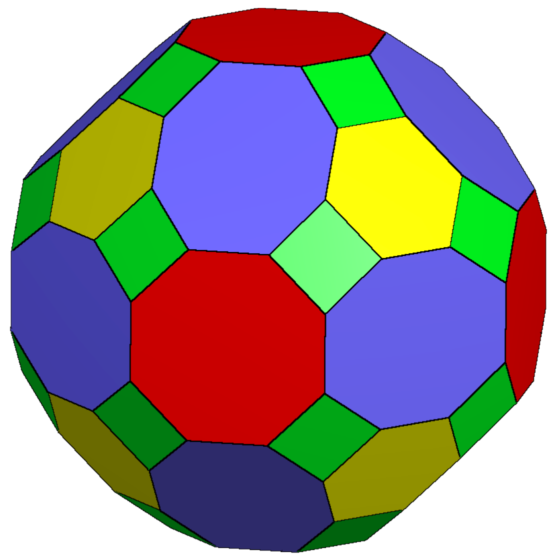
On lattice polytopes, convex matroid optimization, and degree sequences of hypergraphs



Antoine Deza, McMaster

Linear optimization

Given an n-dimensional vector $\boldsymbol{b}$ and an n x d matrix $\boldsymbol{A}$
find, in any, a d-dimensional vector $\boldsymbol{x}$ such that :

$\boldsymbol{Ax} = \boldsymbol{b}$	$\boldsymbol{Ax} = \boldsymbol{b}$ $\boldsymbol{x} \geq 0$
linear algebra	**linear optimization**

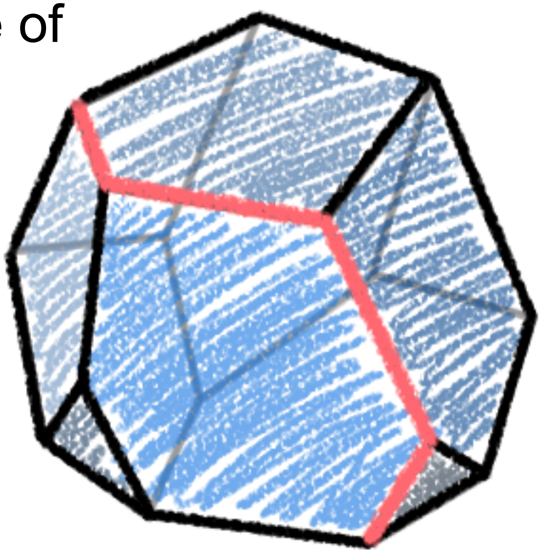
Linear optimization

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** matrix ***A***
find, in any, a ***d***-dimensional vector ***x*** such that :

$\boldsymbol{Ax = b}$	$\boldsymbol{Ax \leq b}$
linear algebra	**linear optimization**

*Can linear optimization be solved in **strongly polynomial*** time? is listed by Smale (Fields Medal 1966) as one of the top mathematical problems for the XXI century

Strongly polynomial *:* algorithm ***independent*** from the ***input data length*** and polynomial in ***n*** and ***d****.*

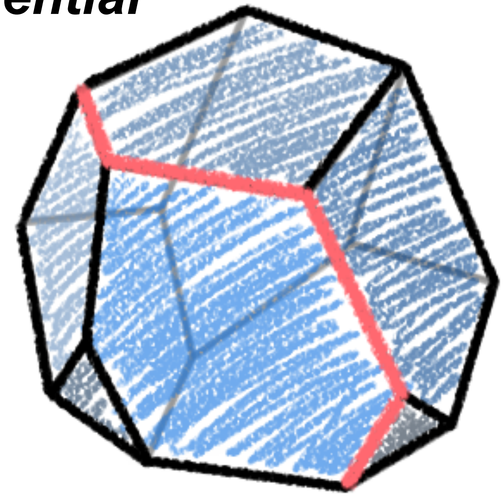


Linear optimization algorithms
simplex methods

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** (full row-rank) matrix ***A*** and a ***d***-dimensional cost vector ***c***, solve : $\{ \max \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \}$

Simplex methods (Dantzig 1947): pivot-based, combinatorial, ***not proven to be polynomial***, efficient in practice

- start from a ***feasible basis***
- use a ***pivot rule***
- find an optimal solution after a ***finite number*** of iterations
- most known pivot rules are known to be ***exponential*** (worst case); ***efficient*** implementations exist

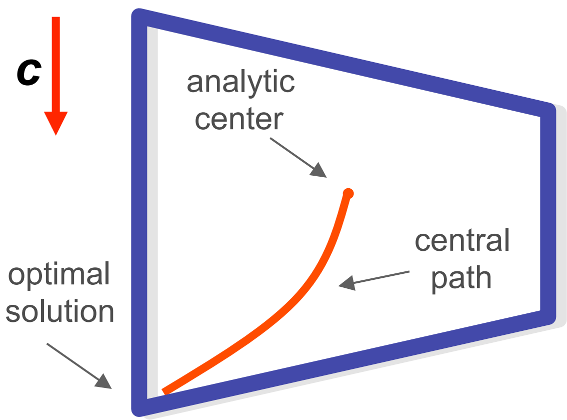


Linear optimization algorithms
(central path following) interior point methods

Given an n-dimensional vector $\boldsymbol{b}$ and an n x d (full row-rank) matrix $\boldsymbol{A}$ and a d-dimensional cost vector $\boldsymbol{c}$, solve : $\{ \max \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \}$

Interior Point Methods :
path-following, ***polynomial***, efficient in practice

- start from the ***analytic center***
- follow the ***central path***
- converge to an optimal solution in $O(\sqrt{n}L)$ iterations
  (L: input data length)



$$\max \quad \boldsymbol{c}^{\mathrm{T}} x - \mu \sum_i \ln(b - Ax)_i$$

μ : central path parameter
$x \in \boldsymbol{P} : Ax \leq b$

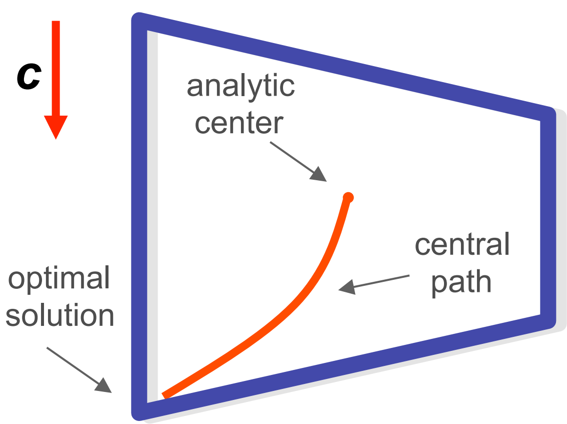
Linear optimization diameter and curvature

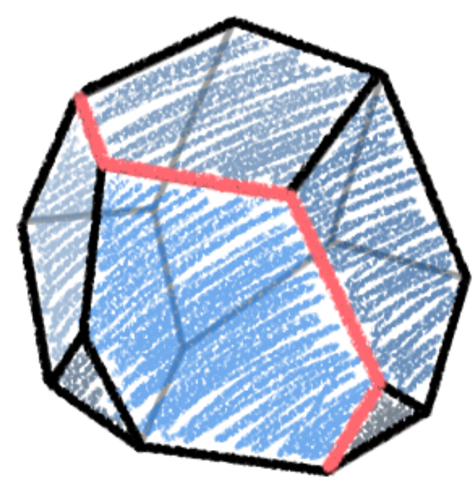
Diameter (*of a polytope*) :

lower bound for the number of iterations for *pivoting* **simplex methods**

Curvature (*of the central path associated to a polytope*) :

large curvature indicates large number of iterations for *path following* **interior point methods**





