Adding Variables - Speed up by including new binary variables

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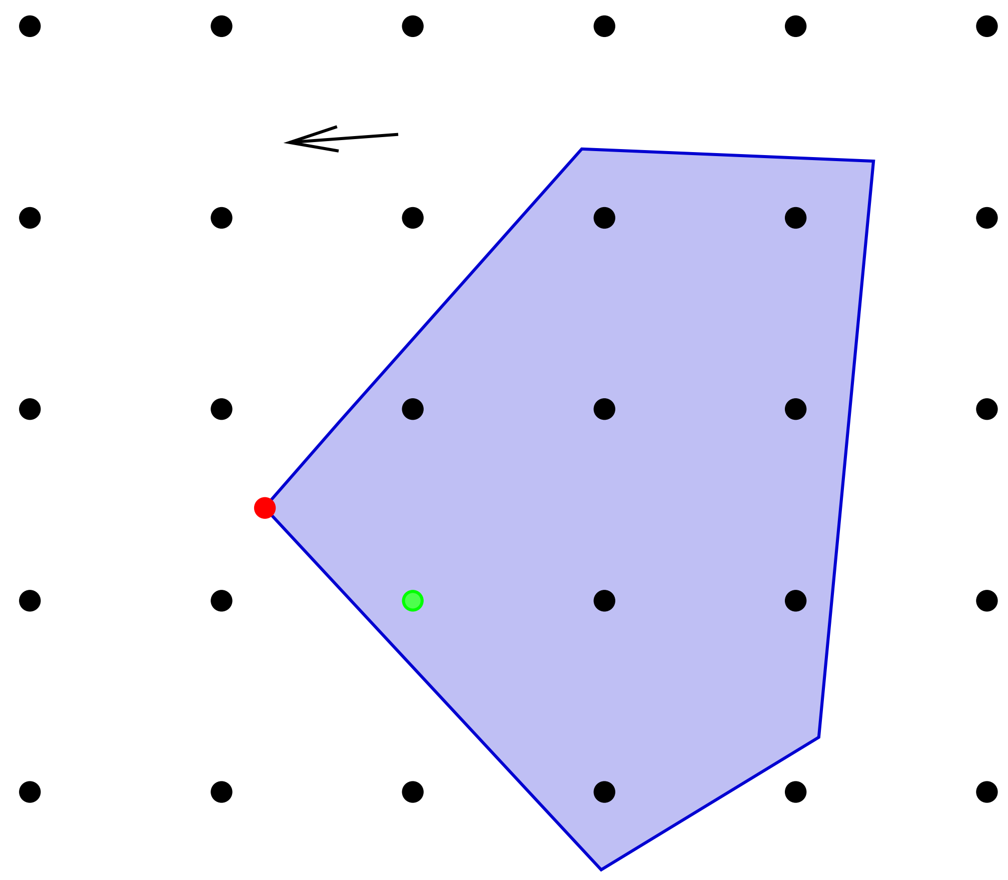
Virginia Tech
RIKEN, 2019

Joint with Sanjeeb Dash and Oktay Gunluk

Mixed Integer Linear Programming

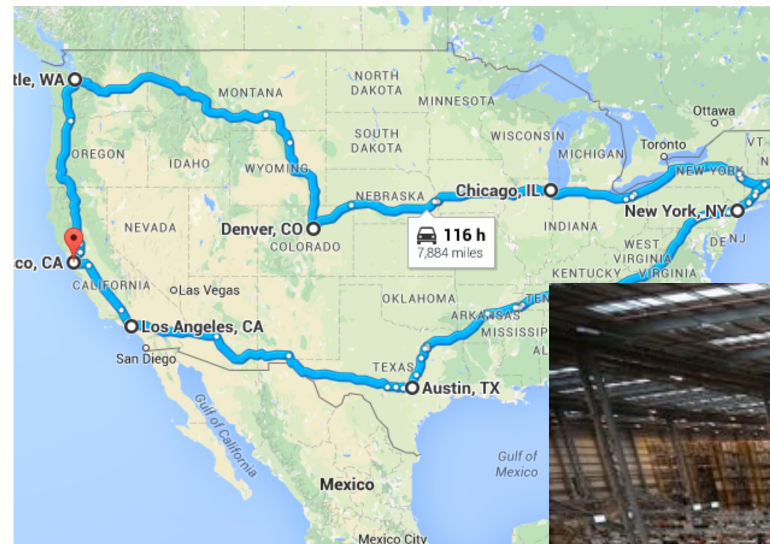
(MILP)

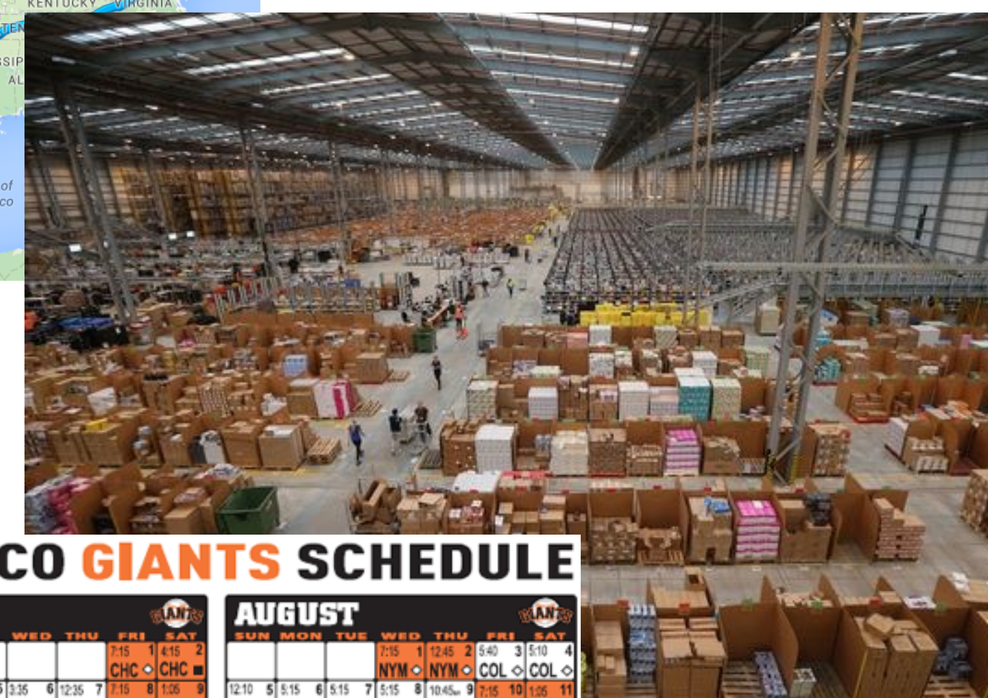
$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \le b \\ & 0 \le x \le u \\ & x \in \mathbb{Z}^n \times \mathbb{R}^d \end{aligned}$$



MIP is very useful

- Supply Chain
- Electric Power
- Finance
- Work Force Management
- Airlines
- Railroads
- Traveling Salesman Problem
- Sports Scheduling
- ….





Mixed Integer Linear Programming

(MILP)

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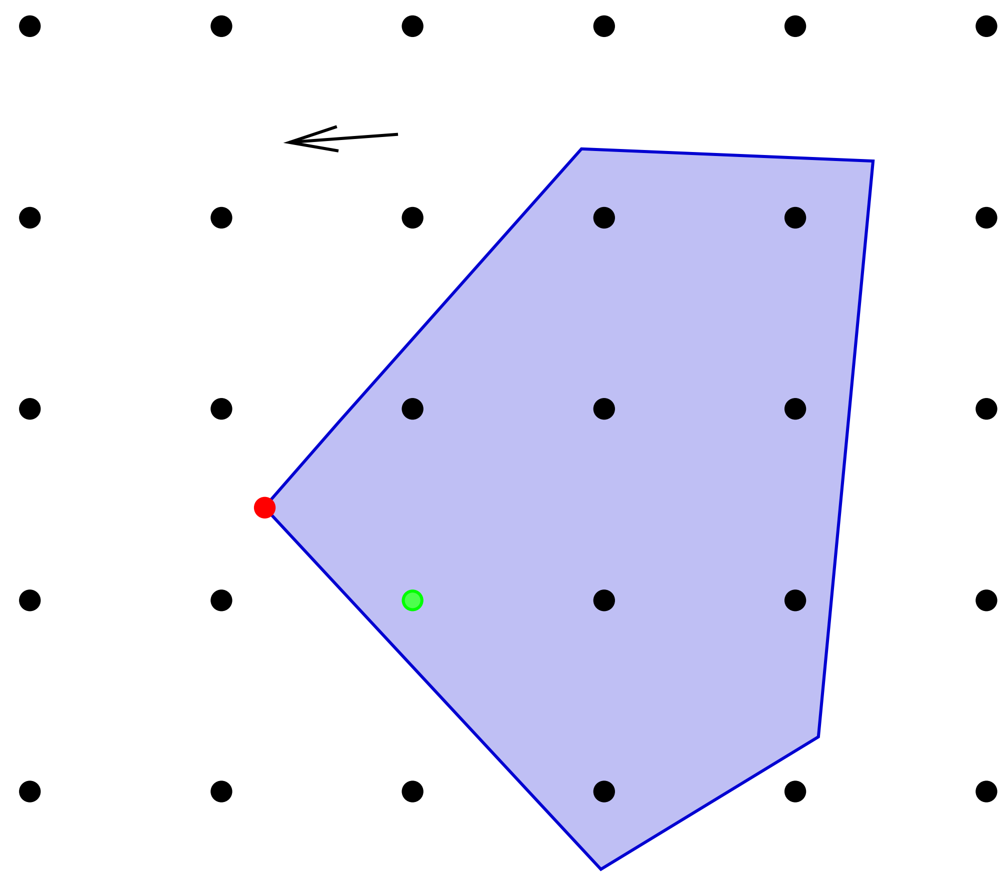


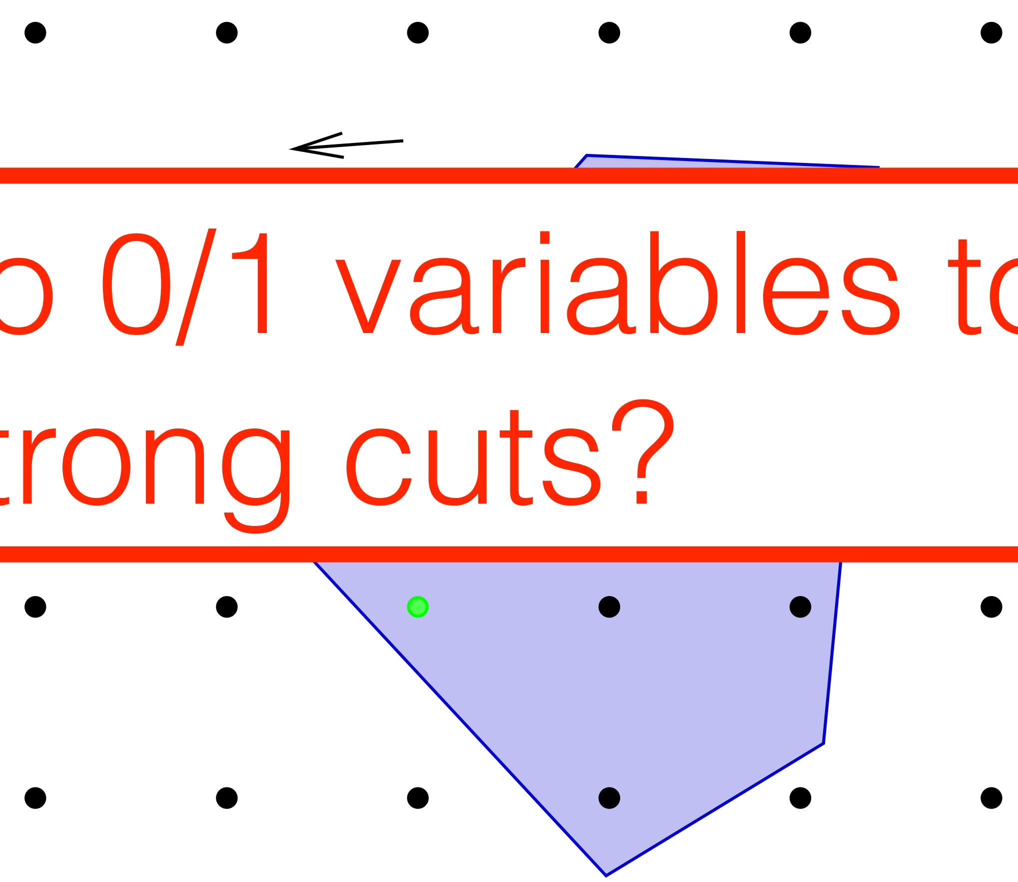
Mixed Integer Linear Programming

(MILP)

How to convert to 0/1 variables to produce strong cuts?

