^{11}	p^{12}	…
…	p^{13}	p^{12}	p^{11}	p^{10}	p^9	p^{10}	p^{11}	p^{12}	p^{13}	…

p: golden ratio ($p^2 + p = 1$)

Initial total value < 1

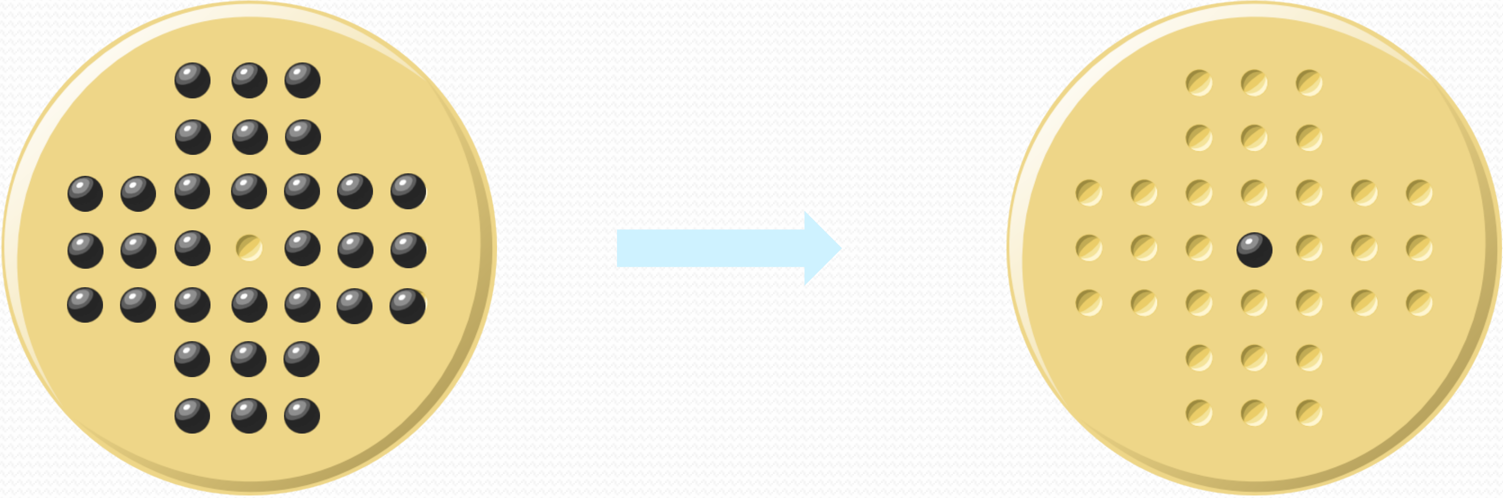
final total value ≥ 1

However many pegs we put on the board, level 5 can not be reached

Assuming we have a *finite* number of pegs initially

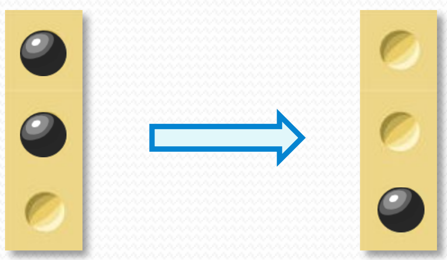
Feasibility problem

Given a board, an initial configuration ***c*** and a final configuration ***c'*** , is there a legal sequence of moves from ***c*** to ***c'*** ?



The peg solitaire problem is *feasible* on the English board, but *infeasible* on the French board.

Feasibility problem - formulation



$$\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}-1\\-1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

- The board has n holes
- A configuration c can be represented by a {0,1}-vector of length n
- A move can be represented by a vector m_i of length n with 3 non-zero entries: two **-1** and one **1**

Feasibility condition:

$$c' - c = \sum_{i=1}^{n-2} m_i, \quad c + \sum_{i=1}^{j} m_i \in \{0,1\}^n, \quad j = 1,2,\ldots,n-2$$

Some Relaxations

Feasibility condition:

$$c' - c = \sum_{i=1}^{n-2} m_i, \quad c + \sum_{i=1}^{j} m_i \in \{0,1\}^n, \quad j = 1,2,\ldots,n-2$$