Linear Optimization

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** matrix ***A***
find, in any, a ***d***-dimensional vector ***x*** such that :

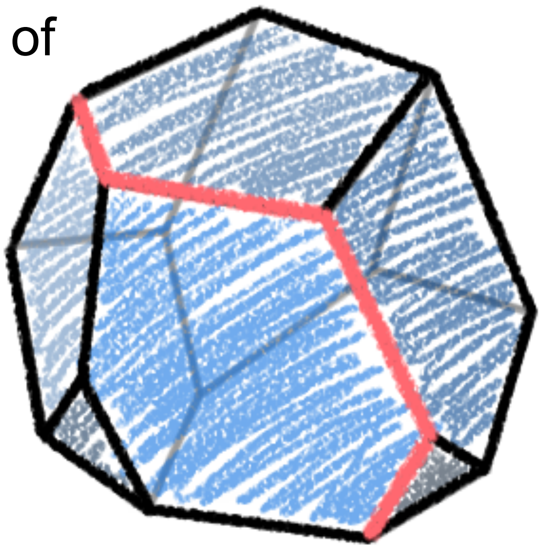
Ax = ***b*** ***Ax*** ≤ ***b***

linear algebra **linear optimization**

Can linear optimization be solved in ***strongly polynomial*** time? is listed by Smale (Fields Medal 1966) as one of the top mathematical problems for the XXI century

Strongly polynomial *:* algorithm ***independent*** from the ***input data length*** and polynomial in ***n*** and ***d.***

Interior point methods are ***not*** strongly polynomial
[Allamigeon, Benchimol, Gaubert, Joswig 2018]



(tropical counterexample to continuous Hirsch conjecture [Deza-Terlaky-Zinchenko 2008])

Discrete optimization and theoretical physics

- Ising model (spin glasses)
  maxcut, cut and metric polytopes [Deza-Laurent 1997]

- a(***d***) : number of generalized retarded functions in quantum field theory
  (number of real-time Green functions) [Evans 1994]

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a(***d***) = number of regions of the arrangement formed by the 2^{d} -1 hyperplanes with {0,1}-valued normals in dimension ***d***

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d = 2 2^d -1 = 3 hyperplanes

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(0,1)

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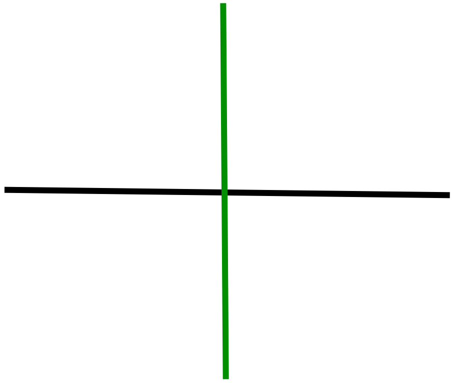
(0,1)
(1,0)



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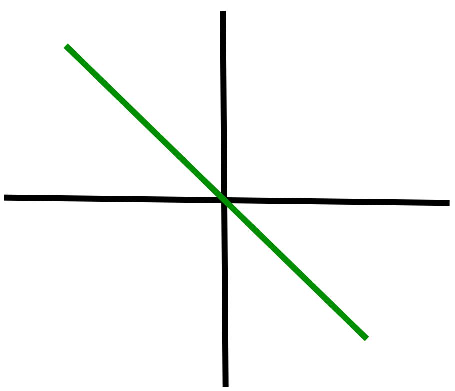
d = 2 $2^d - 1 = 3$ hyperplanes

(0,1)
(1,0)
(1,1)



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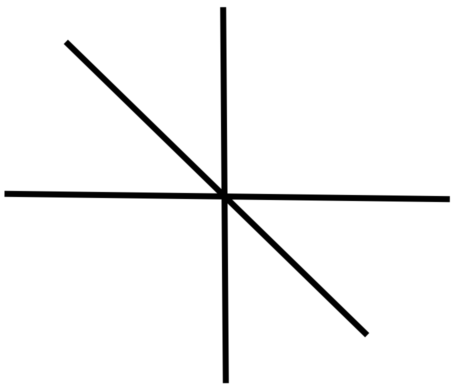

- a(**2**)=6

6 regions

Discrete optimization and theoretical physics

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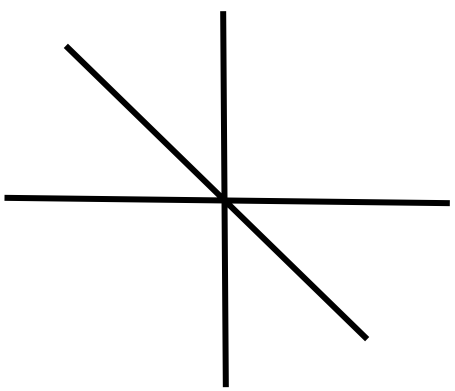
  (number of real-time Green functions) [Evans 1994]

a(d) = number of regions of the arrangement formed by the 2^d -1 hyperplanes with {0,1}-valued normals in dimension d

- is a(d) ≥ d ! [Question by Evans]
- a(d) determined till d = 8
- how to estimate a(d) ?



a(d) regions

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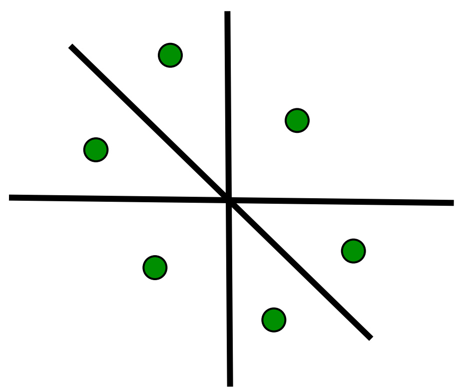

a(*d*) regions => a(*d*) vertices

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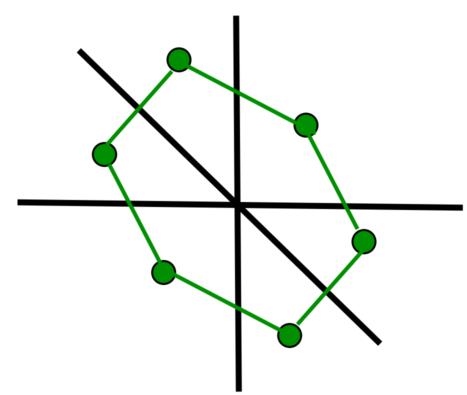
- how to estimate a(***d***) ?
- a(***d***) vertices of the *white whale*



a(**2**) = 6

Discrete optimization and theoretical physics

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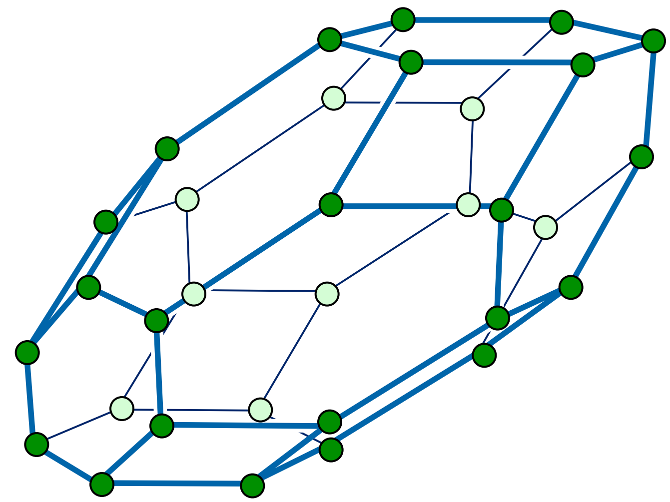
- how to estimate a(***d***) ?
- a(***d***) vertices of the *white whale*