$$0 \leq x \leq u$$

$$x \in \mathbb{Z}^n \times \mathbb{R}^d$$



How to solve MIPs

- Branch and Bound
- Cutting Planes
- Heuristics
- Problem Specific Insights

```
*     0+    0                         63589.3538    37147.3468              41.58%
      0     0    37208.8463    53    63589.3538      Cuts: 22      371    41.49%
      0     0    37240.1690    49    63589.3538      Cuts: 19      400    41.44%
*     0+    0                         43163.7033    37240.1690              13.72%
      0     0    37255.0918    58    43163.7033      Cuts: 15      416    13.69%
      0     0    37274.4440    62    43163.7033      Cuts: 11      434    13.64%
      0     0    37281.6812    43    43163.7033      Cuts: 15      455    13.63%
      0     0    37288.9797    53    43163.7033      Cuts: 13      468    13.61%
      0     0    37292.4372    48    43163.7033       Cuts: 4      475    13.60%
      0     0    37294.8200    55    43163.7033       Cuts: 7      483    13.60%
*     0+    0                         41713.5846    37294.8200              10.59%
*     0+    0                         40468.1795    37294.8200               7.84%
*     0+    0                         38816.4629    37294.8200               3.92%
      0     2    37297.8778    54    38816.4629    37294.8568      483     3.92%
Elapsed time = 0.34 sec. (99.46 ticks, tree = 0.00 MB, solutions = 10)
*    10+   10                         38040.7778    37373.1880               1.75%
*   410+  273                         38028.0793    37560.1706               1.23%
*   410+  273                         37913.5385    37560.1706               0.93%
*   411+  274                         37913.5385    37560.1706               0.93%
*  1499   420      integral     0    37809.0000    37757.2612    15053     0.14%

Implied bound cuts applied:  60
Flow cuts applied:  46
Mixed integer rounding cuts applied:  106
Flow path cuts applied:  29
Lift and project cuts applied:  4
Gomory fractional cuts applied:  60

Root node processing (before b&c):
  Real time             =    0.34 sec. (99.20 ticks)
Parallel b&c, 4 threads:
  Real time             =    0.38 sec. (175.07 ticks)