$\Downarrow$ relax 0-1 condition

$$c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{Z}^+$$

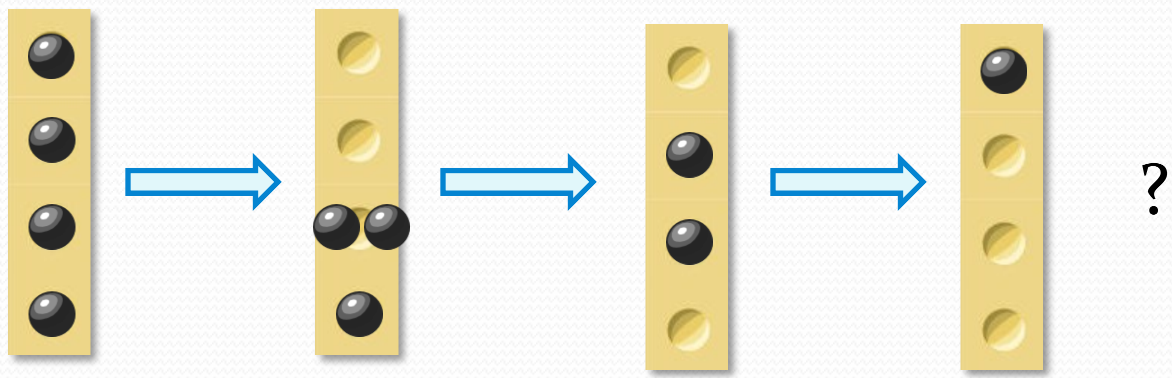
relax non-negativity $\swarrow$

$$c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{Z}$$

$\searrow$ relax integrality

$$c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{R}^+$$

Relaxations: non-negative integers



$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \begin{pmatrix}-1\\-1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\-1\\-1\end{pmatrix} + \begin{pmatrix}1\\-1\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

Feasibility condition: $c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{Z}^+$

Relaxations

Feasibility condition:

$$c' - c = \sum_{i=1}^{n-2} m_i, \quad c + \sum_{i=1}^{j} m_i \in \{0,1\}^n, \quad j = 1, 2, \ldots, n-2$$

$\Downarrow$ **relax 0-1 condition**

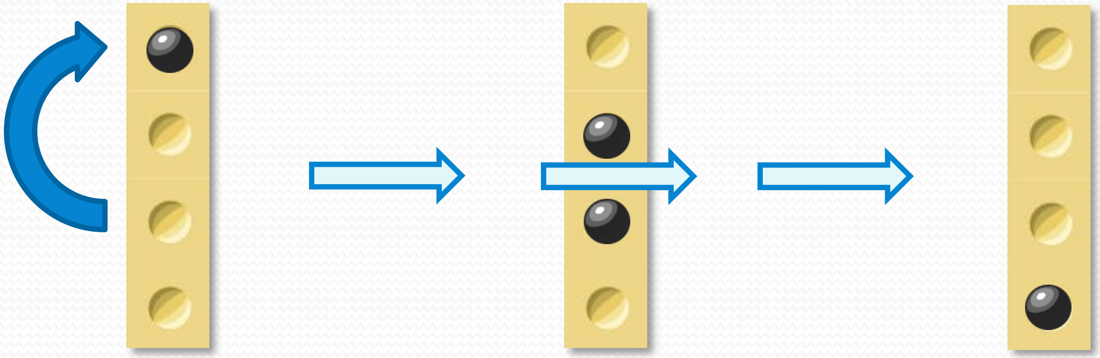
$$c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{Z}^+$$

relax non-negativity $\swarrow$ $\searrow$ **relax integrality**

$$c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{Z}$$

$$c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{R}^+$$

Relaxations: integer game



?

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + (-1)\begin{pmatrix}1\\-1\\-1\\0\end{pmatrix} + (1)\begin{pmatrix}0\\-1\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Feasibility condition: $c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{Z}$

Relaxations

Feasibility condition:

$$c' - c = \sum_{i=1}^{n-2} m_i, \quad c + \sum_{i=1}^{j} m_i \in \{0,1\}^n, \quad j = 1,2,\ldots,n-2$$