a(**3**) = 32

Berlin Mathematical School

BMS Friday Colloquium

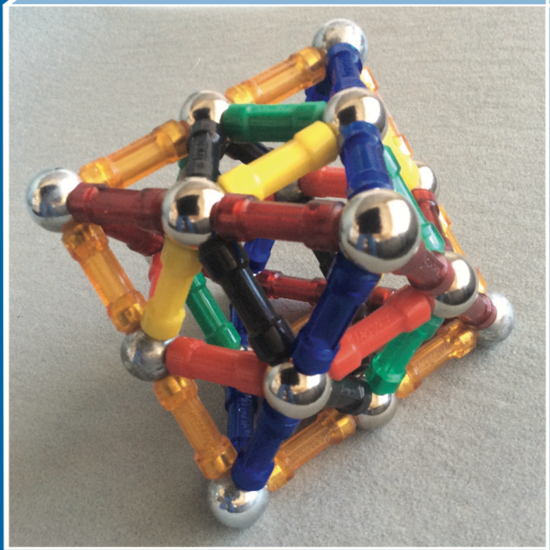
Friday 27 April 2018 at 14:15

Tea & Cookies starting at 13:00

BMS Loft, Urania, An der Urania 17, 10787 Berlin

Louis J. Billera

(Cornell University)



In pursuit of a white whale: On the real linear algebra of vectors of zeros and ones

We are interested in the real linear relations (the real *matroid*) on the set of all 0-1 n-vectors. This fundamental combinatorial object is behind questions arising over the past 50 years in a variety of fields, from economics, circuit theory and integer programming to quantum physics, and has connections to an 1893 problem of Hadamard. Yet there has been little real progress on some of the most basic questions.

Some applications seek the number of regions in $\mathbf{R}^n$ that are determined by the $2^n - 1$ linear hyperplanes having 0–1 normals. This number, asymptotically 2^{n^2}, can be obtained exactly from the characteristic polynomial of the geometric lattice of all real subspaces spanned by these 0–1 vectors. These polynomials are known only through $n = 7$, while the number of regions is known through $n = 8$.

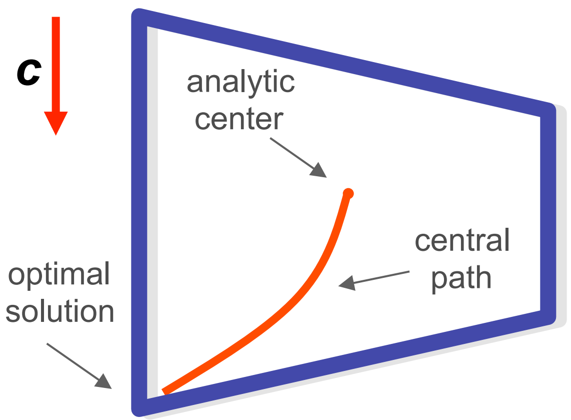
Linear optimization diameter and curvature

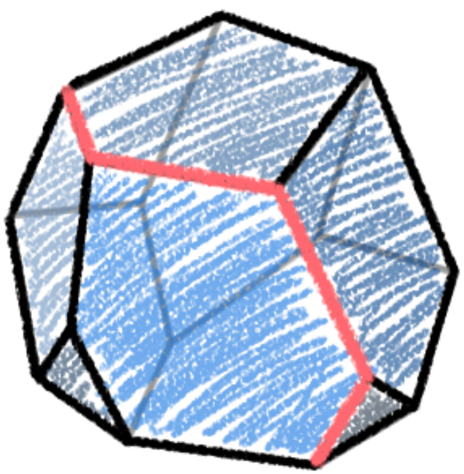
Diameter (*of a polytope*) :

lower bound for the number of iterations for *pivoting* **simplex methods**

Curvature (*of the central path associated to a polytope*) :

large curvature indicates large number of iterations for *path following* **interior point methods**





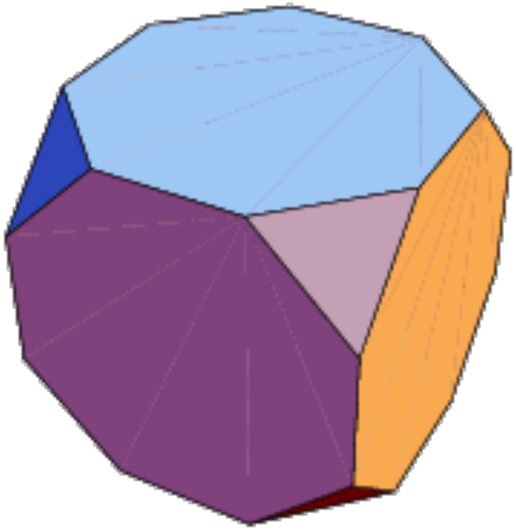
Lattice polytopes with large diameter

lattice (d,k)-polytope : convex hull of points drawn from $\{0,1,\ldots,k\}^d$

diameter $\delta(P)$ of polytope P : smallest number such that **any two vertices** of P can be connected by a **path with at most** $\delta(P)$ **edges**

$\delta(d,k)$: largest diameter over all **lattice** (d,k)-polytopes

ex. $\delta(3,3) = 6$ and is achieved by a ***truncated cube***



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$\delta(d,k)$: largest diameter over all **lattice** (d,k)-polytopes

- $\delta(P)$: lower bound for the worst case number of iterations required by *pivoting methods* (simplex) to optimize a linear function over P
- *Hirsch conjecture* : $\delta(P) \leq n - d$ (n number of inequalities) was ***disproved*** [Santos 2012]

($\delta(P) \leq (n - d)^{\log d - \ldots}$ [Kalai-Kleitman 1992, Todd 2014, Sukegawa 2019])

Lattice polytopes with large diameter

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- *Hirsch conjecture* : $\delta(P) \leq n - d$ (n number of inequalities) was ***disproved*** [Santos 2012]

  - ✓ no polynomial upper bound known for $\delta(P)$
  - ✓ best current bound [Sukegawa 2019]

Lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

upper bounds :

$\delta(d,1) \le d$		[Naddef 1989]
$\delta(2,k) = O(k^{2/3})$		[Balog-Bárány 1991]
$\delta(2,k) = 6(k/2\pi)^{2/3} + O(k^{1/3}\log k)$		[Thiele 1991] [Acketa-Žunić 1995]
$\delta(d,k) \le kd$		[Kleinschmid-Onn 1992]
$\delta(d,k) \le kd - \lceil d/2 \rceil$	for $k \ge 2$	[Del Pia-Michini 2016]
$\delta(d,k) \le kd - \lceil 2d/3 \rceil - (k-3)$	for $k \ge 3$	[Deza-Pournin 2018]

Lattice polytopes with large diameter

$\delta(d,k)$: largest ***diameter*** of a convex hull of points drawn from $\{0,1,\ldots,k\}^d$

lower bounds :

$\delta(d,1) \geq d$ [Naddef 1989]

$\delta(d,2) \geq \lfloor 3d/2 \rfloor$ [Del Pia-Michini 2016]

$\delta(d,k) = \Omega(k^{2/3}\, d)$ [Del Pia-Michini 2016]

$\delta(d,k) \geq \lfloor (k+1)d/2 \rfloor$ for $k < 2d$ [Deza-Manoussakis-Onn 2018]