  Sync time (average)   =    0.07 sec.
  Wait time (average)   =    0.12 sec.
                          ------------
Total (root+branch&cut) =    0.73 sec. (274.27 ticks)

Solution status : Optimal Objective function value = 37809  Objective best value = 37809   Optimality Gap = 0
Time = 1.89754
```

```
*     0+    0                    63589.3538    37147.3468             41.58%
      0     0    37208.8463    53    63589.3538      Cuts: 22     371    41.49%
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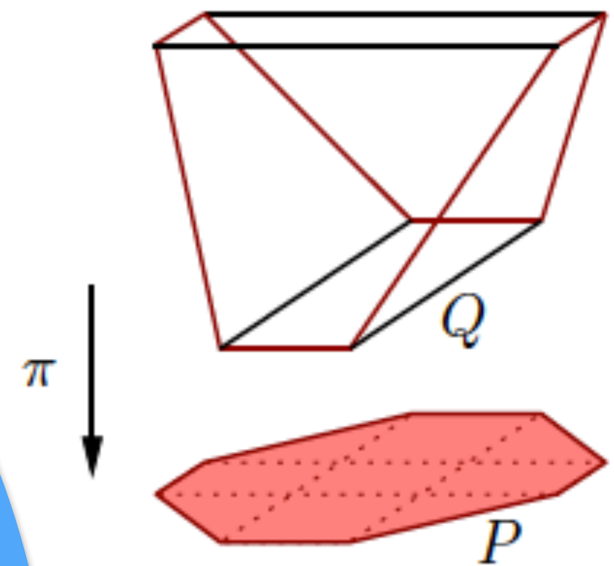
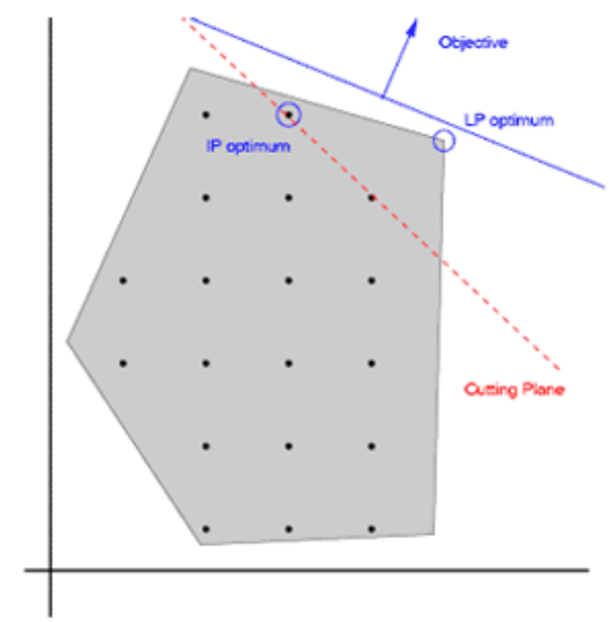
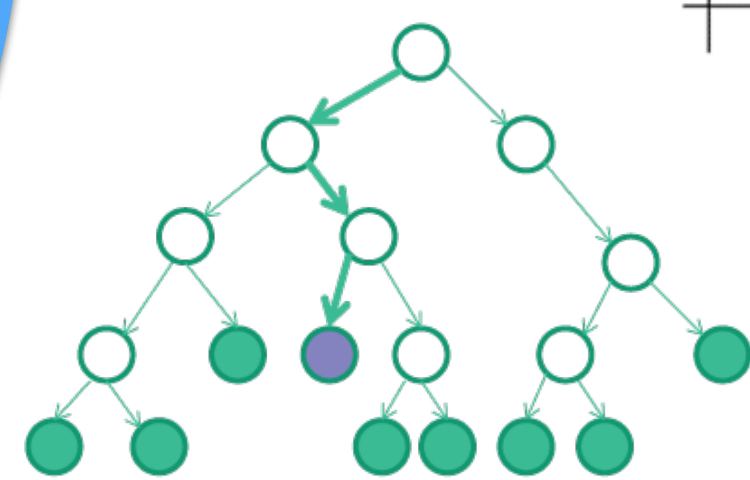
Outline

- Binary Reformulations
- Cutting Planes
- Comparing Formulations
- Compare Branching







Binary Reformulations

Binary Reformulation

Exchange general integer variables for 0/1 variables

(MILP)

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \le b \\ & 0 \le x \le u \\ & x \in \mathbb{Z}^n \times \mathbb{R}^d \end{aligned}$$

→

(MBLP)

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & A'(x,z) \le b' \\ & 0 \le x \le u \\ & x \in \mathbb{R}^n \times \mathbb{R}^d \\ & z \in \{0,1\}^q \end{aligned}$$

Binary Reformulation

Exchange general integer variables for 0/1 variables

$$0 \leq x \leq {\color{blue}u}$$
$$x \in \mathbb{Z}$$

Binary Reformulation

Exchange general integer variables for 0/1 variables

$$0 \leq x \leq u$$
$$x \in \mathbb{Z}$$

→

(Log Reformulation)

$$x = \sum_{j=0}^{\lfloor \log(u+1) \rfloor} 2^j z_j$$

$$z \in \{0,1\}^{\lfloor \log(u+1) \rfloor}$$

$$23 = 2^4 \cdot 1 + 2^3 \cdot 0 + 2^2 \cdot 1 + 2^1 \cdot 1 + 2^0 \cdot 1$$

Binary Reformulation

Exchange general integer variables for 0/1 variables

$$0 \leq x \leq u$$
$$x \in \mathbb{Z}$$

→

(Full Reform.)

$$x = \sum_{j=1}^{u} j z_j$$
$$z_1 + z_2 + \cdots + z_u \leq 1$$
$$z \in \{0,1\}^u$$

$$23 = 1 \cdot 0 + 2 \cdot 0 + \cdots + 22 \cdot 0 + 23 \cdot 1 + 24 \cdot 0 + \ldots$$

Binary Reformulation

Exchange general integer variables for 0/1 variables

$$0 \le x \le u$$
$$x \in \mathbb{Z}$$

→

(Unary Reform.)

$$x = \sum_{j=1}^{u} z_j$$

$$z \in \{0,1\}^u$$