$\Downarrow$ **relax 0-1 condition**

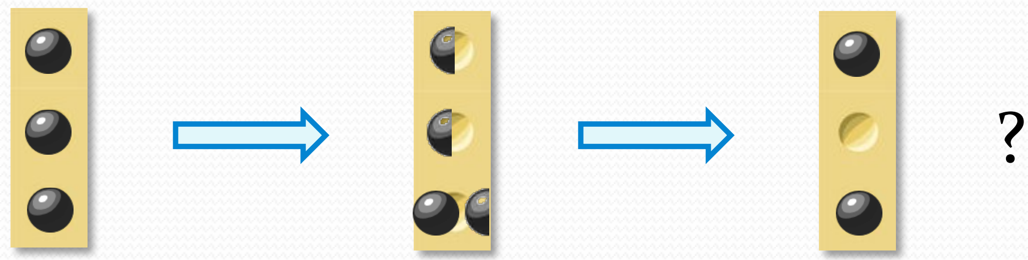
$$c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{Z}^+$$

relax non-negativity $\swarrow$ $\searrow$ **relax integrality**

$$c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{Z}$$

$$c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{R}^+$$

Relaxations: fractional game



$$\begin{pmatrix}1\\1\\1\end{pmatrix} + 0.5\begin{pmatrix}-1\\-1\\1\end{pmatrix} + 0.5\begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

Feasibility condition:

$$c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{R}^+$$

Geometric interpretation

Feasibility conditions :

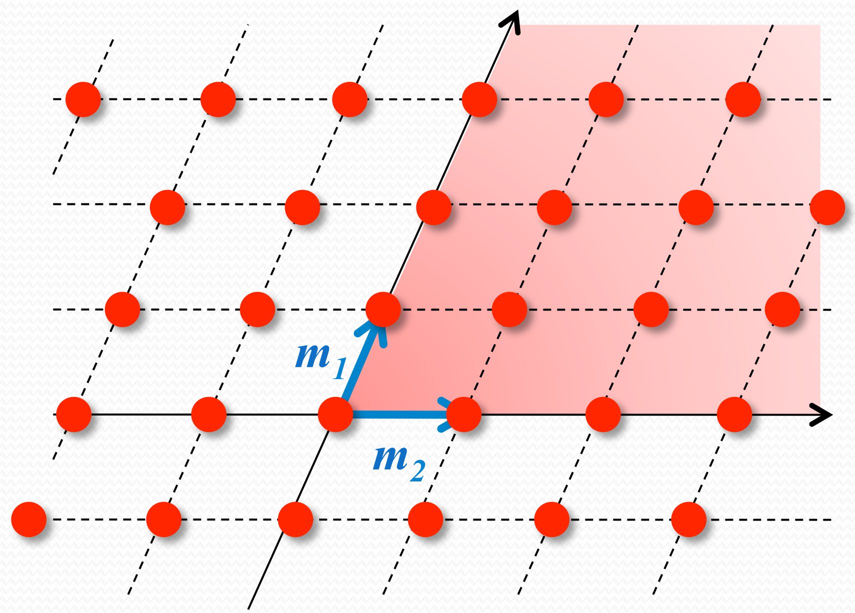
0-1 $\quad c' - c = \sum_{i=1}^{n-2} m_i, \quad c + \sum_{i=1}^{j} m_i \in \{0,1\}^n, \quad j = 1, 2, \ldots, n-2$

Natural $\quad c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{Z}^+$

Integer $\quad c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{Z}$

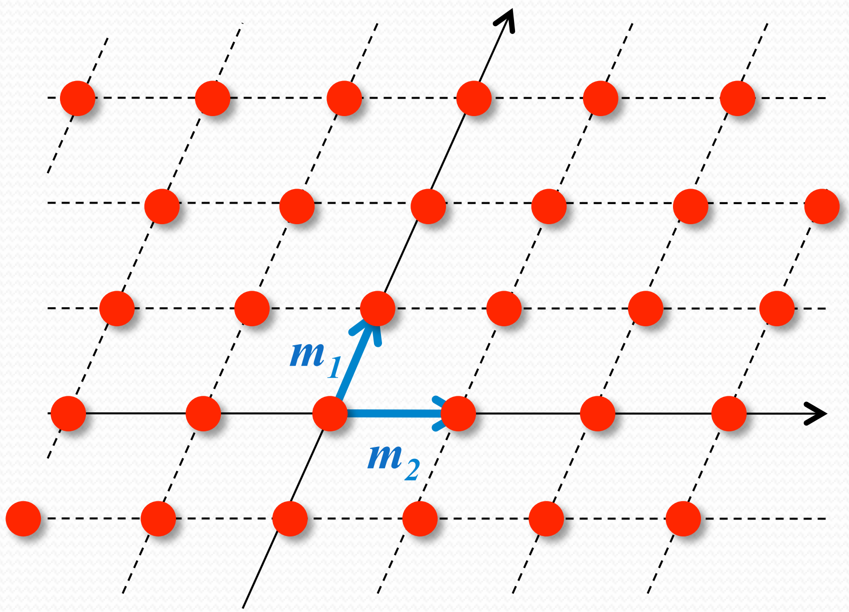
Fractional $\quad c' - c = \sum_{m \in M} \lambda_m m, \quad \lambda_m \in \mathbb{R}^+$