Lattice polytopes with large diameter

$\delta(d,k)$		*k*								
		1	2	3	4	5	6	7	8	9
d	2	2								
	3	3								
	4	4								
	5	5								

$\delta(d,1) = d$

[Naddef 1989]

Lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3								
	4	4								
	5	5								

$\delta(d,1) = d$ [Naddef 1989]

$\delta(2,k)$: close form [Thiele 1991] [Acketa-Žunić 1995]

Lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4							
	4	4	6							
	5	5	7							

$\delta(d,1) = d$ [Naddef 1989]

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$\delta(d,2) = \lfloor 3d/2 \rfloor$ [Del Pia-Michini 2016]

Lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	7	9				
	4	4	6	8						
	5	5	7							

$\delta(d,1) = d$ [Naddef 1989]

$\delta(2,k)$: close form [Thiele 1991] [Acketa-Žunić 1995]

$\delta(d,2) = \lfloor 3d/2 \rfloor$ [Del Pia-Michini 2016]

$\delta(4,3)=8$, $\delta(3,4)=7$, $\delta(3,5)=9$ [Deza-Pournin 2018], [Chadder-Deza 2017]

Lattice polytopes with large diameter

$\delta(d,k)$		k = 1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	7	9	10			
	4	4	6	8						
	5	5	7	10						

$\delta(d,1) = d$ [Naddef 1989]

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$\delta(4,3)=8$, $\delta(3,4)=7$, $\delta(3,5)=9$ [Deza-Pournin 2018], [Chadder-Deza 2017]

$\delta(5,3)=10$, $\delta(3,6)=10$ [Deza-Deza-Guan-Pournin 2019]

Lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	7	9	10	11+	12+	13+
	4	4	6	8	10+	12+	14+	16+	17+	18+
	5	5	7	10	12+	15+	17+	20+	22+	25+

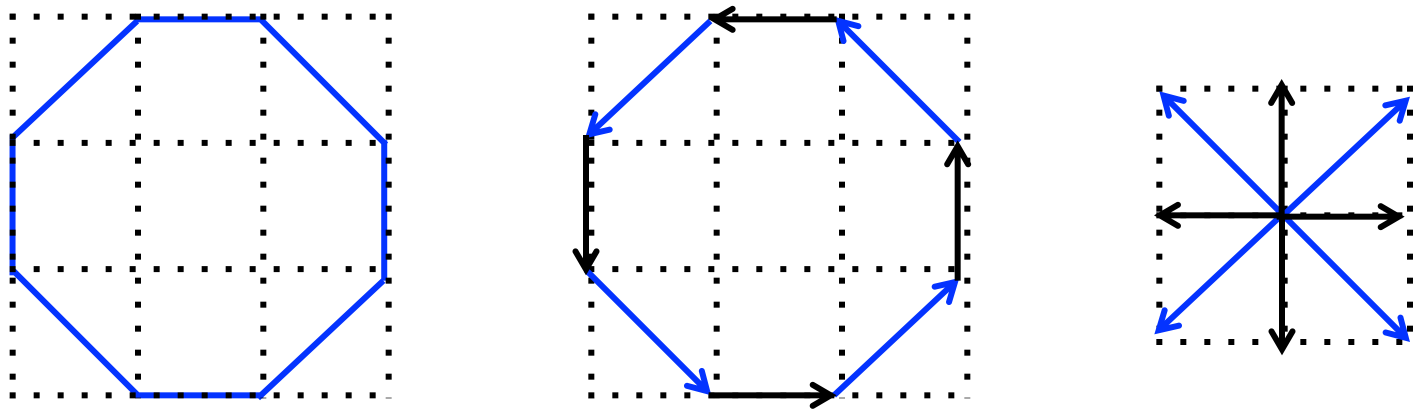
- Conjecture [Deza-Manoussakis-Onn 2018] $\delta(d,k) \le \lfloor (k+1)d/2 \rfloor$

and $\delta(d,k)$ is achieved, up to translation, by a *Minkowski sum of primitive lattice vectors*. The conjecture holds for all known entries of $\delta(d,k)$

Lattice polygons with large diameter

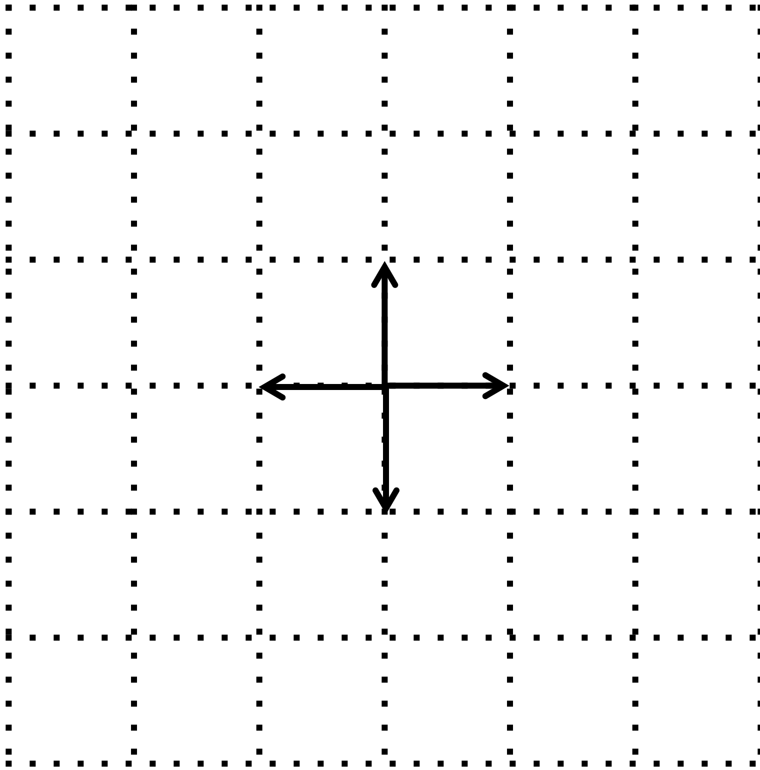
Q. What is δ(2,***k***) : largest diameter of a polygon which vertices are drawn form the ***k*** x ***k*** grid?

A polygon can be associated to a set of vectors (*edges*) *summing up to zero*, and *without a pair of positively multiple vectors*



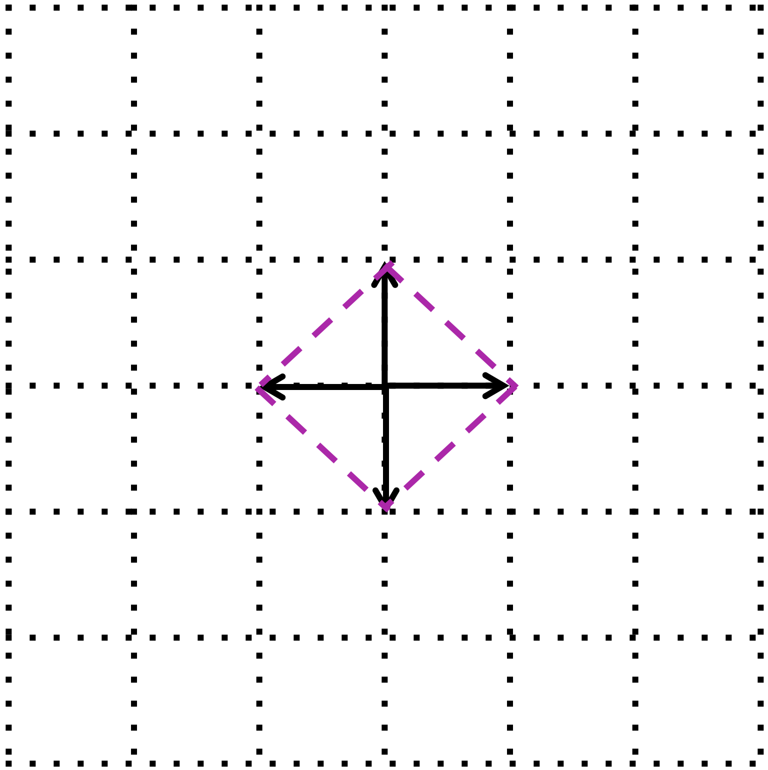
δ(2,3) = 4 is achieved by the 8 vectors : (±1,0), (0,±1), (±1,±1)