$$23 = 1 + 1 + 0 + 0 + 0 + 1 + 1 + 0 + 1 + 0 + 1 + 1 + 1 \ldots$$

Binary Reformulation

Exchange general integer variables for 0/1 variables

$$0 \leq x \leq u$$
$$x \in \mathbb{Z}$$

→

(Unary Reform.)

$$x = \sum_{j=1}^{u} z_j$$
$$z_1 \geq z_2 \geq \ldots \geq z_u$$
$$z \in \{0,1\}^u$$

$$23 = 1 + 1 + \cdots + 1 + 1 + 1 + 1 + 0 + 0 + 0 + 0 + 0 \ldots$$

Which one to choose?

Log

$$x = \sum_{j=0}^{\lfloor \log(u+1) \rfloor} 2^j z_j$$

$$z \in \{0,1\}^{\lfloor \log(u+1) \rfloor}$$

Glover '75

Unary

$$x = \sum_{j=1}^{u} z_j$$

$$z_1 \geq z_2 \geq \ldots \geq z_u$$

$$z \in \{0,1\}^u$$

Roy '07
Bonami, Margot '15

Full

$$x = \sum_{j=1}^{u} j z_j$$

$$z_1 + z_2 + \cdots + z_u \leq 1$$

$$z \in \{0,1\}^u$$

Miller '63
Glover '75
Sherali, Adams '99
Owen and Mehrotra '02
Angulo, Van-vyve '17

Which one to choose?

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$$x = \sum_{j=0}^{\lfloor \log(u+1) \rfloor} 2^j z_j$$

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Glover ’75

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$$z_1 \geq z_2 \geq \ldots \geq z_u$$

$$z \in \{0,1\}^u$$

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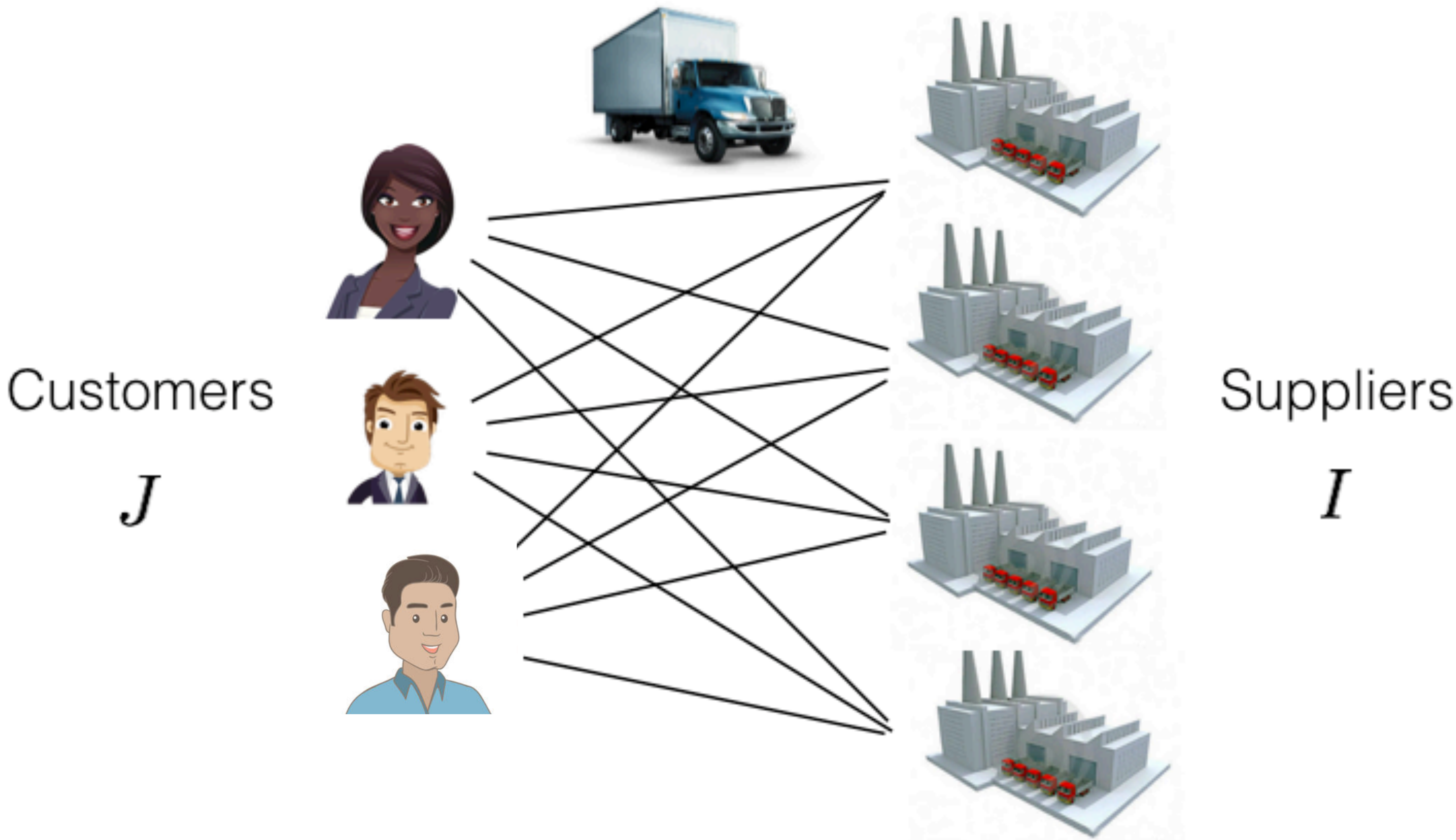
Miller ’63
Glover ’75
Sherali, Adams ’99
Owen and Mehrotra ’02
Angulo, Van-vyve '17

Are there other options to consider?

A motivating experiment

Angulo, Van Vyve '17:
Fixed Charged Transportation Problem

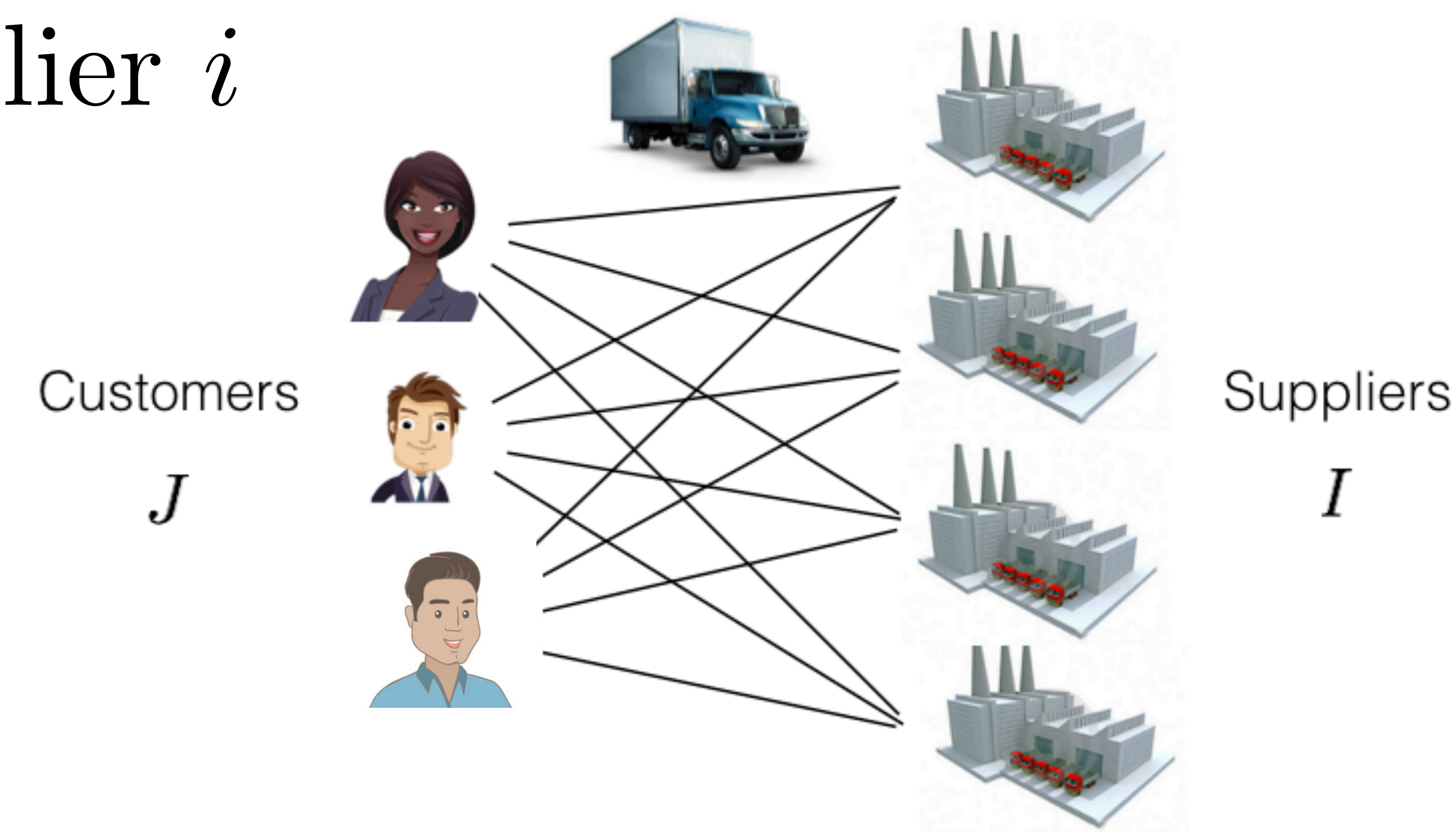


Angulo, Van Vyve '17:
Fixed Charged Transportation Problem

x_{ij} quantity supplier $i \rightarrow$ customer j

d_j demand of customer j

c_i capacity of supplier i

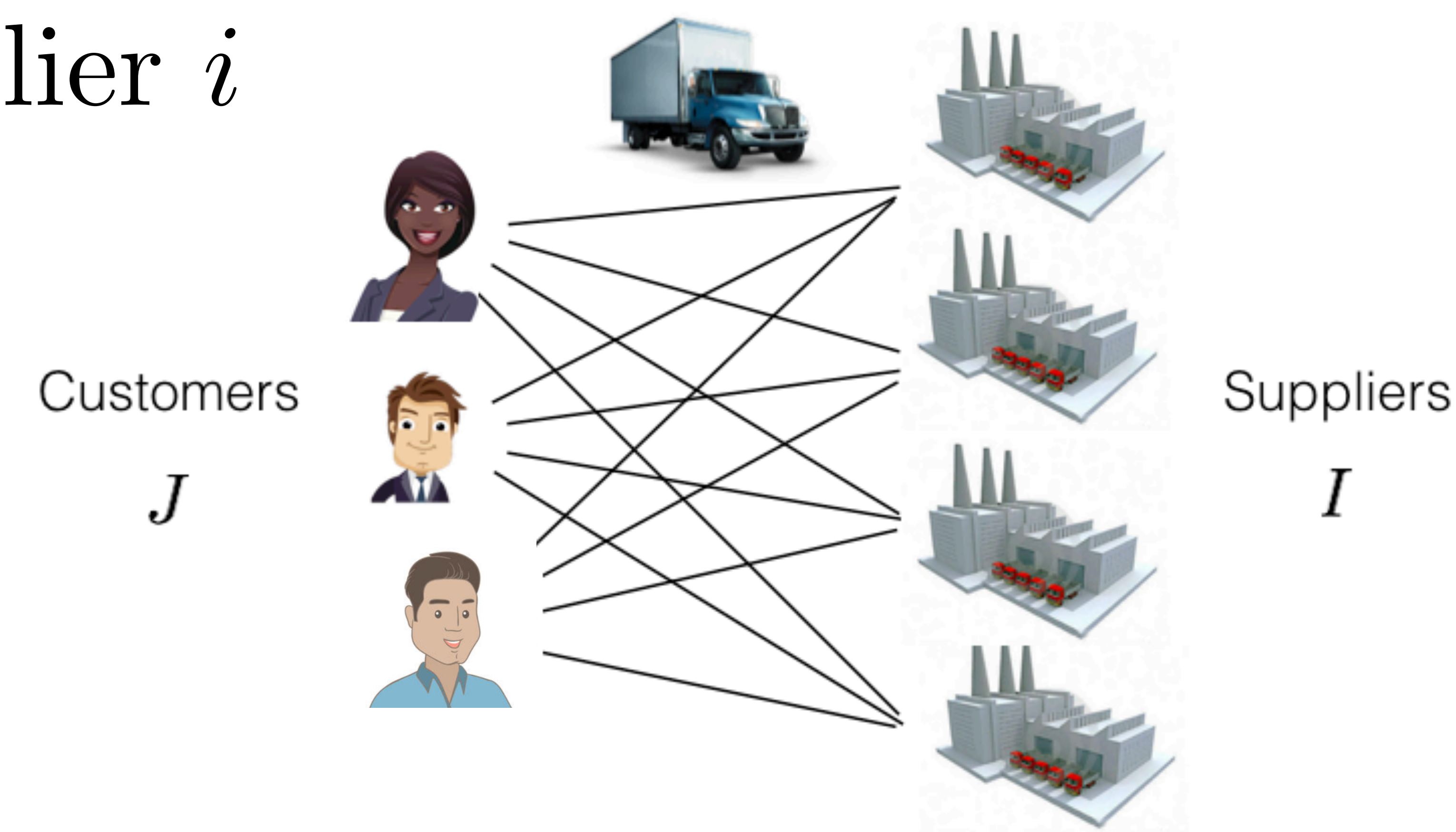


Angulo, Van Vyve '17:
Fixed Charged Transportation Problem

x_{ij} quantity supplier $i \to$ customer j
d_j demand of customer j
c_i capacity of supplier i

$$\sum_i x_{ij} = d_j$$

$$\sum_j x_{ij} \leq c_i$$



Angulo, Van Vyve '17:
Fixed Charged Transportation Problem

$$\min \quad p^\top x + q^\top y$$

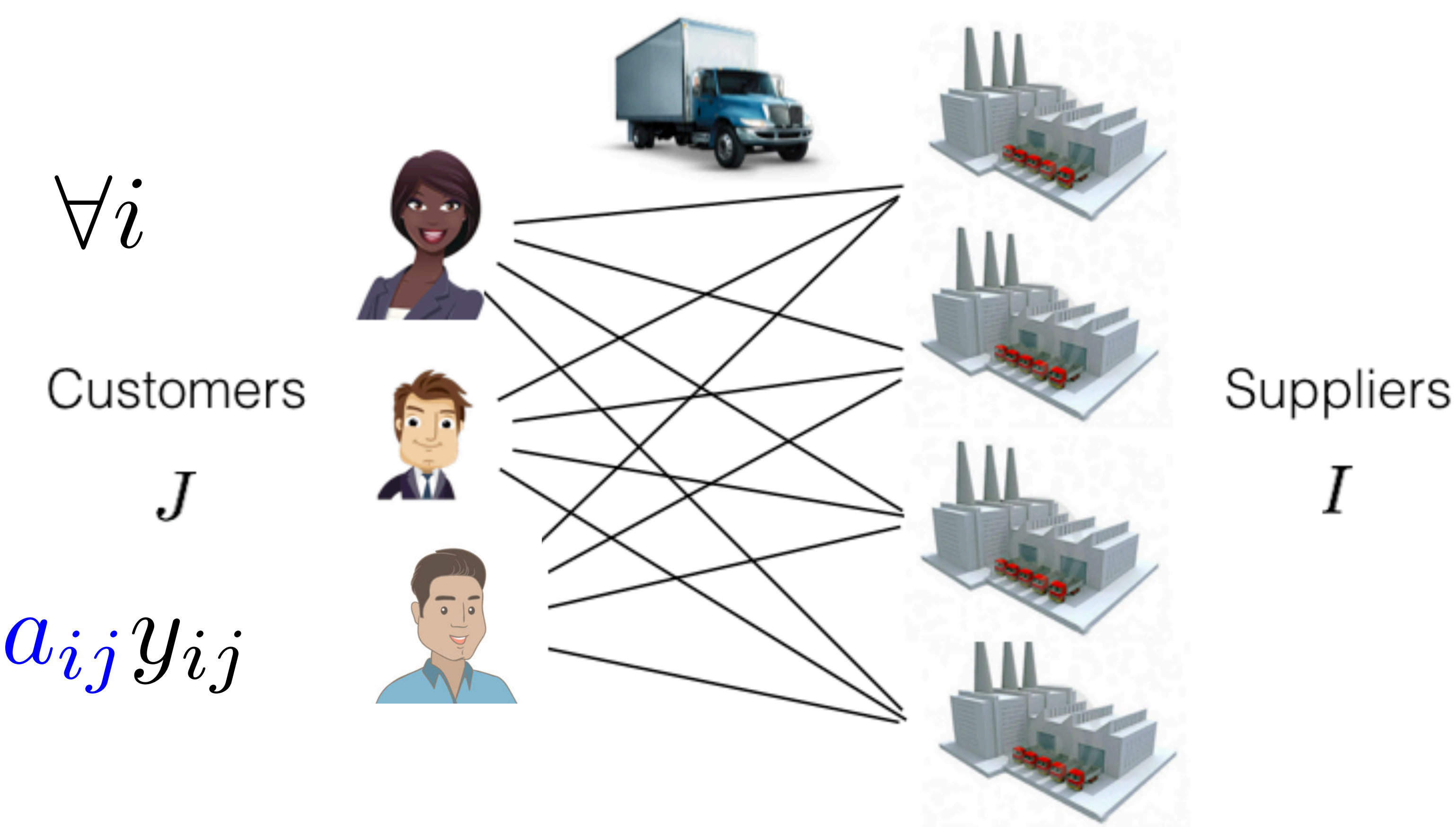
$$\sum_i x_{ij} = d_j \quad \forall j$$

$$\sum_j x_{ij} \leq c_i \quad \forall i$$

$$\forall i, j \quad y_{ij} \leq x_{ij} \leq a_{ij} y_{ij}$$