A given game is feasible if $c' - c$ is in a certain range :



Geometric interpretation

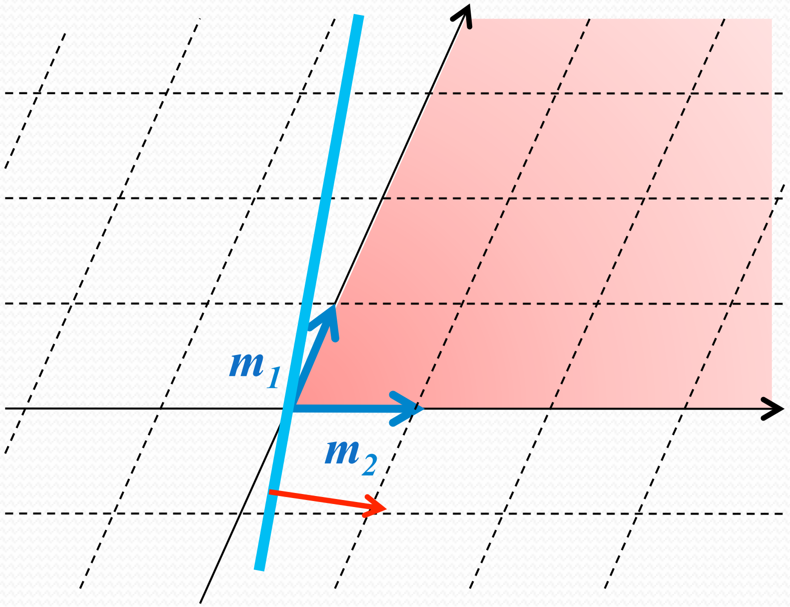
Solitaire lattice : the set of all *integer* combinations of moves