Lattice polygons with large diameter



$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

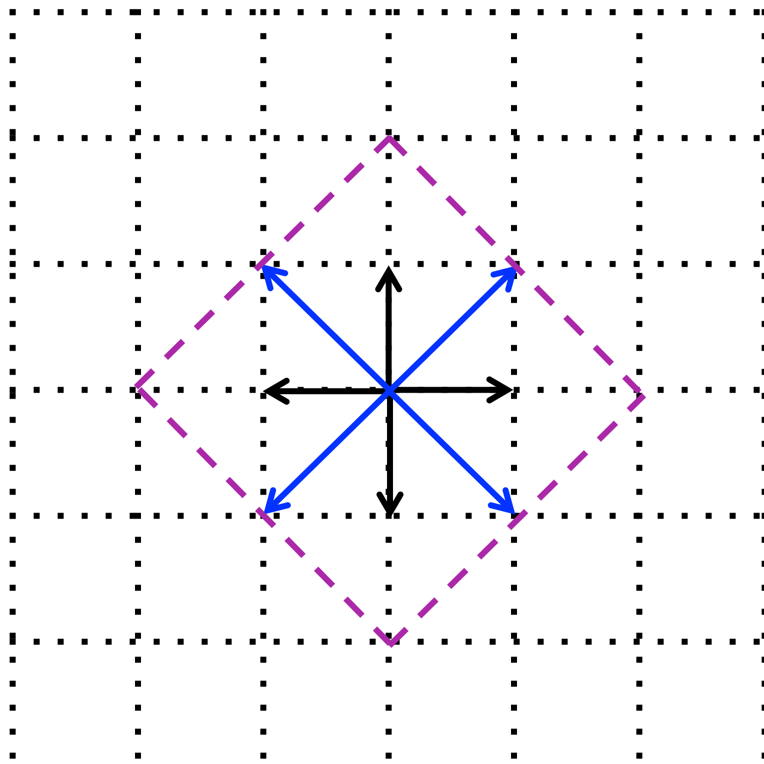
Lattice polygons with large diameter



$||x||_1 \leq 1$

$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

Lattice polygons with large diameter

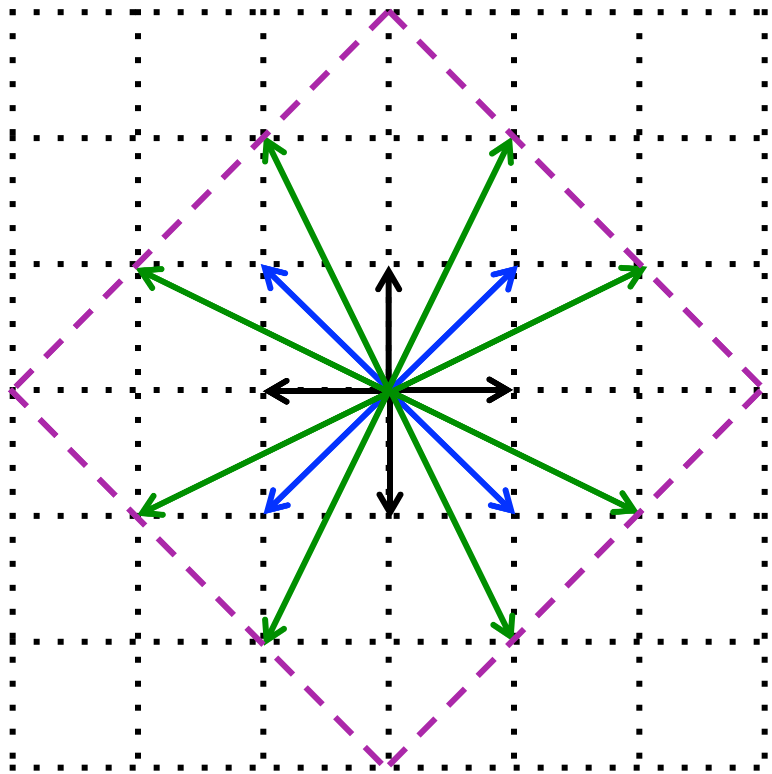


$||x||_1 \leq 2$

$\delta(2,2) = 2$; vectors : $(\pm 1,0)$, $(0,\pm 1)$

$\delta(2,3) = 4$; vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$

Lattice polygons with large diameter



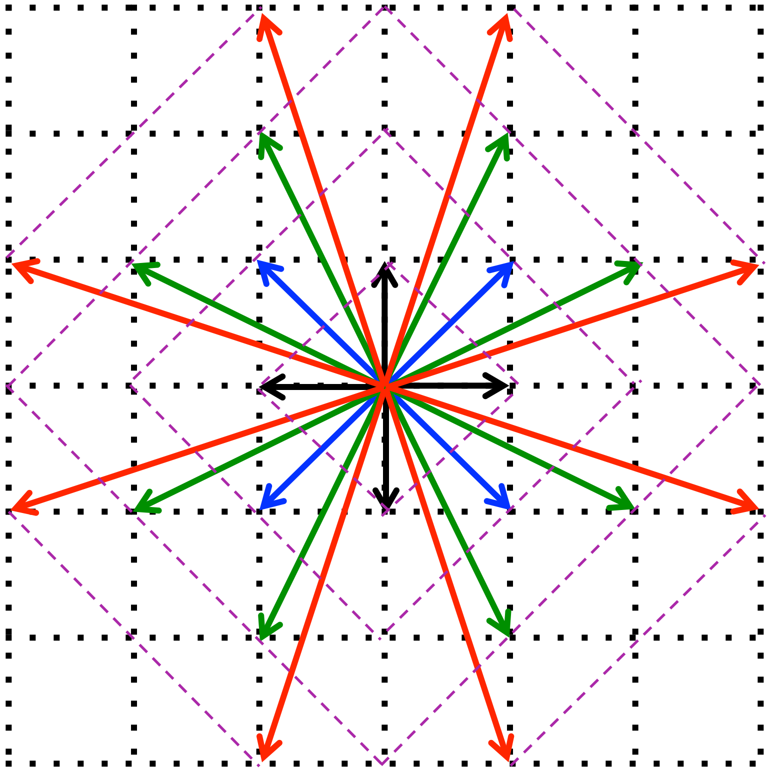
$||x||_1 \le$ **3**

$\delta(2,2) = 2$; vectors : $(\pm1,0)$, $(0,\pm1)$

$\delta(2,3) = 4$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$

$\delta(2,9) = 8$; vectors : $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$, $(\pm1,\pm2)$, $(\pm2,\pm1)$