$$x_{ij} \in \mathbb{Z},$$

$$y_{ij} \in \{0, 1\}$$



Angulo, Van Vyve '17:
Fixed Charged Transportation Problem

Full

$$x = \sum_{j=1}^{u} j z_j$$

$$z_1 + z_2 + \cdots + z_u \leq 1$$

$$z \in \{0,1\}^u$$

$$\min \quad p^\top x + q^\top y$$

$$\sum_i x_{ij} = d_j \quad \forall j$$

$$\sum_j x_{ij} \leq c_i \quad \forall i$$

$$\forall i, j$$

$$0 \leq x_{ij} \leq a_{ij} y_{ij}$$

$$x_{ij} \in \mathbb{Z},$$

$$y_{ij} \in \{0,1\}$$

Angulo, Van Vyve '17:
Fixed Charged Transportation Problem

$$\min \quad p^\top x + q^\top y$$

$$\sum_i x_{ij} = d_j \quad \forall j$$

$$\sum_j x_{ij} \le c_i \quad \forall i$$

$$\forall i, j$$

$$0 \le x_{ij} \le a_{ij} y_{ij}$$

$$x_{ij} \in \mathbb{Z},$$

$$y_{ij} \in \{0, 1\}$$

$$x_{ij} = \sum_{k=1}^{a_{ij}} k z_{i,j,k},$$

$$\sum_k z_{i,j,k} \le 1$$

$$z_{i,j,k} \in \{0, 1\} \ \forall \ k$$

Full

$$x = \sum_{j=1}^{u} j z_j$$

$$z_1 + z_2 + \cdots + z_u \le 1$$

$$z \in \{0, 1\}^u$$

Angulo, Van Vyve '17:
Fixed Charged Transportation Problem

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$$x_{ij} = \sum_{k=1}^{a_{ij}} k z_{i,j,k},$$
$$\sum_k z_{i,j,k} \leq 1$$
$$z_{i,j,k} \in \{0,1\} \ \forall \ k$$
$$y_{ij} \in \{0,1\}$$

Substitute

$$\min \quad \hat{p}^\top z + q^\top y$$
$$\sum_i \sum_{k=1}^{a_{ij}} k z_{i,j,k} = d_j \quad \forall j$$
$$\sum_j \sum_{k=1}^{a_{ij}} k z_{i,j,k} \leq c_i \quad \forall i$$
$$0 \leq \sum_{k=1}^{a_{ij}} k z_{i,j,k} \leq a_{ij} y_{ij}$$
$$\sum_k z_{i,j,k} \leq 1$$
$$z_{i,j,k} \in \{0,1\} \ \forall \ k$$
$$y_{ij} \in \{0,1\}$$

Angulo, Van Vyve '17:
Fixed Charged Transportation Problem

$$\begin{aligned}
\min \quad & p^\top x + q^\top y \\
& \sum_i x_{ij} = d_j \quad \forall j \\
& \sum_j x_{ij} \le c_i \quad \forall i \\
& 0 \le x_{ij} \le a_{ij} y_{ij} \\
& x_{ij} = \sum_{k=1}^{a_{ij}} k z_{i,j,k}, \\
& \sum_k z_{i,j,k} \le 1 \\
& z_{i,j,k} \in \{0,1\} \ \forall\ k \\
& y_{ij} \in \{0,1\}
\end{aligned}$$

Substitute

$$\begin{aligned}
\min \quad & \hat{p}^\top z + q^\top y \\
& \sum_i \sum_{k=1}^{a_{ij}} k z_{i,j,k} = d_j \quad \forall j \\
& \sum_j \sum_{k=1}^{a_{ij}} k z_{i,j,k} \le c_i \quad \forall i \\