Rule-of-Three (almost) amounts to *lattice feasibility*

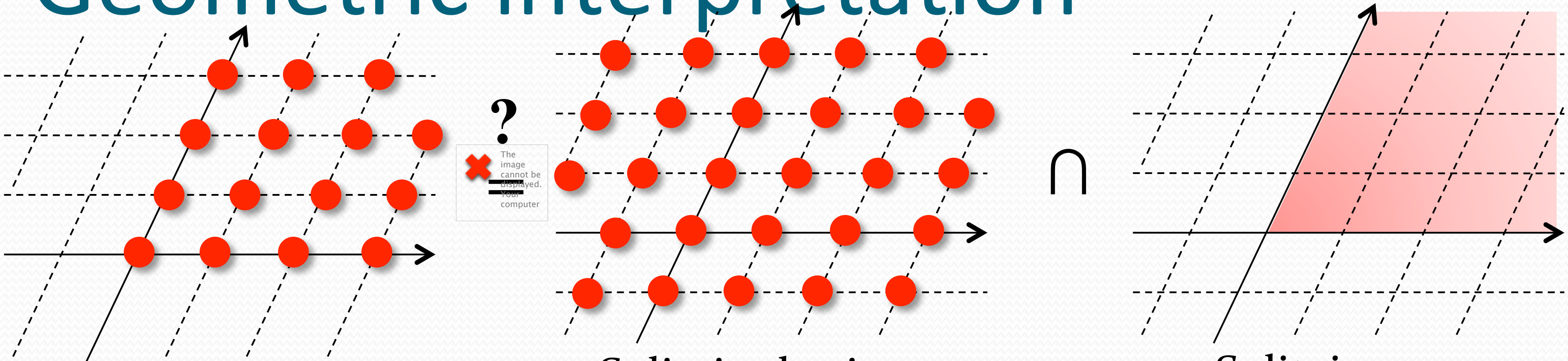
Geometric interpretation

Solitaire cone : the set of *non-negative* combinations of all legal moves



Pagoda functions (in particular facets) amounts to *cone feasibility*

Geometric interpretation



Solitaire integer cone | Solitaire lattice | Solitaire cone

$$v = \begin{pmatrix}1\\1\\8\end{pmatrix},\quad m_1 = \begin{pmatrix}3\\0\\3\end{pmatrix},\quad m_2 = \begin{pmatrix}0\\3\\3\end{pmatrix},\quad m_3 = \begin{pmatrix}0\\0\\3\end{pmatrix},\quad m_4 = \begin{pmatrix}2\\2\\1\end{pmatrix}$$

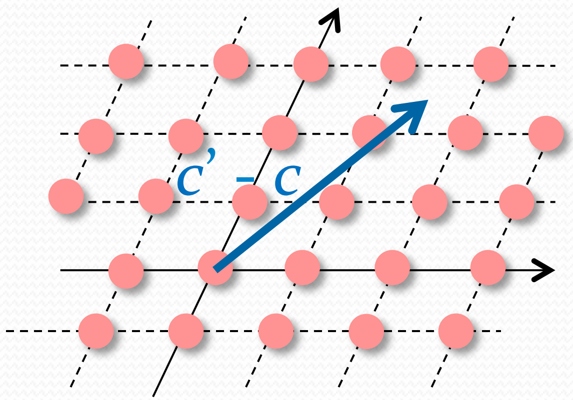
v is ***in*** the solitaire lattice : $v = m_1 + m_2 + m_3 - m_4$

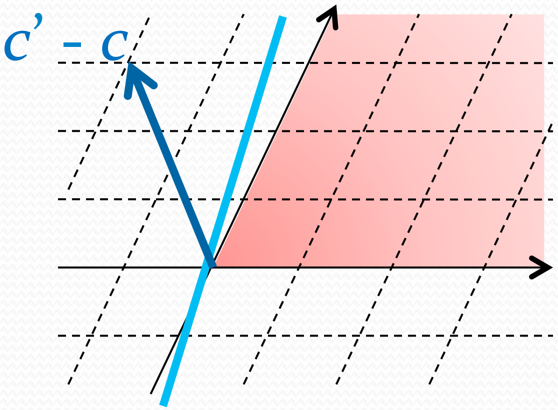
v is ***in*** the solitaire cone : $v = \frac{1}{3}m_1 + \frac{1}{3}m_2 + 2m_3 + 0m_4$

v is ***not in*** the solitaire integer cone

Geometric interpretation

Infeasibility of the original game is *implied* by infeasibility of any *relaxation*





Peg solitaire infeasible on French board :
$c' - c$ is ***not in*** the solitaire ***lattice***

Solitaire army infeasible at level 5:
$c' - c$ ***is not*** in the solitaire ***cone***

Geometric and combinatorial properties of the solitaire cone and lattice

- English board: 33-dimensional cone, 76 moves, 9.2 million facets (estimated) -- question raised by Donald Knuth, Günter Ziegler

  [Avis, Deza: *Mathematical Programming* 2002]
- Lattice criterion vs Rule-of-Three

  [Deza, Onn: *Graphs and Combinatorics* 2002]
- Upper & lower bounds on the number of facets (exponential in the dimension) for toric boards, characterization of {0,1}-facets, incidence, adjacency and diameter (rectangular boards)

  [Avis, Deza: *Discrete Applied Mathematics* 2001]

  [Avis, Deza, Onn : *IEICE Transactions* 2000]

Geometric and combinatorial properties of the solitaire cone and lattice

- Equivalence with a (dual) metric cone for a generalized solitaire game; related NP-completeness [Avis, Deza 2001]
- Metric/cut analogue for the relaxation of the solitaire cone by its {0,1}-valued facets [Avis, Deza 2001]
- The feasibility of 0-1 game is NP-complete on the *n* by *n* board, even if the final position contains exactly one peg [Uehara-Iwata 1990]; this indicates that easily checked necessary and sufficient conditions for feasibility are unlikely to exist

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Thank You