Lattice polygons with large diameter

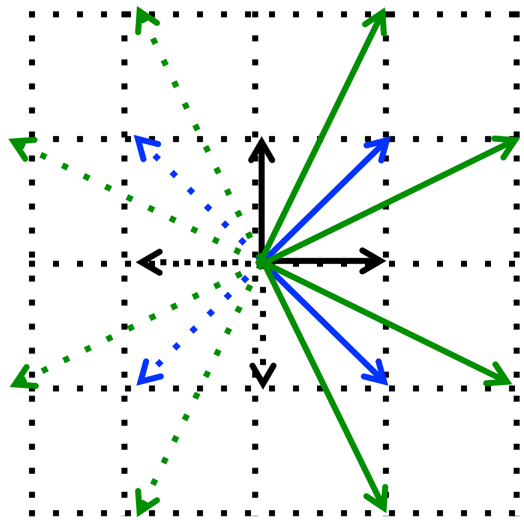


$\|x\|_1 \leq p$

$\delta(2,k) = 2\sum_{i=1}^{p} \varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

φ(*p*) : ***Euler totient function*** counting positive integers less or equal to *p* relatively prime with *p*
φ(1) = φ(2) = 1, φ(3) = φ(4) = 2,…

Primitive polygons



$||x||_1 \leq p$

$H_1(2,p)$: Minkowski sum generated by $\{x \in \mathbb{Z}^2 : ||x||_1 \leq p,\ \gcd(x)=1,\ x \succeq 0\}$

$H_1(2,p)$ has diameter $\delta(2,k) = 2\sum_{i=1}^{p} \varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

Ex. $H_1(2,2)$ generated by (1,0), (0,1), (1,1), (1,-1) (fits, *up to translation*, in 3x3 grid)

$x \succeq 0$: first nonzero coordinate of x is nonnegative

Primitive zonotopes

$$H_q(d,p) : \text{Minkowski } (x \in \mathbb{Z}^d : ||x||_q \le p, \ \gcd(x)=1, \ x \succeq 0)$$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

Given a set G of $\boldsymbol{m}$ vectors (generators),
Minkowski (G) : convex hull of all the 2^m ***subsums*** of the $\boldsymbol{m}$ vectors in G

- ***Primitive zonotopes*****:** Minkowski sum generated by ***short integer*** vectors which are ***pairwise linearly independent***

- *Note*: convex hull of all the ***signed*** subsums of the vectors of $H_q(d,p)$ *is a generalization of the permutahedron of type* B_d

Primitive zonotopes

$H_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p})$: Minkowski $(x \in \mathbb{Z}^{\boldsymbol{d}} : ||x||_{\boldsymbol{q}} \leq \boldsymbol{p}$, $\gcd(x)=1$, $x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $H_{\boldsymbol{q}}(\boldsymbol{d}, 1)$: $[0, 1]^{\boldsymbol{d}}$ cube for $\boldsymbol{q} \neq \infty$

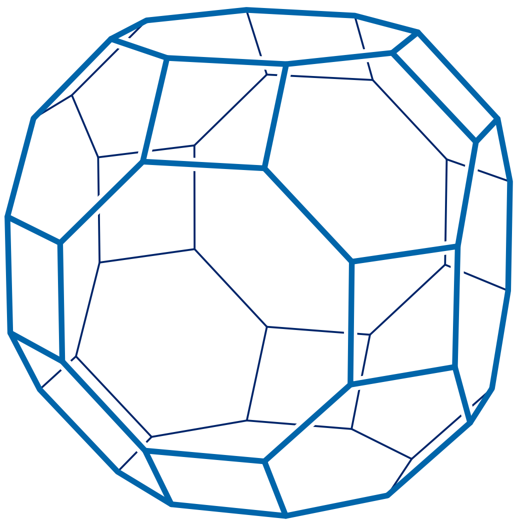
Primitive zonotopes

$$H_q(\mathbf{d},\mathbf{p}) : \text{Minkowski } (x \in \mathbb{Z}^{\mathbf{d}} : ||x||_q \leq \mathbf{p},\ \gcd(x)=1,\ x \succeq 0)$$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $H_1(\mathbf{d},2)$: permutahedron of type $B_{\mathbf{d}}$ *(up to a homothety)*

  $H_1(\mathbf{3},2)$: great rhombicuboctahedron

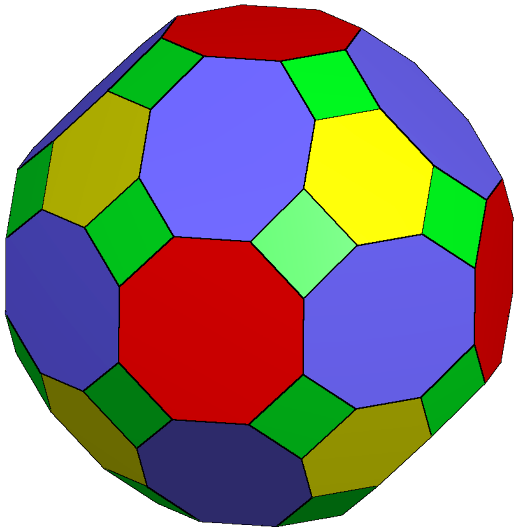


Primitive zonotopes

$$H_{\boldsymbol{q}}(\boldsymbol{d},\boldsymbol{p}) : \text{Minkowski } (x \in \mathbb{Z}^{\boldsymbol{d}} : ||x||_{\boldsymbol{q}} \leq \boldsymbol{p},\ \gcd(x)=1,\ x \succeq 0)$$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- $H_\infty(3,1)$: truncated small rhombicuboctahedron



Positive primitive zonotopes

$H_q(\boldsymbol{d},\boldsymbol{p})$: Minkowski $(x \in \mathbb{Z}^{\boldsymbol{d}} : ||x||_q \leq \boldsymbol{p}$, gcd$(x)$=1, $x \succeq 0)$

$x \succ 0$: first nonzero coordinate of x is nonnegative

$H_q(\boldsymbol{d},\boldsymbol{p})^{+}$: Minkowski $(x \in \mathbb{Z}_{+}^{\boldsymbol{d}} : ||x||_q \leq \boldsymbol{p}$, gcd$(x)$=1$)$

Positive primitive zonotopes

$$H_q(d,p)$$: Minkowski $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succ 0$: first nonzero coordinate of x is nonnegative

$$H_q(d,p)^+$$: Minkowski $(x \in \mathbb{Z}_+^d : ||x||_q \leq p, \gcd(x)=1)$

- $H_1(d,2)^+$: Minkowski sum permutahedron + unit cube *(graphical zonotope)*
- $H_\infty(d,1)^+$: *white whale* *(hypergraphical zonotope)*

$$a(d) = |H_\infty(d,1)^+|$$

number $a(d)$ of generalized retarded functions in quantum field theory is equal to the number of vertcies of $H_\infty(d,1)^+$

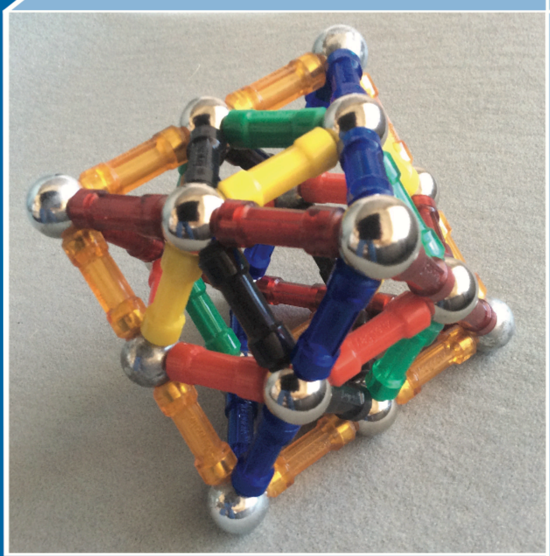
Berlin Mathematical School

BMS Friday Colloquium

Friday 27 April 2018 at 14:15

Tea & Cookies starting at 13:00

BMS Loft, Urania, An der Urania 17, 10787 Berlin



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Louis J. Billera

(Cornell University)

$H_\infty(n,1)^+$

In pursuit of a white whale: On the real linear algebra of vectors of zeros and ones

We are interested in the real linear relations (the real *matroid*) on the set of all 0-1 n-vectors. This fundamental combinatorial object is behind questions arising over the past 50 years in a variety of fields, from economics, circuit theory and integer programming to quantum physics, and has connections to an 1893 problem of Hadamard. Yet there has been little real progress on some of the most basic questions.