& 0 \le \sum_{k=1}^{a_{ij}} z_{i,j,k} \le y_{ij} \\
& \sum_k z_{i,j,k} \le 1 \\
& z_{i,j,k} \in \{0,1\} \ \forall\ k \\
& y_{ij} \in \{0,1\}
\end{aligned}$$

Example

Geometric mean of running times for 5 instances with CPLEX 12.6.1
(600 sec time limit)

Angulo, Van Vyve '17:
Fixed Charged Problem

$$
\begin{aligned}
\min \quad & \sum_{ij} q_{ij} y_{ij} \\
& \sum_{i} x_{ij} = d_j \quad \forall j \\
& \sum_{j} x_{ij} \le c_i \quad \forall i \\
& y_{ij} \le x_{ij} \le a_{ij} y_{ij} \quad \forall i < j \\
& x_{ij} \in \mathbb{Z}, y_{ij} \in \{0,1\} \quad \forall i < j
\end{aligned}
$$

Before Substitution:

Problem $n, \max\{c_i\}$	P	F(P)	U(P)	AvV
30, 10	4.9	12.0	23.0	10.1
40, 10	75.2	77.0	261.5	48.1
40, 20	469.0	471.1	600	434.1

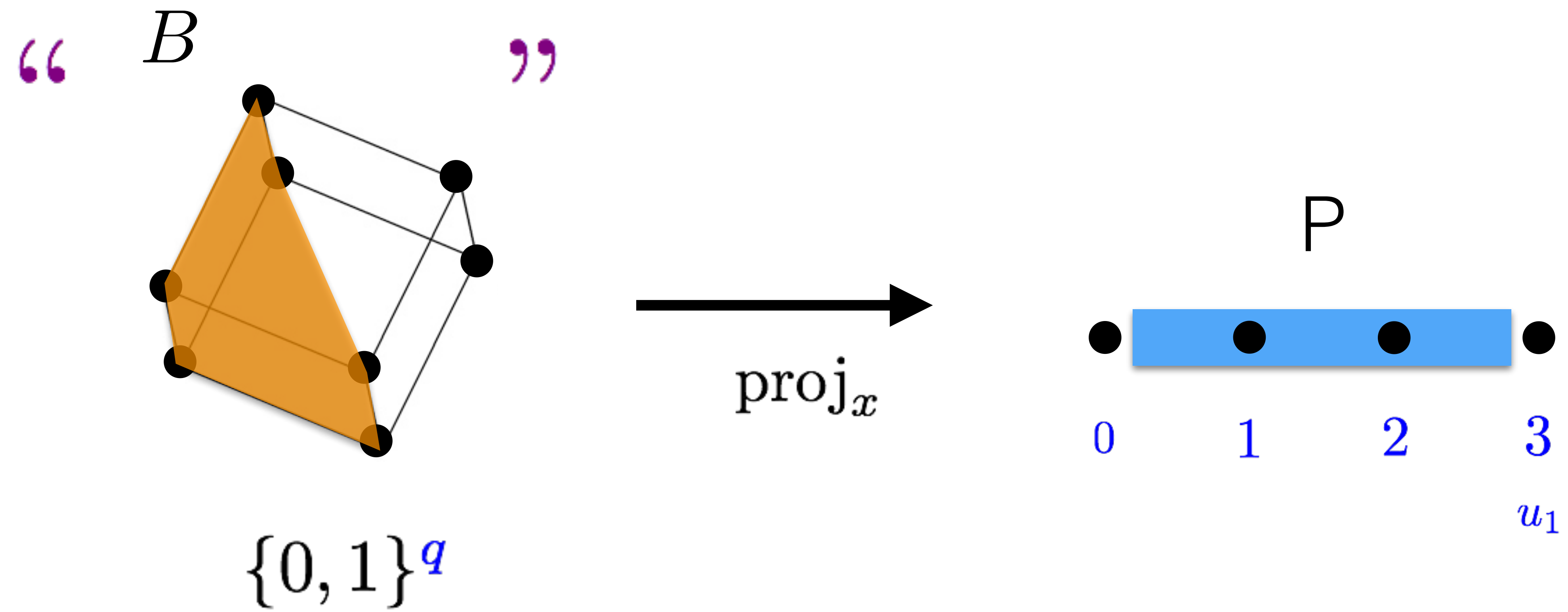
After Substitution:

Problem $n, \max\{c_i\}$	P	F(P)	U(P)	AvV
30, 10	4.9	2.5	600	0.9
40, 10	75.2	7.1	600	2.4
40, 20	469.0	23.8	600	6.3

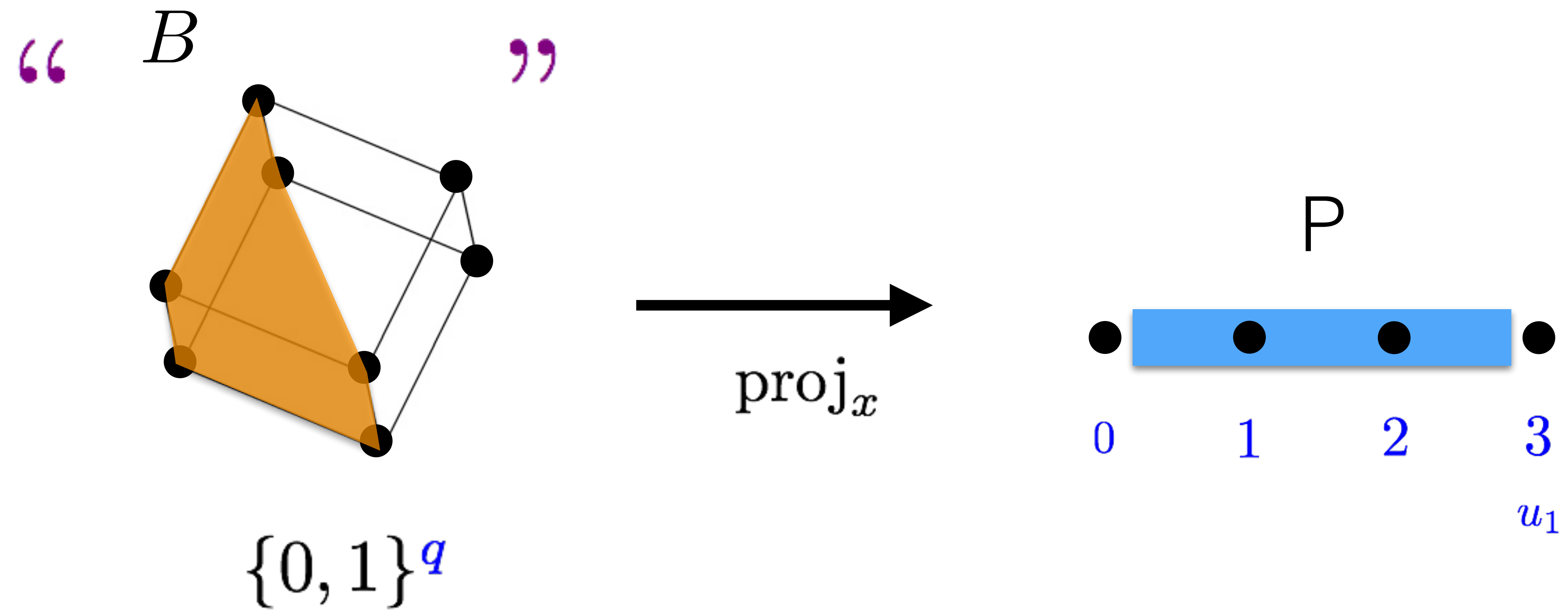
*Computational analysis by Bonami, Dash, Lodi, Tramontani

What is a binarization?

Binarization



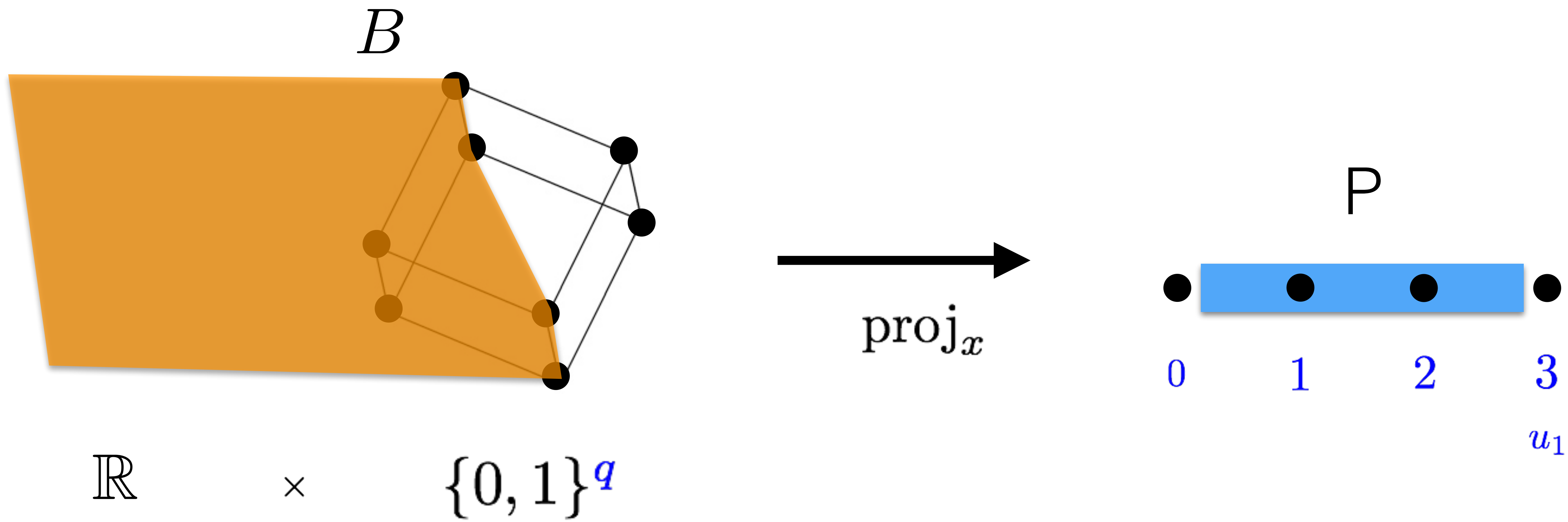
Binarization



- $P = \{x \in \mathbb{R}^n : Ax \leq b,\ 0 \leq x_i \leq u\}$.
- A *binarization* is a polytope $B \subseteq \mathbb{R} \times [0,1]^q$ with

  $[0,u] \cap \mathbb{Z} \subseteq \text{proj}_{x_i}(B \cap (\mathbb{R} \times \{0,1\}^q))$.

Binarization



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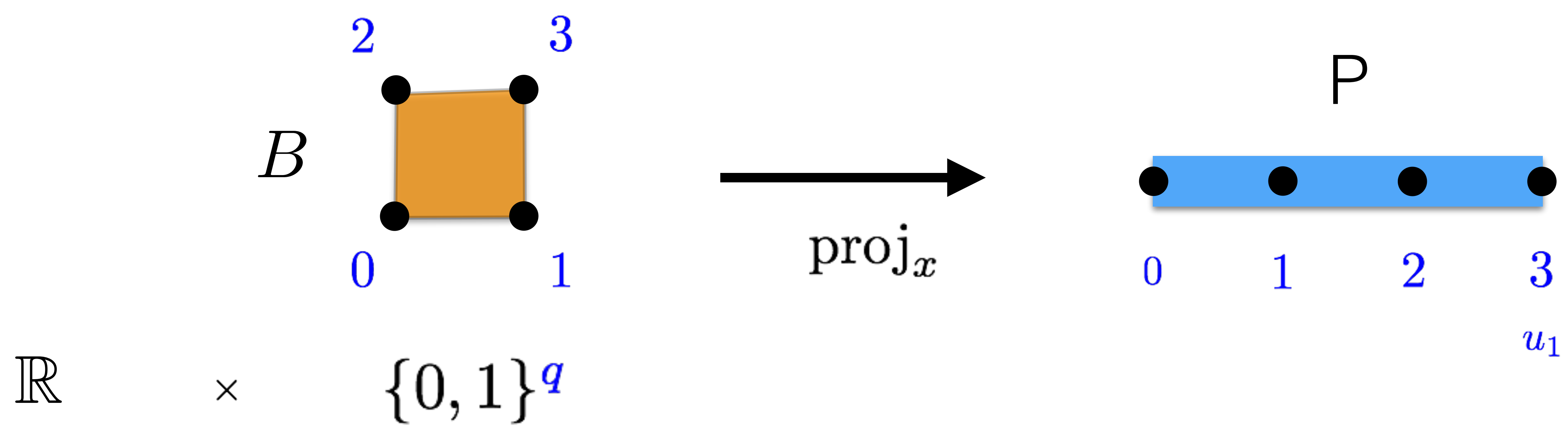
Binarization

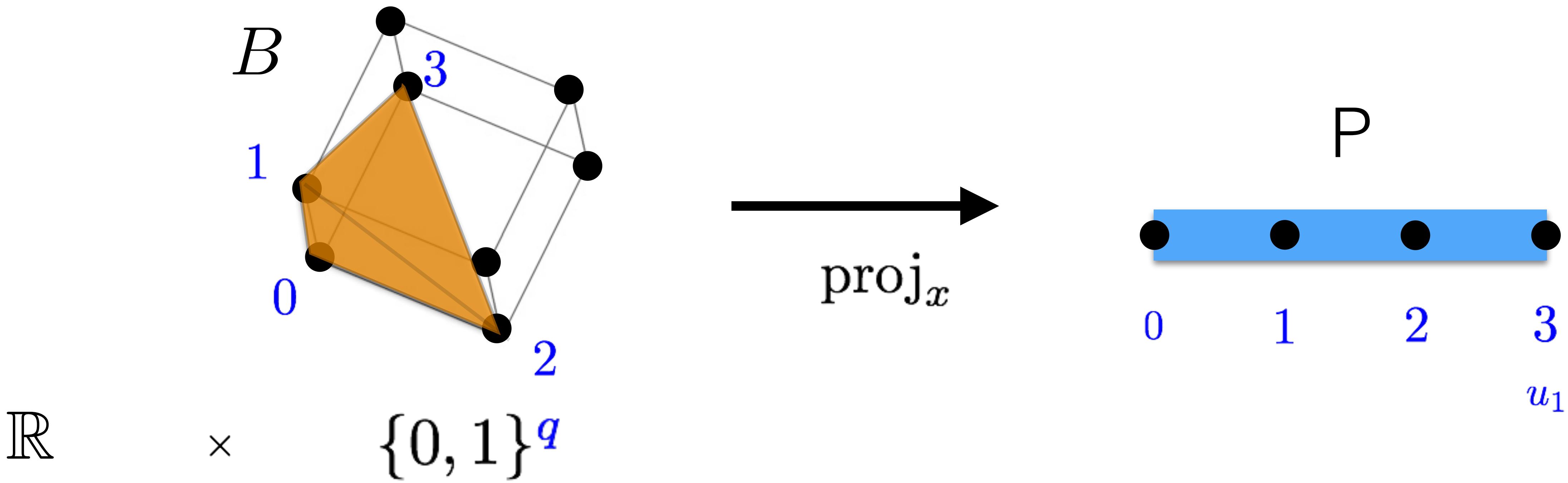


Log

$$x = \sum_{j=0}^{\lfloor \log(u+1) \rfloor} 2^j z_j \leq u$$

$$z \in [0,1]^{\lfloor \log(u+1) \rfloor}$$

Binarization



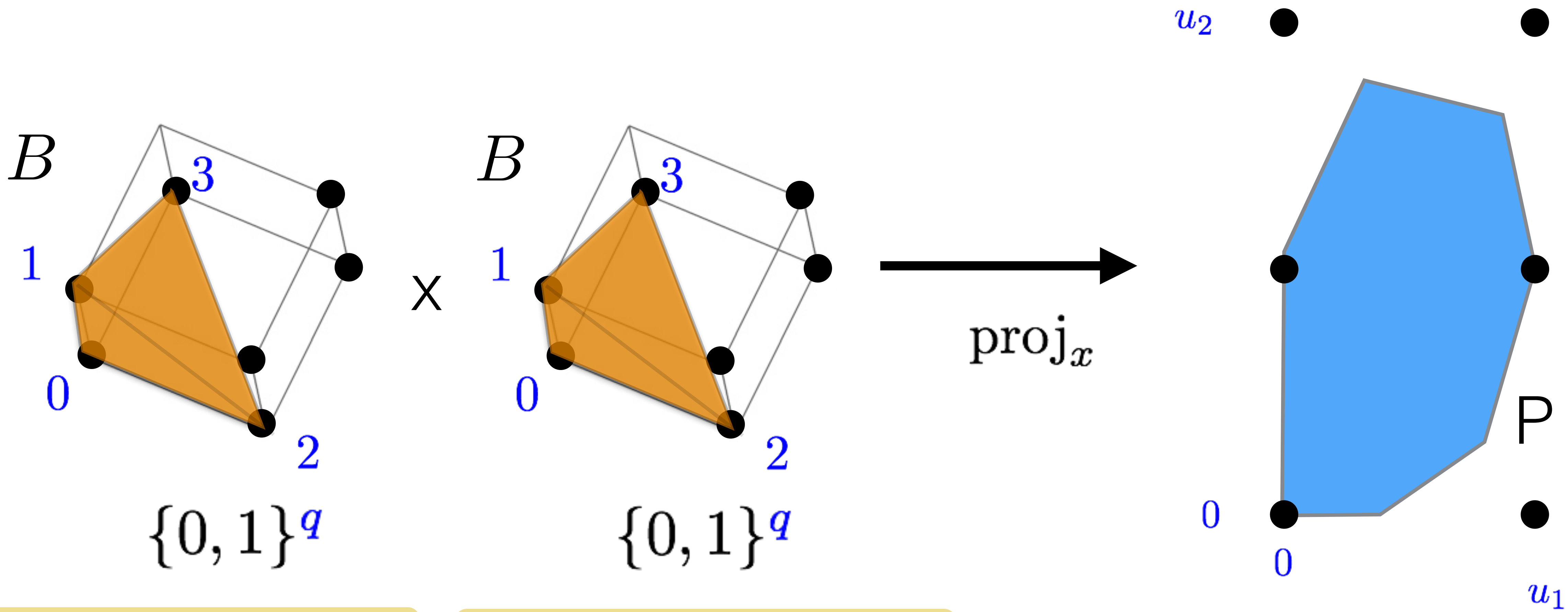
Full

$$x = \sum_{j=1}^{u} j z_j$$

$$z_1 + z_2 + \cdots + z_u \leq 1$$

$$z \in [0, 1]^u$$

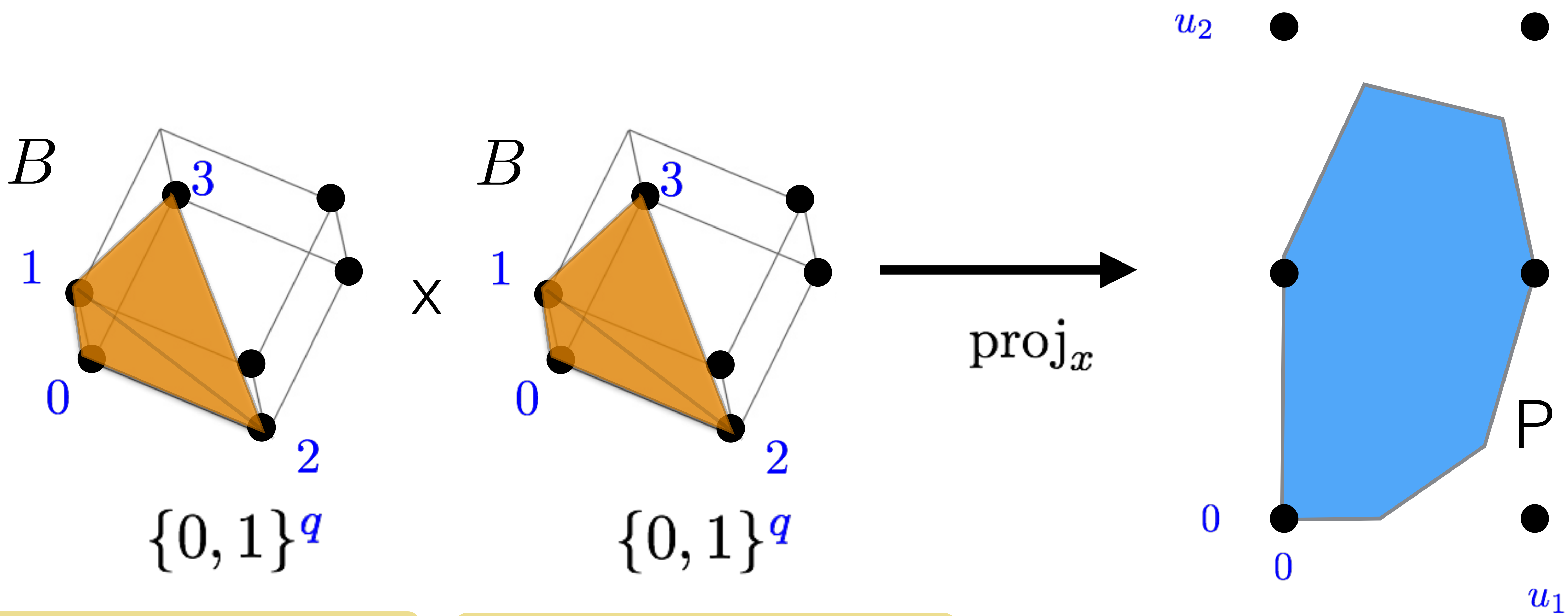
Binarization



$$x_1 = \sum_{j=1}^{3} j z_j^1$$

$$x_2 = \sum_{j=1}^{3} j z_j^2$$

Binarization



$x_1 = \sum_{j=1}^{3} j z_j^1$

$x_2 = \sum_{j=1}^{3} j z_j^2$

Given a binarization $\mathcal{B}$ and a polytope P,
the *binary extended formulation* is

$$P_{\mathcal{B}} = \{(x,z) \in \mathbb{R}^n \times [0,1]^q : x \in P, (x_i, z^i) \in B^i\}$$

Binarization

Binary Extended Formulation

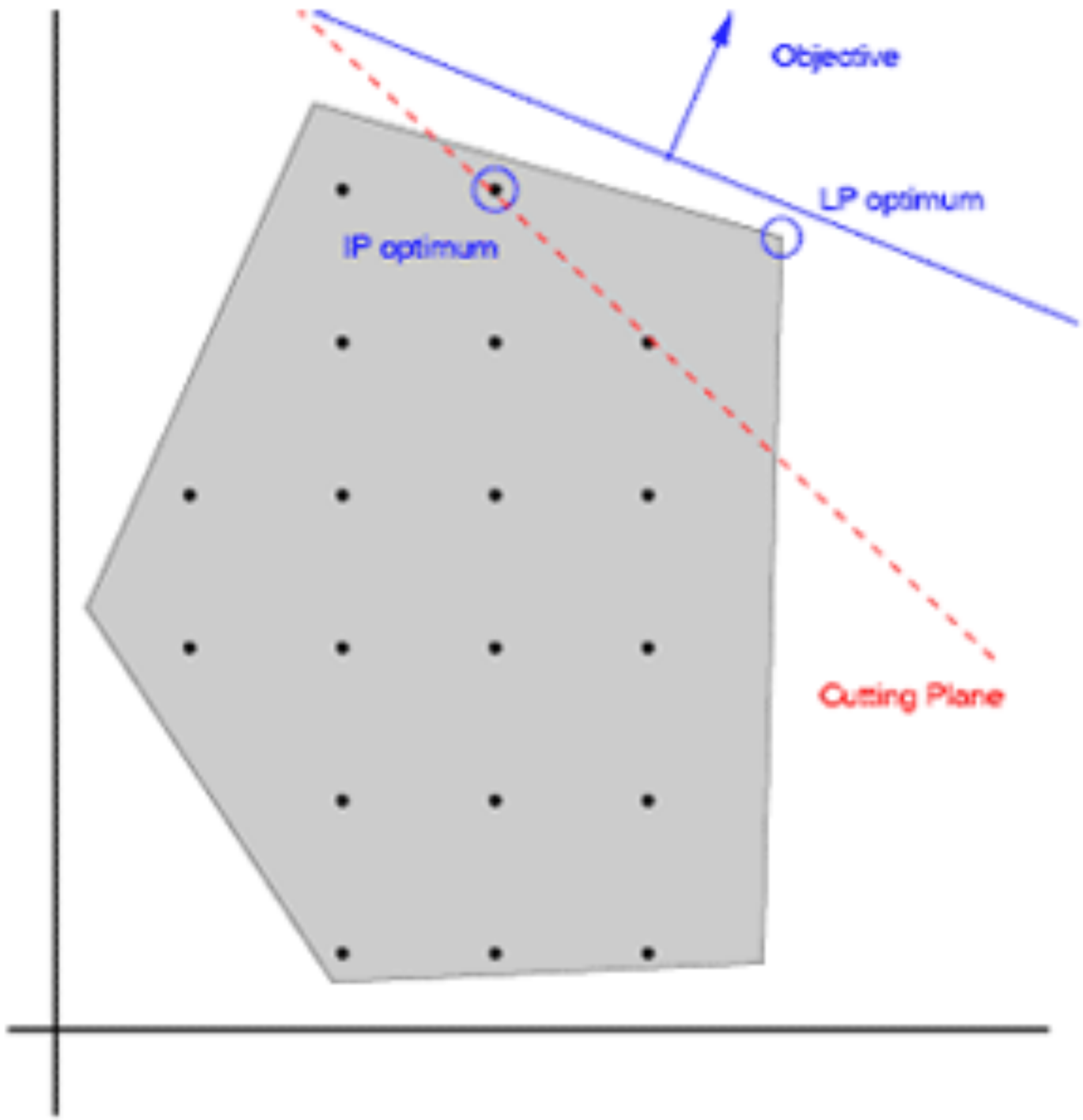
Given a binarization $\mathcal{B}$ and a polytope P, the *binary extended formulation* is

$$P_{\mathcal{B}} = \{(x, z) \in \mathbb{R}^n \times [0, 1]^q : x \in P, (x_i, z^i) \in B^i\}$$

- $\text{proj}_x(P_{\mathcal{B}}) = P$
- $\text{proj}_x(P_{\mathcal{B}} \cap \mathbb{R}^n \times \{0, 1\}^q) = P \cap \mathbb{Z}^n$

Cutting Planes

Cutting Planes



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      0     0    37281.6812    43    43163.7033     Cuts: 15      455   13.63%
      0     0    37288.9797    53    43163.7033     Cuts: 13      468   13.61%
      0     0    37292.4372    48    43163.7033      Cuts: 4      475   13.60%
      0     0    37294.8200    55    43163.7033      Cuts: 7      483   13.60%
*     0+    0                    41713.5846   37294.8200           10.59%
*     0+    0                    40468.1795   37294.8200            7.84%
*     0+    0                    38816.4629   37294.8200            3.92%
      0     2    37297.8778    54    38816.4629   37294.8568      483    3.92%
Elapsed time = 0.34 sec. (99.46 ticks, tree = 0.00 MB, solutions = 10)
*    10+   10                    38040.7778   37373.1880            1.75%
*   410+  273                    38028.0793   37560.1706            1.23%
*   410+  273                    37913.5385   37560.1706            0.93%
*   411+  274                    37913.5385   37560.1706            0.93%
*  1499   420      integral     0    37809.0000   37757.2612    15053    0.14%

Implied bound cuts applied:  60
Flow cuts applied:  46
Mixed integer rounding cuts applied:  106
Flow path cuts applied:  29
Lift and project cuts applied:  4
Gomory fractional cuts applied:  60

Root node processing (before b&c):
  Real time             =    0.34 sec. (99.20 ticks)
Parallel b&c, 4 threads:
  Real time             =    0.38 sec. (175.07 ticks)
  Sync time (average)   =    0.07 sec.
  Wait time (average)   =    0.12 sec.
                          ------------
Total (root+branch&cut) =    0.73 sec. (274.27 ticks)

Solution status : Optimal Objective function value = 37809  Objective best value = 37809   Optimality Gap = 0