Some applications seek the number of regions in $\mathbf{R}^n$ that are determined by the $2^n - 1$ linear hyperplanes having 0–1 normals. This number, asymptotically 2^{n^2}, can be obtained exactly from the characteristic polynomial of the geometric lattice of all real subspaces spanned by these 0–1 vectors. These polynomials are known only through $n = 7$, while the number of regions is known through $n = 8$.

Primitive zonotopes

- lattice polytopes with *large diameter*

$H_q(d,p)$: Minkowski $(x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0$: first nonzero coordinate of x is nonnegative

- For $k < 2d$, Minkowski sum of a subset of the generators of $H_1(d,2)$ is, up to translation, a lattice (d,k)-polytope with diameter $\lfloor (k+1)d/2 \rfloor$

Lattice polytopes with large diameter

$\delta(d,k)$		k								
		1	2	3	4	5	6	7	8	9
d	2	2	3	4	4	5	6	6	7	8
	3	3	4	6	7	9	10	11+	12+	13+
	4	4	6	8	10+	12+	14+	16+	17+	18+
	5	5	7	10	12+	15+	17+	20+	22+	25+

- Conjecture [Deza-Manoussakis-Onn 2018] $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$

and $\delta(d,k)$ is achieved, up to translation, by a *Minkowski sum of primitive lattice vectors*. The conjecture holds for all known entries of $\delta(d,k)$

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

$H_\infty(d,1)^+$

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A034997 Number of Generalized Retarded Functions in Quantum Field Theory. 1

2, 6, 32, 370, 11292, 1066044, 347326352, 419172756930 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

COMMENTS a(d) is the number of parts into which d-dimensional space (x_1,...,x_d) is split by a set of (2^d - 1) hyperplanes c_1 x_1 + c_2 x_2 + ...+ c_d x_d =0 where c_j are 0 or +1 and we exclude the case with all c=0.

Also, a(d) is the number of independent real-time Green functions of Quantum Field Theory produced when analytically continuing from Euclidean time/energy (d+1 = number of energy/time variables). These are also known as Generalized Retarded Functions.

The numbers up to d=6 were first produced by T. S. Evans using a Pascal program, strictly as upper bounds only. M. van Eijck wrote a C program using a direct enumeration of hyperplanes which confirmed these and produced the value for d=7. Kamiya et al. showed how to find these numbers and some associated polynomials using more sophisticated methods, giving results up to d=7. T. S. Evans added the last number on Aug 01 2011 using an updated version of van Eijck's program, which took 7 days on a standard desktop computer.

REFERENCES Björner, Anders. "Positive Sum Systems", in Bruno Benedetti, Emanuele Delucchi, and Luca Moci, editors, Combinatorial Methods in Topology and Algebra. Springer International Publishing, 2015. 157-171.

M. van Eijck, Thermal Field Theory and Finite-Temperature Renormalisation Group, PhD thesis, Univ. Amsterdam, 4th Dec. 1995.

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$H_\infty(d,1)^+$

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LINKS

Table of n, a(n) for n=1..8.

L. J. Billera, J. T. Moore, C. D. Moraites, Y. Wang and K. Williams, Maximal unbalanced families, arXiv preprint arXiv:1209.2309 [math.CO], 2012. - From N. J. A. Sloane, Dec 26 2012

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T. S. Evans, N-point finite temperature expectation values at real times, Nuclear Physics B 374 (1992) 340-370.

T. S. Evans, What is being calculated with Thermal Field Theory?, arXiv:hep-ph/9404262 and in "Particle Physics and Cosmology: Proceedings of the Ninth Lake Louise Winter School", World Scientific, 1995 (ISBN 9810221002).

H. Kamiya, A. Takemura and H. Terao, Ranking patterns of unfolding models of codimension one, Advances in Applied Mathematics 47 (2011) 379 - 400.

Vertices of primitive zonotopes

Sloane OEI sequences

$H_\infty(\boldsymbol{d},1)^{+}$ vertices : A034997 = number of generalized retarded functions in quantum Field theory (determined till $\boldsymbol{d}$ =8)

$H_\infty(\boldsymbol{d},1)$ vertices : A009997 = number of regions of hyperplane arrangements with {-1,0,1}-valued normals in dimension $\boldsymbol{d}$ (determined till $\boldsymbol{d}$ =7)

Estimating the number of vertices of $H_\infty(\boldsymbol{d},1)^{+}$ (*white whale*)

$\boldsymbol{d}(\boldsymbol{d}-1)/2 \leq \log_2 | H_\infty(\boldsymbol{d},1)^{+} | \leq \boldsymbol{d}^{\,2}$ [Billera et al 2012]

$\boldsymbol{d}(\boldsymbol{d}-1)/2 \leq \log_2 | H_\infty(\boldsymbol{d},1)^{+} | \leq \boldsymbol{d}(\boldsymbol{d}-3)$ [Deza-Pournin-Rakotonarivo 2019]

Computational determination of the number of vertices of primitive zonotopes

Sloane OEI sequences

$H_\infty(\boldsymbol{d},1)^+$ vertices : A034997 = number of generalized retarded functions in quantum Field theory (determined till $\boldsymbol{d}=8$)

$H_\infty(\boldsymbol{d},1)$ vertices : A009997 = number of regions of hyperplane arrangements with {-1,0,1}-valued normals in dimension $\boldsymbol{d}$ (determined till $\boldsymbol{d}=7$)

Estimating the number of vertices of $H_\infty(\boldsymbol{d},1)$ (*matroid optimization*)

$d \leq \log_3 | H_\infty(d,1) | \leq d(d-1)$ [Melamed-Onn 2014]

$d \log d \leq \log_3 | H_\infty(d,1) | \leq d(d-1)$ [Deza-Onn-Manoussakis 2018]

$d(d-1)/2 \leq \log_3 | H_\infty(d,1) | \leq d(d-2)$ [Deza-Pournin-Rakotonarivo 2019]

Convex Matroid Optimization

The optimal solution of max { **f**(**W**x) : $x \in$ **S**} is attained at a vertex of the projection integer polytope in $\mathbb{R}^{d}$: conv(**WS**) = **W**conv(**S**)

S : set of feasible point in $\mathbb{Z}^{n}$ (in the talk **S** $\in \{0,1\}^{n}$)
W : integer d x n matrix (**W** is {0,1,…, p}-valued)
f : convex function from $\mathbb{R}^{d}$ to $\mathbb{R}$

Q. What is the maximum number **v**(d,n) of vertices of conv(**WS**) when **S** $\in \{0,1\}^{n}$ and **W** is a {0,1}-valued d x n matrix ?

obviously v(d,n) ≤ |**WS**| = $O(n^{d})$
in particular v(2,n) = $O(n^{2})$, and v(2,n) = $\Omega(n^{0.5})$

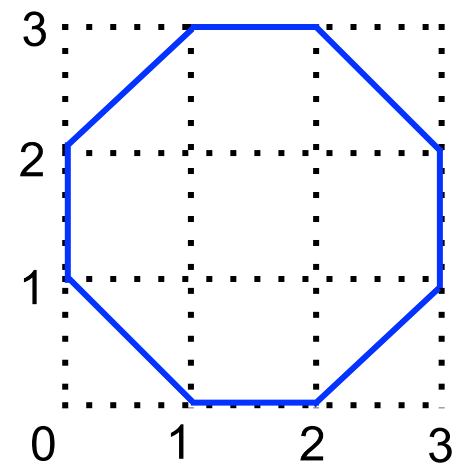
Convex Matroid Optimization

[Melamed-Onn 2014] Given matroid **S** of order ***n*** and {0,1,…,***p***}-valued ***d*** x ***n*** matrix **W**, the maximum number **m**(***d***,***p***) of vertices of conv(**WS**) is independent of ***n*** and **S**

Ex: maximum number **m**(2,1) of vertices of a planar projection conv(**WS**) of matroid **S** by a binary matrix **W** is attained by the following matrix and uniform matroid of rank 3 and order 8:

$$\mathbf{W} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{S} = U(3,8) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$



conv(**WS**)

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in particular v(2,n) = O(n^2), and v(2,n) = Ω($n^{0.5}$)

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Convex Matroid Optimization

[Melamed-Onn 2014] Given matroid **S** of order n and $\{0,1,\ldots,p\}$-valued d x n matrix **W**, the maximum number $\mathbf{m}(d,p)$ of vertices of conv(**WS**) is independent of n and **S**

[Deza-Manoussakis-Onn 2018] Given matroid **S** of order n, $\{0,1,\ldots,p\}$-valued d x n matrix **W**, maximum number $\mathbf{m}(d,p)$ of vertices of conv(**WS**) is equal to the number of vertices of $H_\infty(d,p)$

$$\mathbf{m}(d,p) = |\ H_\infty(d,p)\ |$$

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$$\mathbf{m}(d,p) = |\, H_\infty(d,p) \,|$$

[Melamed-Onn 2014]

$$d\, 2^d \le \mathbf{m}(d,1) \le 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i}$$

$24 \le \mathbf{m}(3,1) \le 158$

$64 \le \mathbf{m}(4,1) \le 19840$

$\mathbf{m}(2,1) = 8$

[Deza-Manoussakis-Onn 2018]

$$d!\, 2^d \le \mathbf{m}(d,1) \le 2\sum_{i=0}^{d-1}\binom{(3^d-3)/2}{i} - f(d)$$

$\mathbf{m}(3,1) = \mathbf{96}$

$\mathbf{m}(4,1) = \mathbf{5376}$

$$\mathbf{m}(2,p) = 8\sum_{i=1}^{p}\varphi(i)$$

Convex Matroid Optimization

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[Deza-Manoussakis-Onn 2018] Given matroid **S** of order ***n***, $\{0,1,\ldots,p\}$-valued ***d*** x ***n*** matrix **W**, maximum number $\mathbf{m}(d,p)$ of vertices of conv(**WS**) is equal to the number of vertices of $H_\infty(d,p)$

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[Deza-Pournin-Rakotonarivo 2019]

$$3^{d(d-1)/2} \le \mathbf{m}(d,1) \le 3^{d(d-2)}$$

$\mathbf{m}(3,1) = \mathbf{96}$
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Primitive Zonotopes

(degree sequences)

$\boldsymbol{D_d}$: convex hull of the degree sequences of all hypergraphs on $\boldsymbol{d}$ nodes

$$\boldsymbol{D_d} = H_\infty(\boldsymbol{d},1)^+$$

$\boldsymbol{D_d}(\boldsymbol{k})$: convex hull of the degree sequences of all $\boldsymbol{k}$-uniform hypergraphs on $\boldsymbol{d}$ nodes

Primitive Zonotopes

(degree sequences)

D_d : convex hull of the degree sequences of all hypergraphs on d nodes

$$D_d = H_\infty(d,1)^+$$

$D_d(k)$: convex hull of the degree sequences of all k-uniform hypergraphs on d nodes

Q: check whether $x \in D_d(k) \cap \mathbb{Z}^d$ is the degree sequence of a k-uniform hypergraph. Necessary condition: sum of the coordinates of x is multiple of k.

[Erdős-Gallai 1960]: for k = 2 (graphs) necessary condition is sufficient

[Liu 2013] exhibited counterexamples (holes) for k = 3 (Klivans-Reiner **Q**.)

Primitive Zonotopes

(degree sequences)

$\boldsymbol{D_d}$: convex hull of the degree sequences of all hypergraphs on $\boldsymbol{d}$ nodes

$$\boldsymbol{D_d} = H_\infty(\boldsymbol{d},1)^{+}$$

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[Erdős-Gallai 1960]: for $\boldsymbol{k} = 2$ (graphs) necessary condition is sufficient

[Liu 2013] exhibited counterexamples (holes) for $\boldsymbol{k} = 3$ (Klivans-Reiner **Q**.)

- Answer to Colbourn-Kocay-Stinson **Q.** (1986)
  Deciding whether a given integer sequence is the degree sequence of a 3-hypergraph is NP-complete [Deza-Levin-Meesum-Onn 2018]

Primitive zonotopes, convex matroid optimization, and degree sequences of hypergraphs

$\delta(d,k)$: *largest diameter* over all lattice (d,k)-polytopes

- Conjecture : $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$ and $\delta(d,k)$ is achieved, up to translation, by a *Minkowski sum* of primitive lattice vectors (holds for all known $\delta(d,k)$)

  $\Rightarrow \delta(d,k) = \lfloor (k+1)d/2 \rfloor$ for $k < 2d$

- $\mathbf{m}(d,p) = |\, H_\infty(d,p) \,|$ (*convex matroid optimization complexity*)

- tightening of the *bounds* for $\mathbf{m}(d,1) = |\, H_\infty(d,1)^+ \,|$

- tightening of the *bounds* for $a(d) = |\, H_\infty(d,1)^+ \,|$ (*white whale*)

- Answer to [Colbourn-Kocay-Stinson 1986] question:
  Deciding whether a given integer sequence is the *degree sequence* of a 3-hypergraph is *NP-complete* [Deza-Levin-Meesum-Onn 2018]

Primitive zonotopes, convex matroid optimization, and degree sequences of hypergraphs

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- $\mathbf{m}(d,p) = | H_\infty(d,p) |$ (*convex matroid optimization complexity*)
- tightening of the *bounds* for $\mathbf{m}(d,1) = | H_\infty(d,1)^+ |$
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- Answer to [Colbourn-Kocay-Stinson 1986] question:
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✓ ***thank you***

Algorithmic, combinatorial, and geometric aspects of finear optimization

- ✓ Deza, Levin, Meesum, and Onn: *Hypergraphic degree sequences are hard.* Bulletin of the European Association for Theoretical Computer Science (2019)

- ✓ Deza, Deza, Guan, and Pournin: *Distances between vertices of lattice polytopes*. Optimization Letters (2019)

- ✓ Deza, Pournin, and Rakotonarivo: *Vertices of primitive zonotopes.* Contemporary Mathematics (2019)

- ✓ Deza, Levin, Meesum, and Onn: *Optimization over degree sequences*. SIAM Journal on Discrete Mathematics (2018)

- ✓ Deza, Manoussakis, and Onn: *Primitive zonotopes.* Discrete and Computational Geometry (2018)

- ✓ Deza and Pournin: *Improved bounds on the diameter of lattice polytopes*. Acta Mathematica Hungarica (2018)