Time = 1.89754
```

```
*     0+    0                    63589.3538   37147.3468              41.58%
      0     0    37208.8463    53   63589.3538      Cuts: 22      371    41.49%
      0     0    37240.1690    49   63589.3538      Cuts: 19      400    41.44%
*     0+    0                    43163.7033   37240.1690              13.72%
      0     0    37255.0918    58   43163.7033      Cuts: 15      416    13.69%
      0     0    37274.4440    62   43163.7033      Cuts: 11      434    13.64%
      0     0    37281.6812    43   43163.7033      Cuts: 15      455    13.63%
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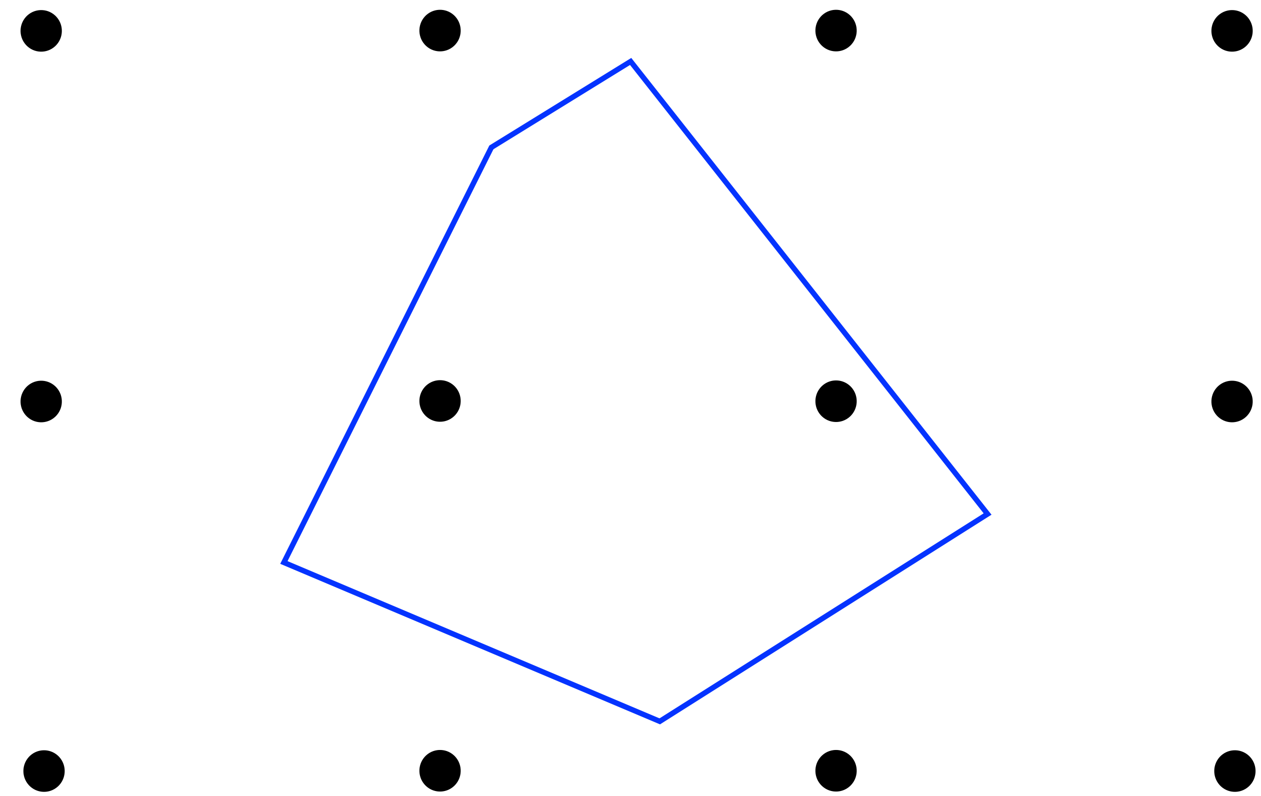
All cuts used are types of split cuts

Split Cuts



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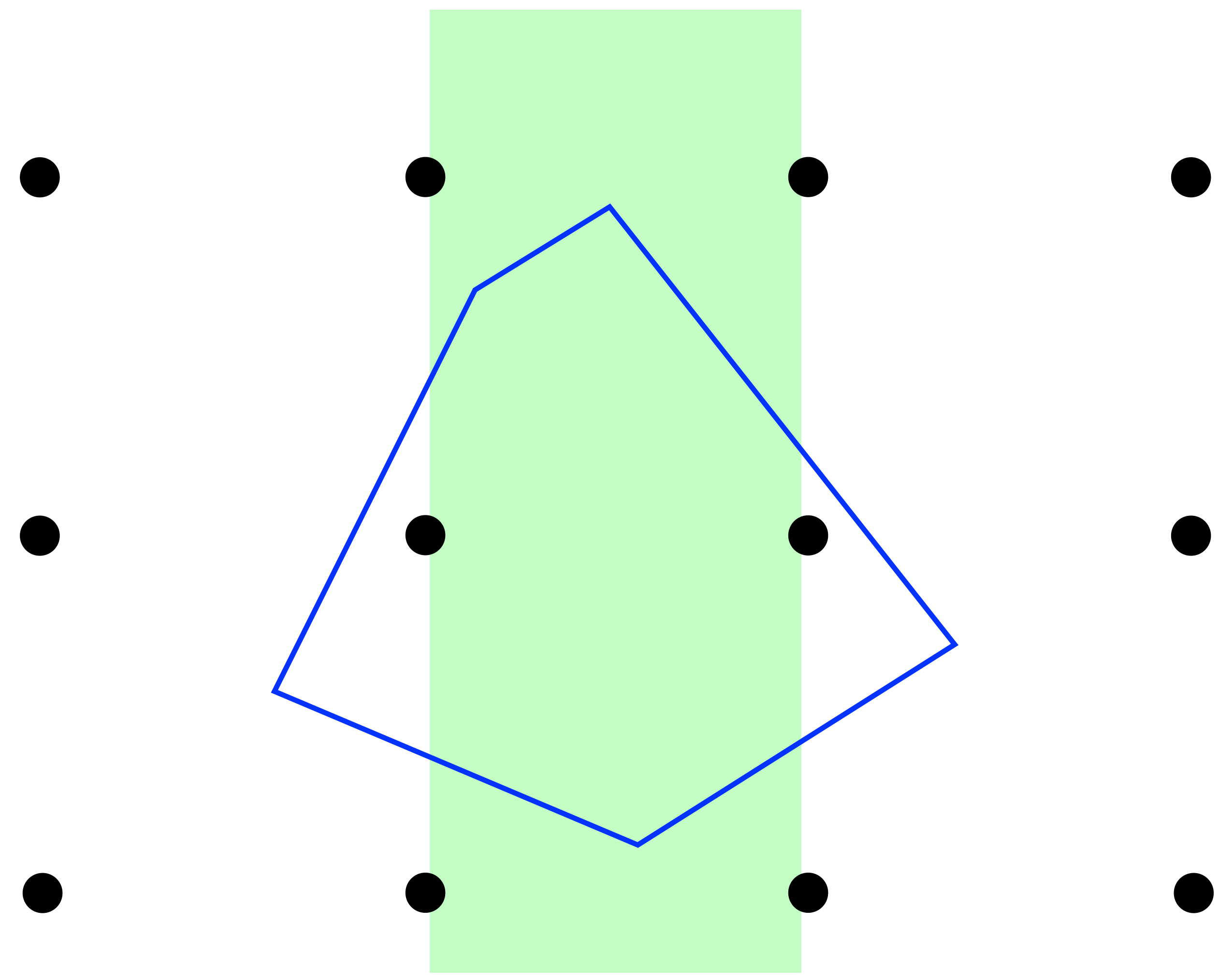
$$SC(P) = \bigcap_{S \in \mathcal{S}} \mathrm{conv}(P \setminus S).$$

Split Cuts



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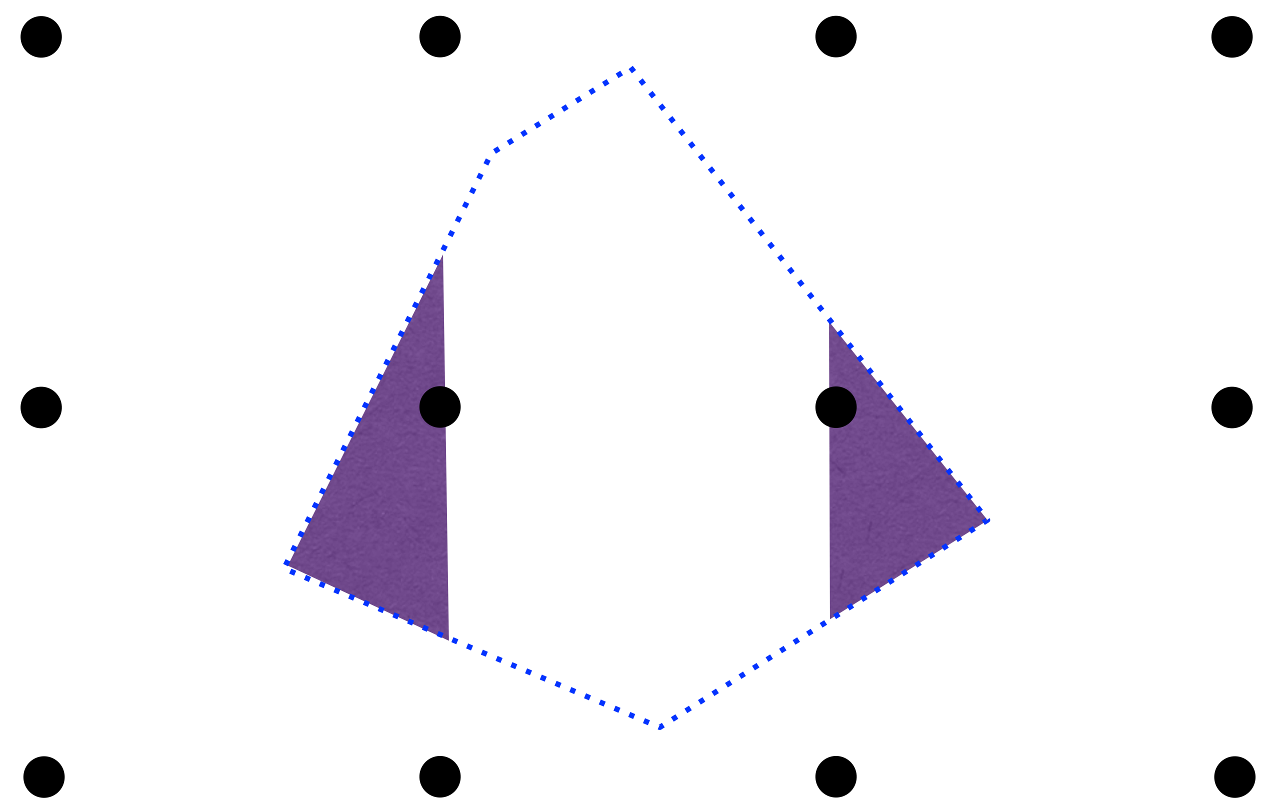

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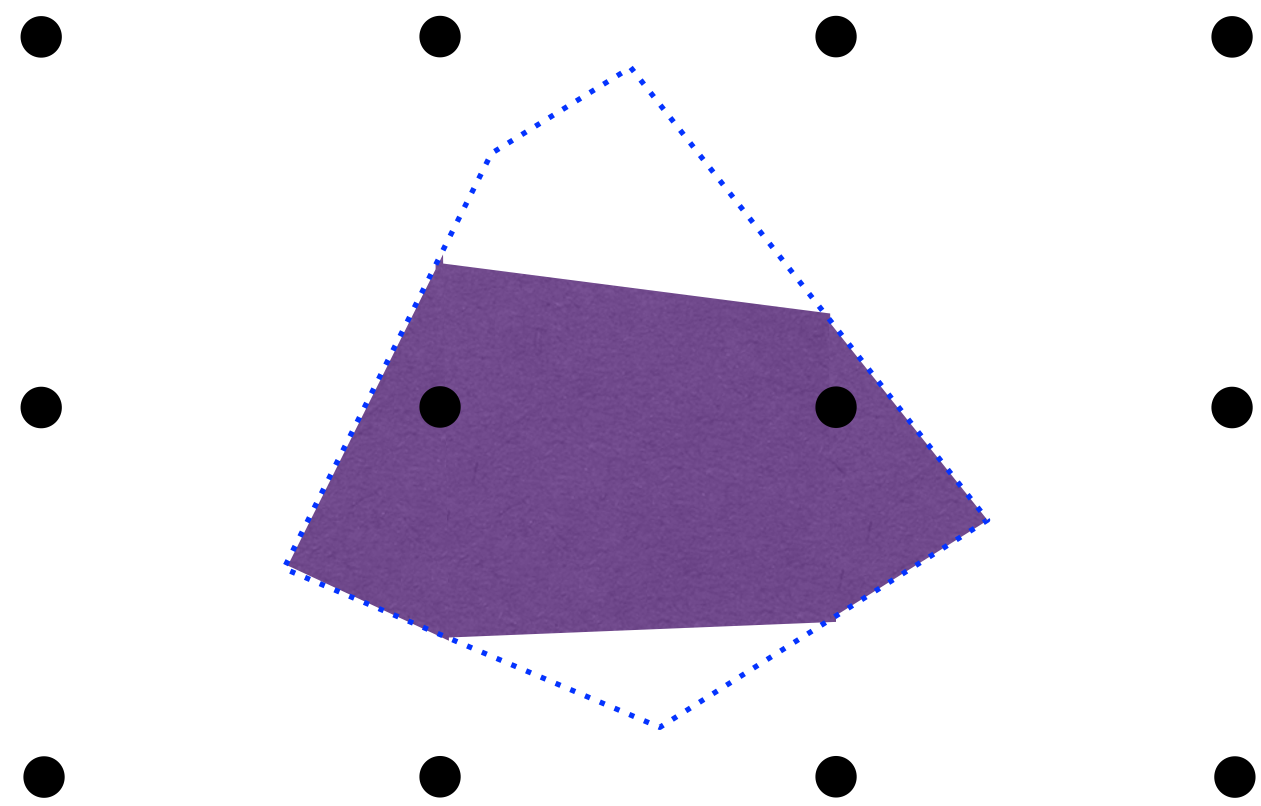

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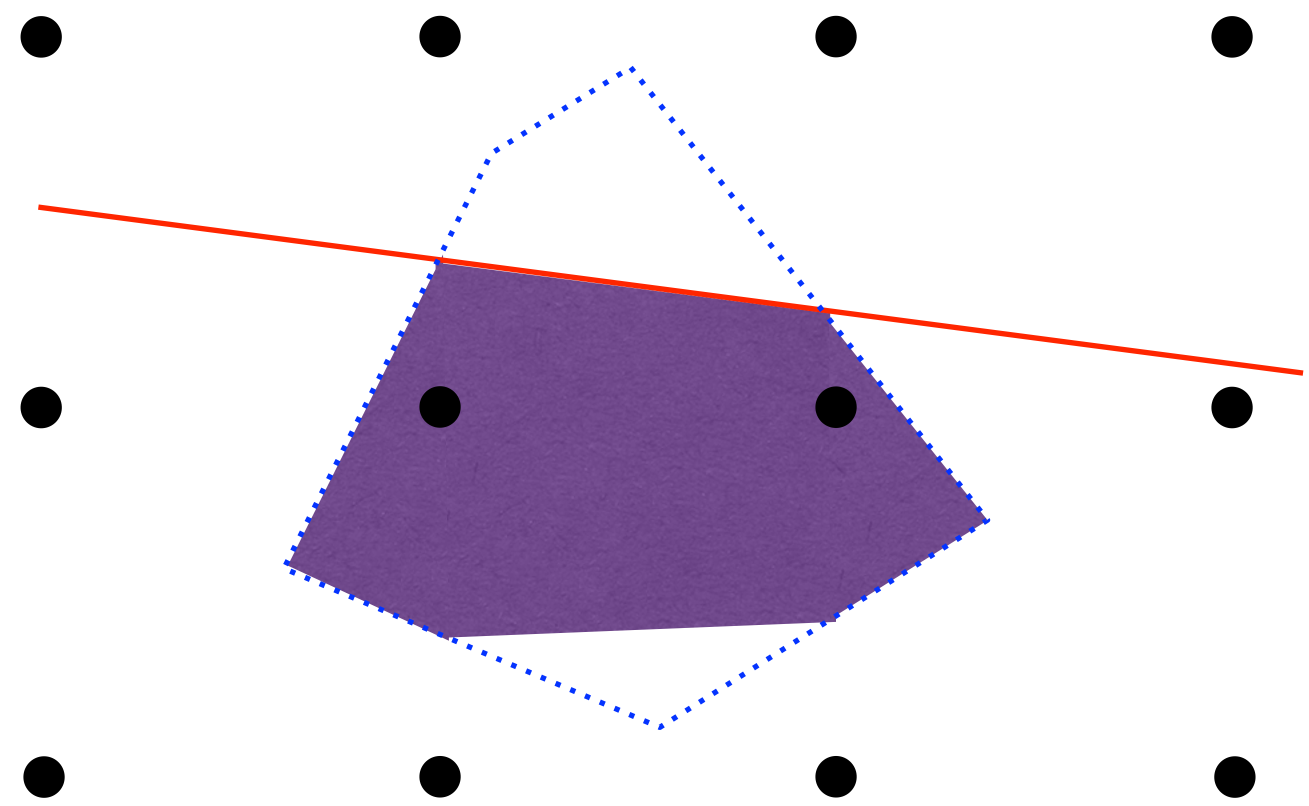

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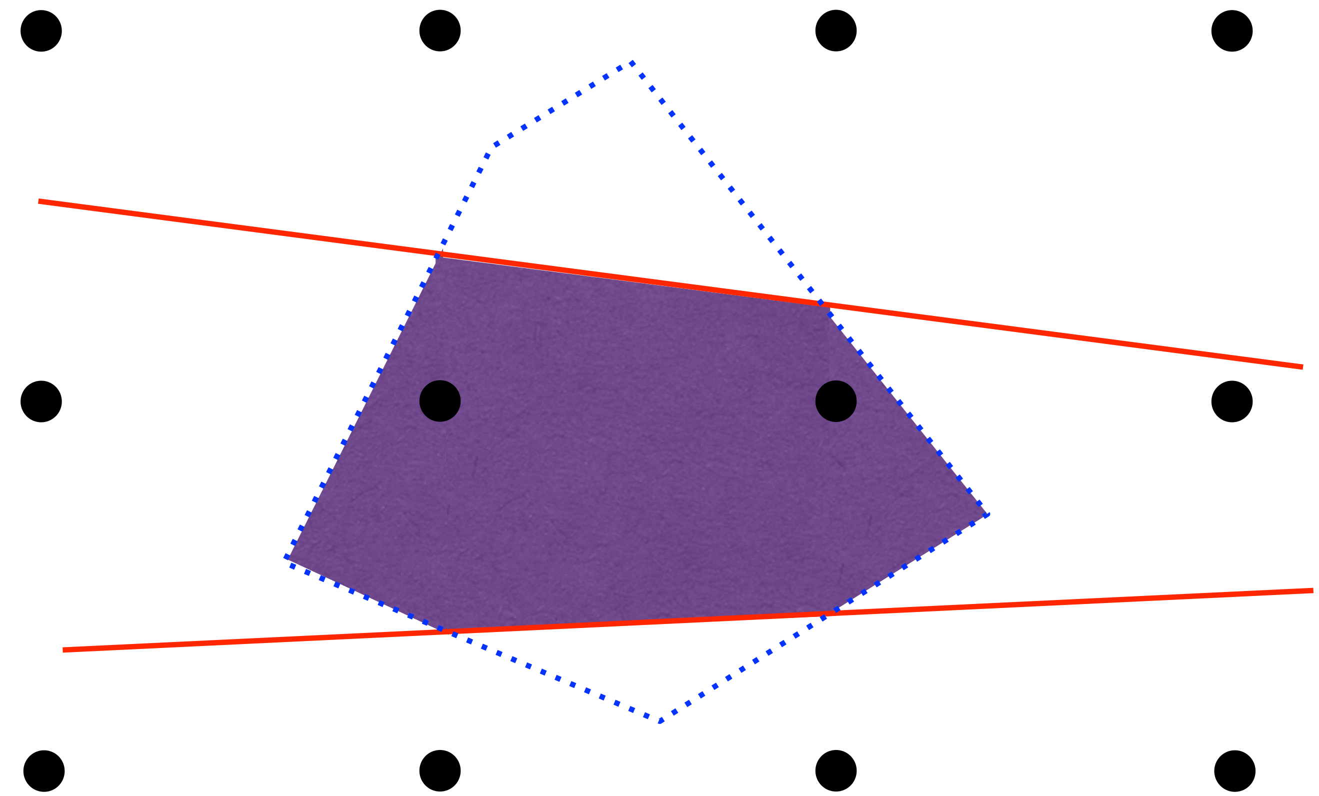

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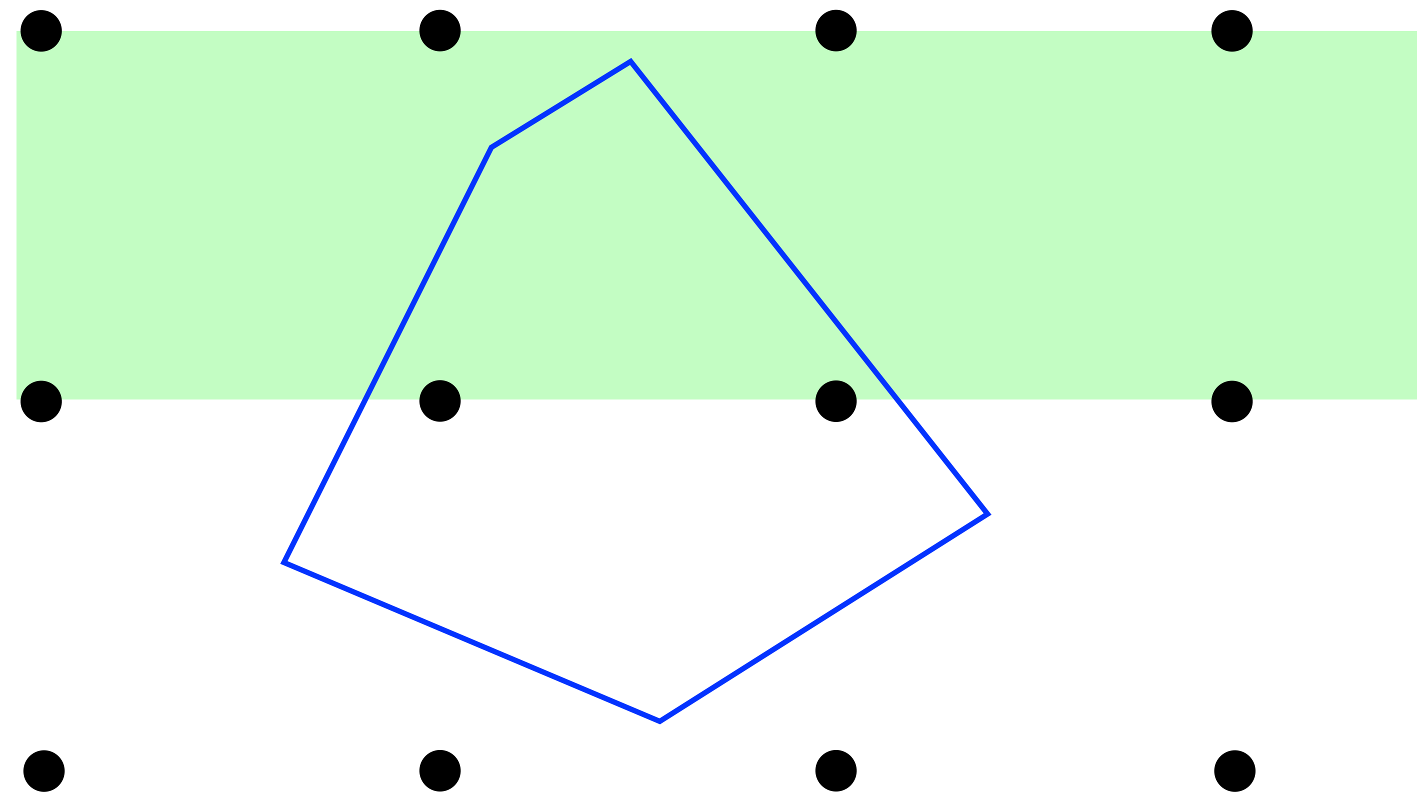

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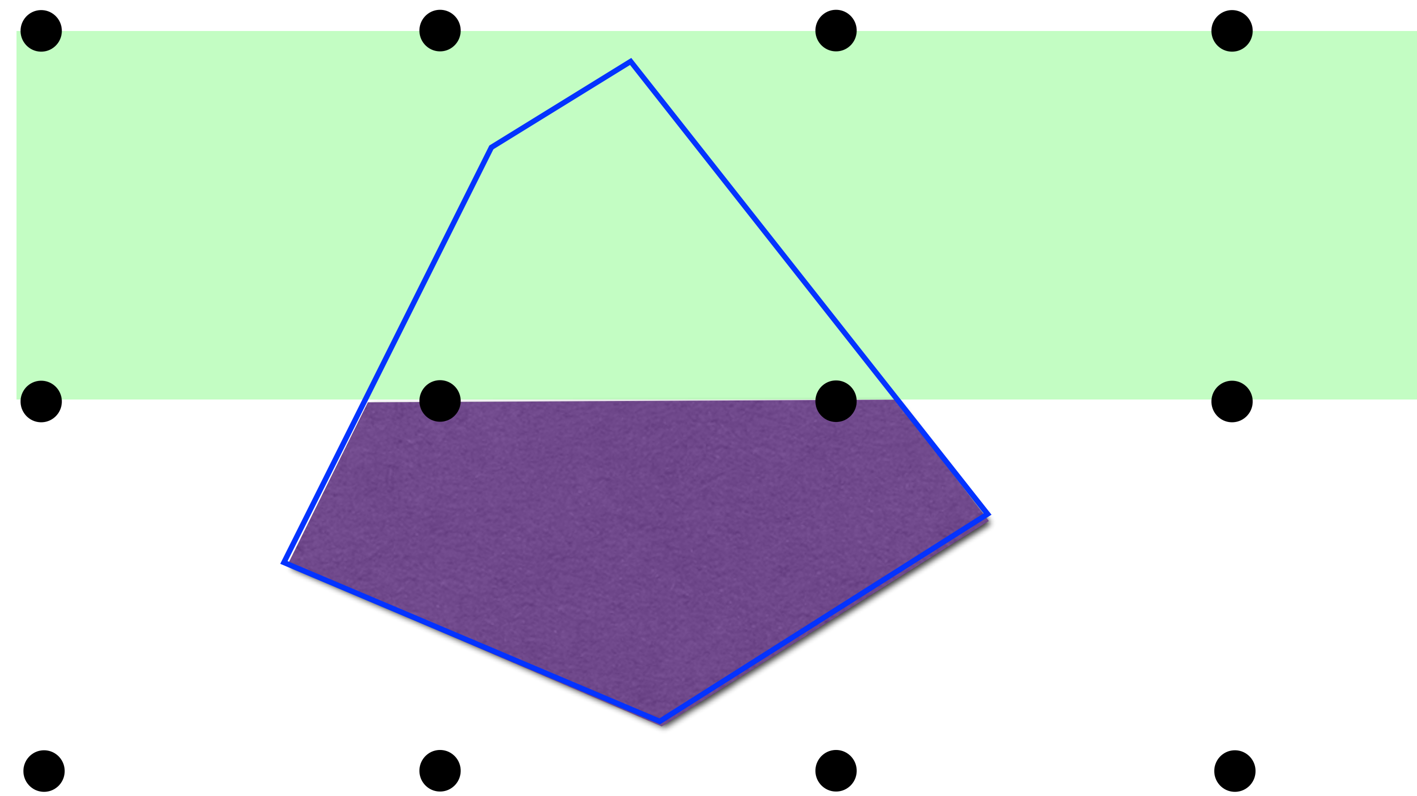

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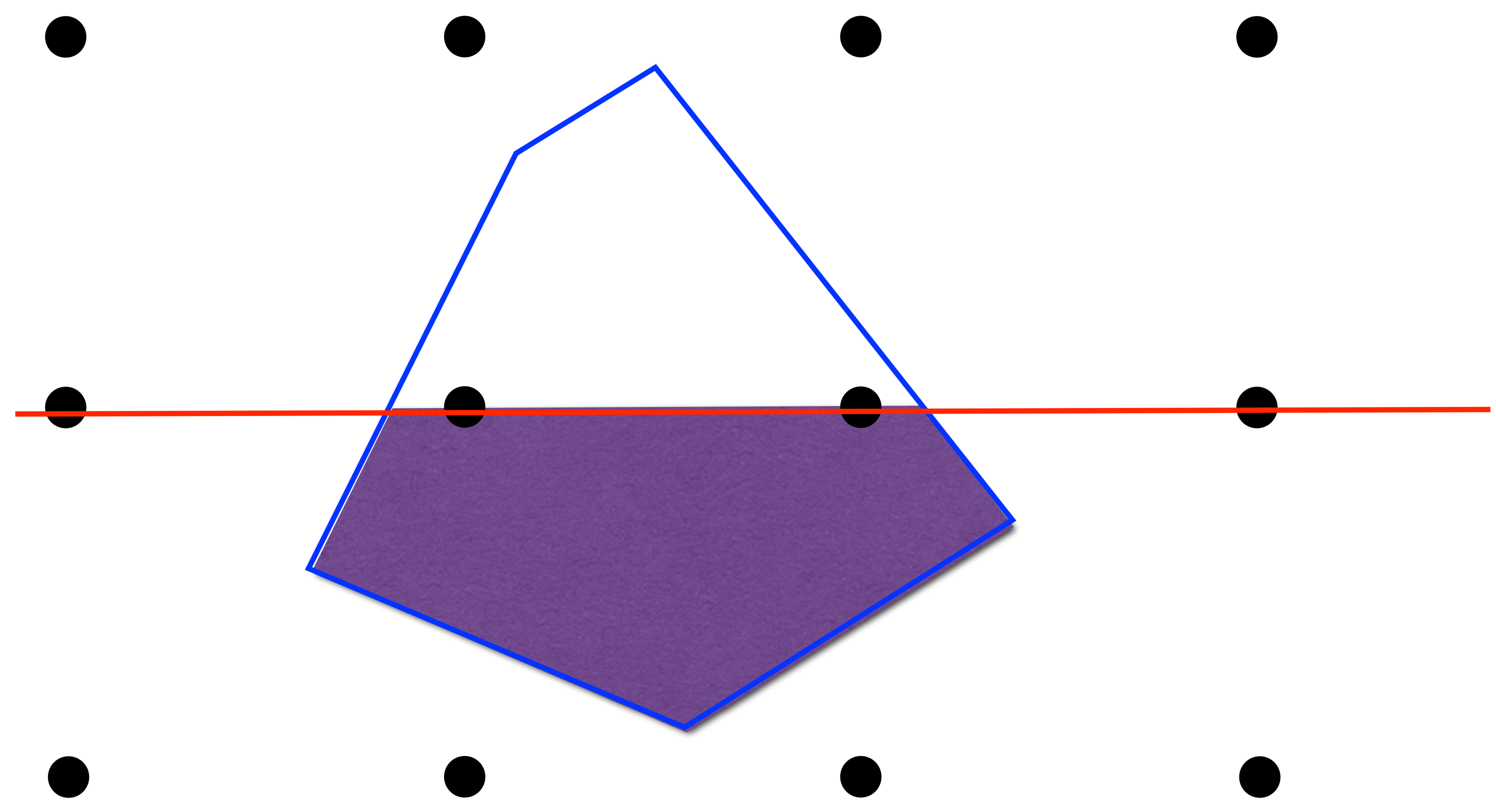

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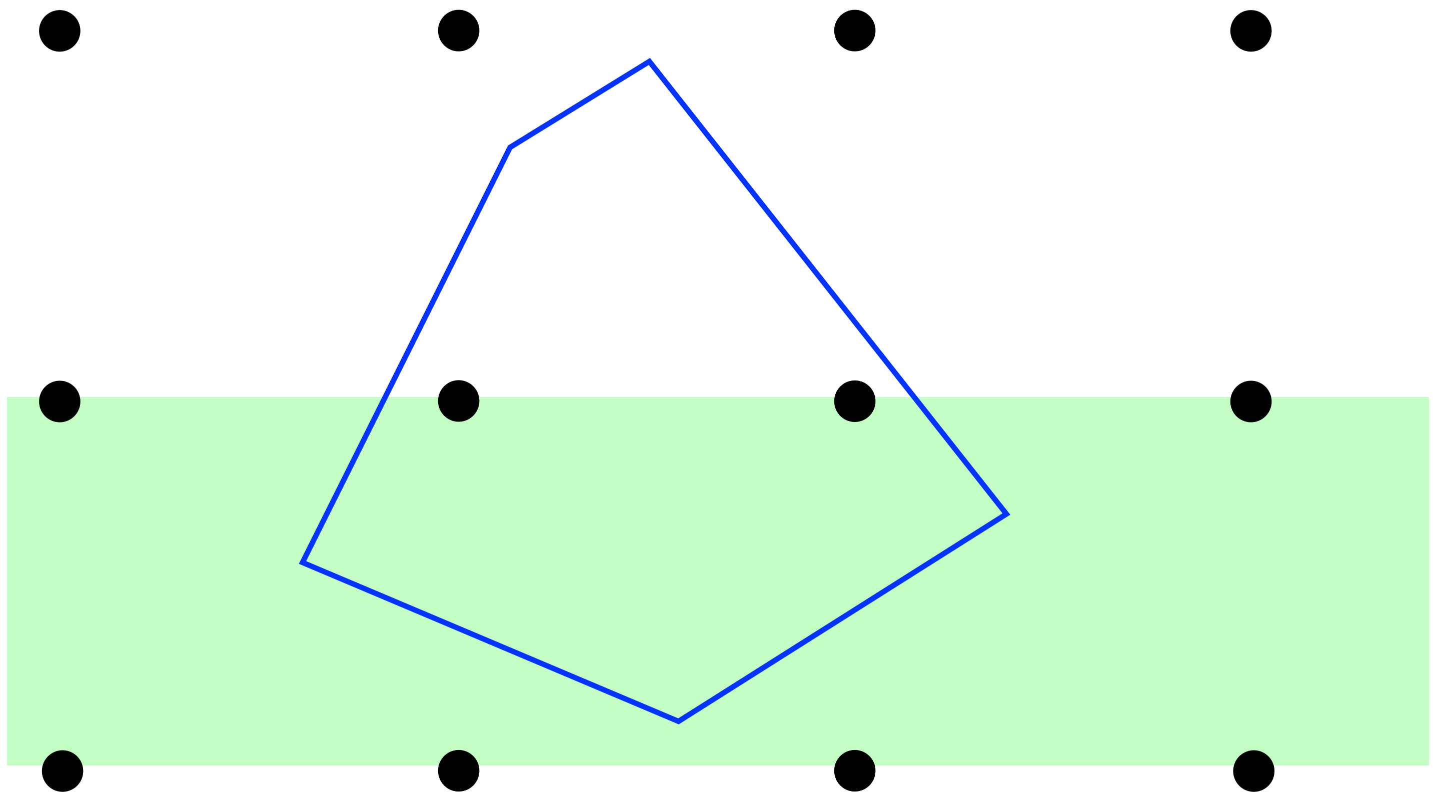

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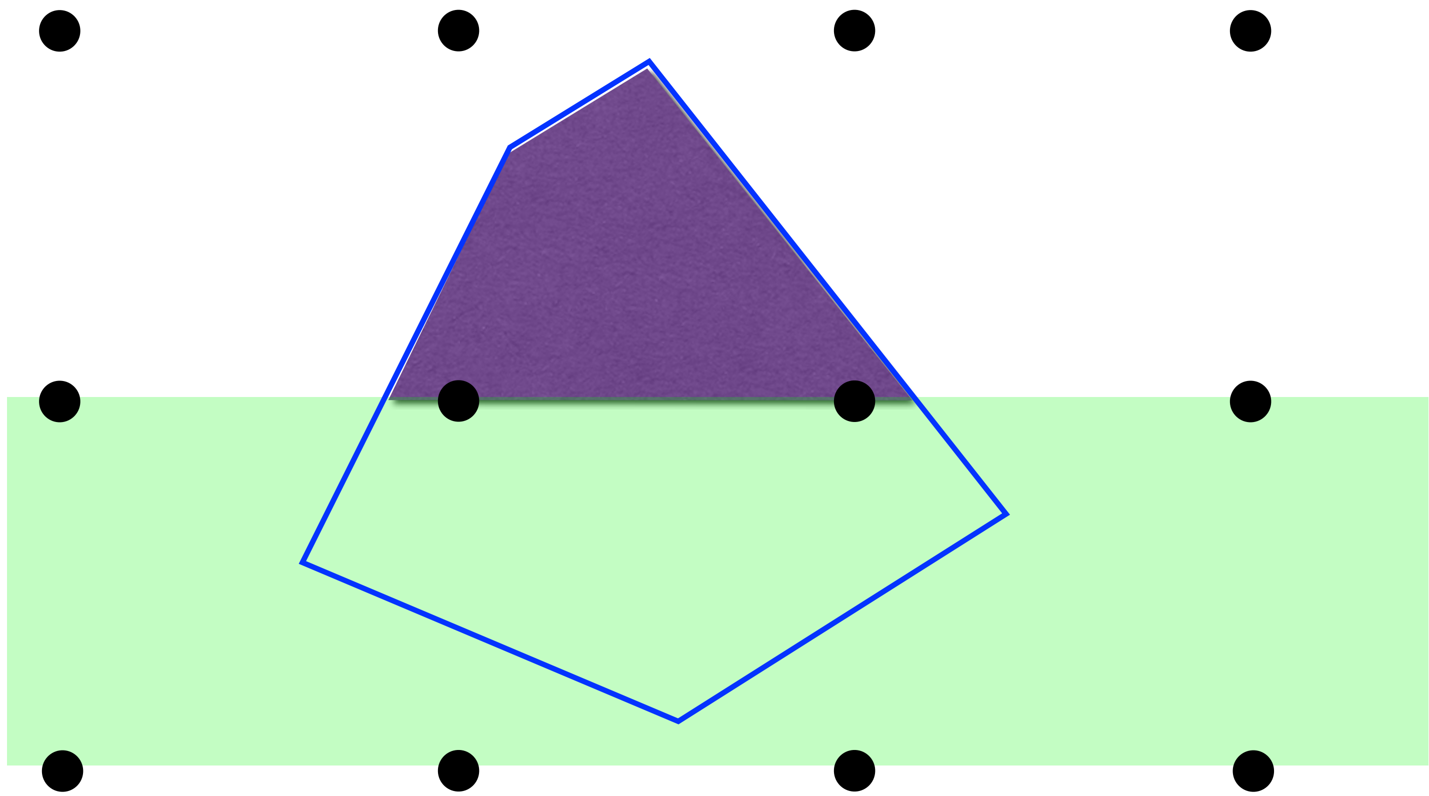

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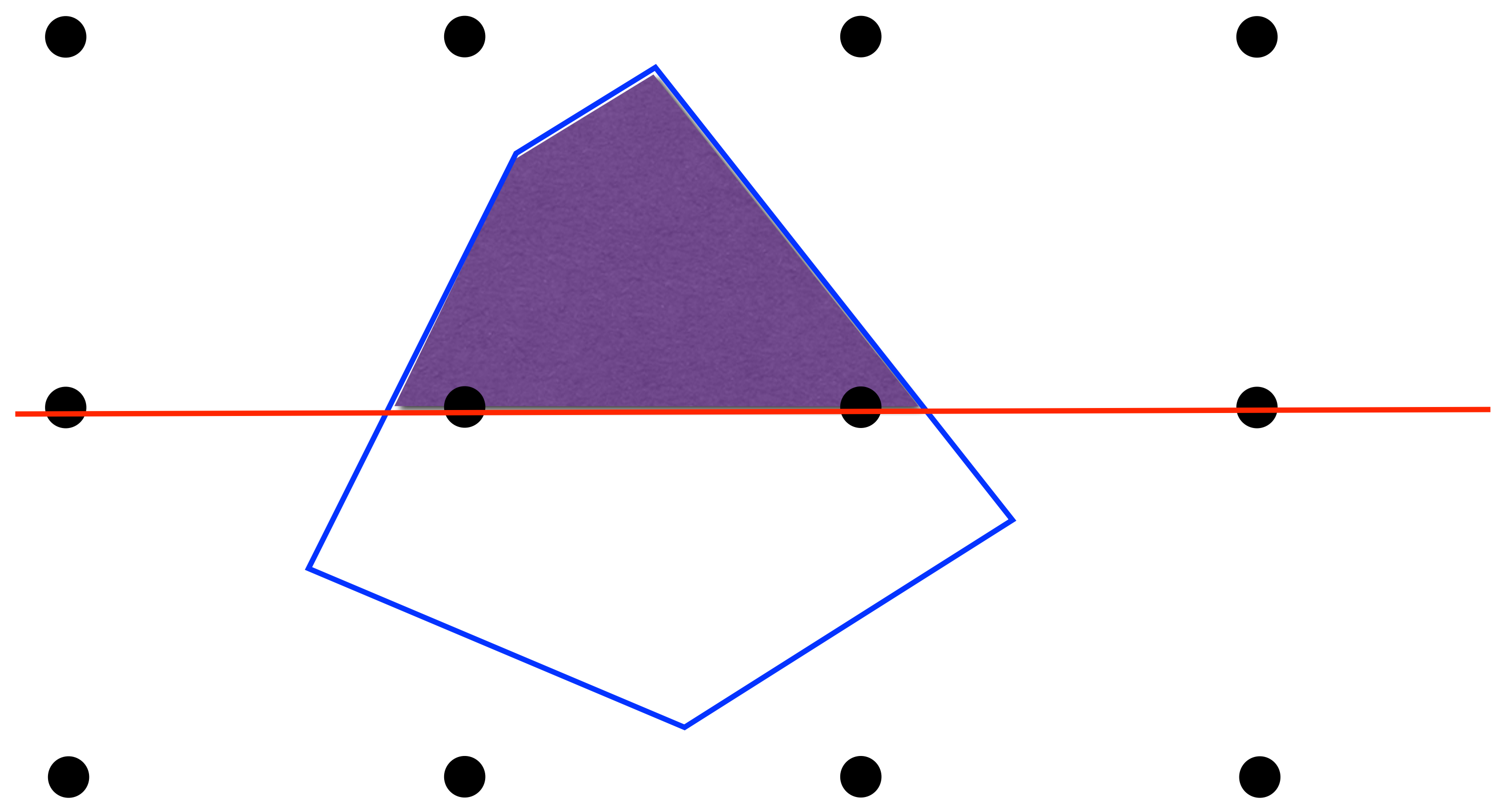

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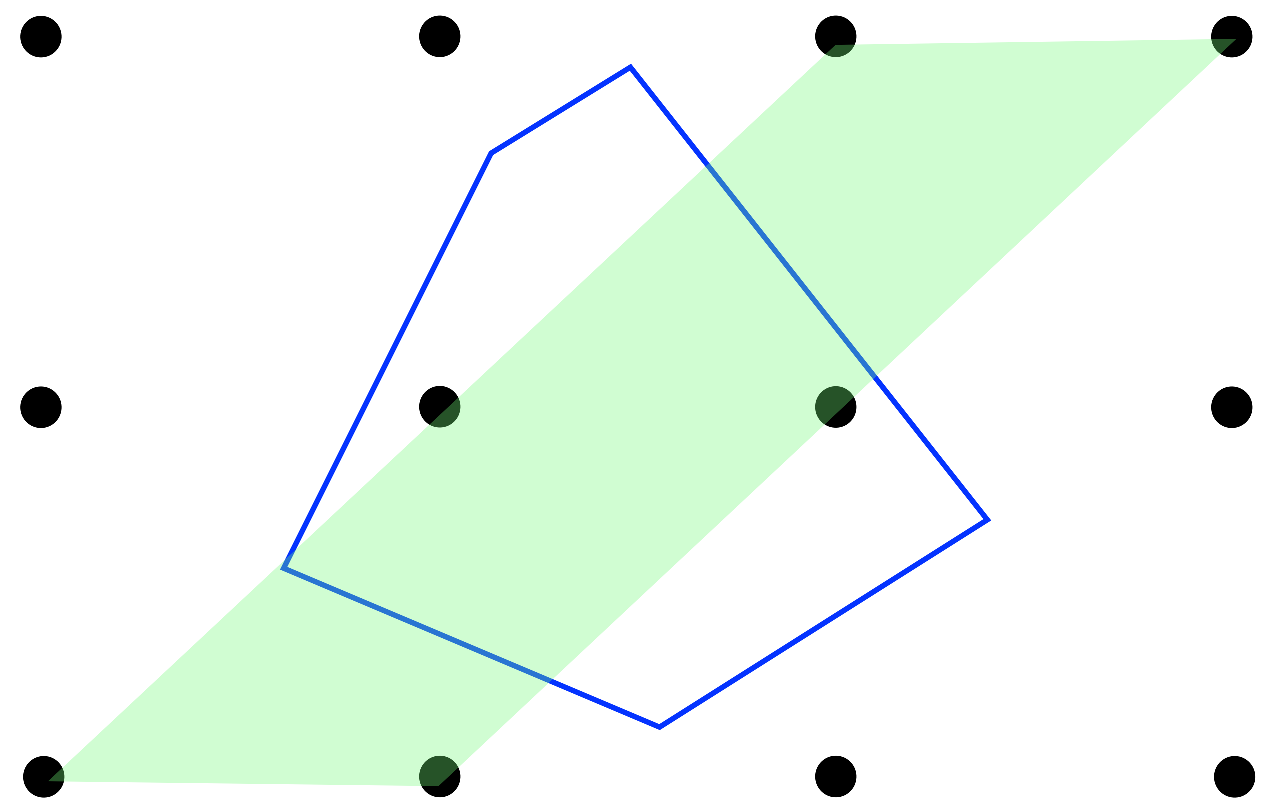

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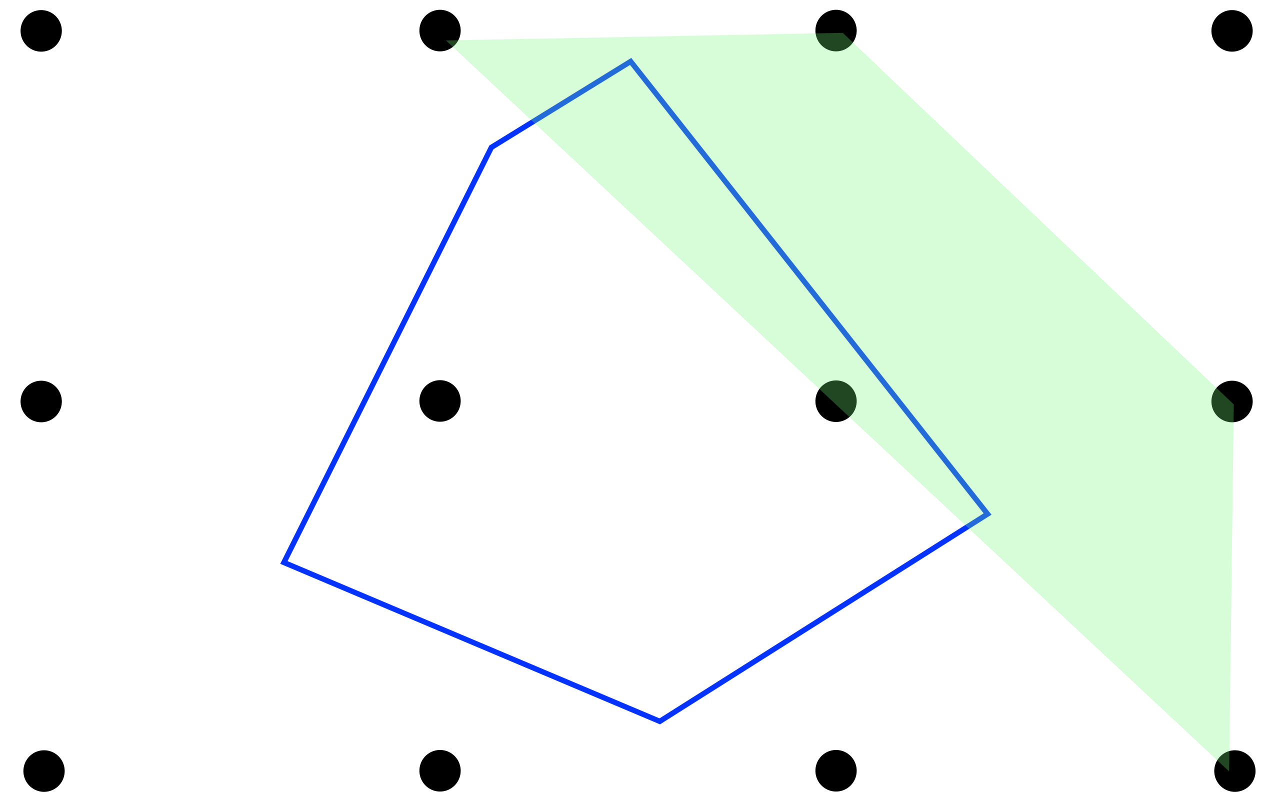

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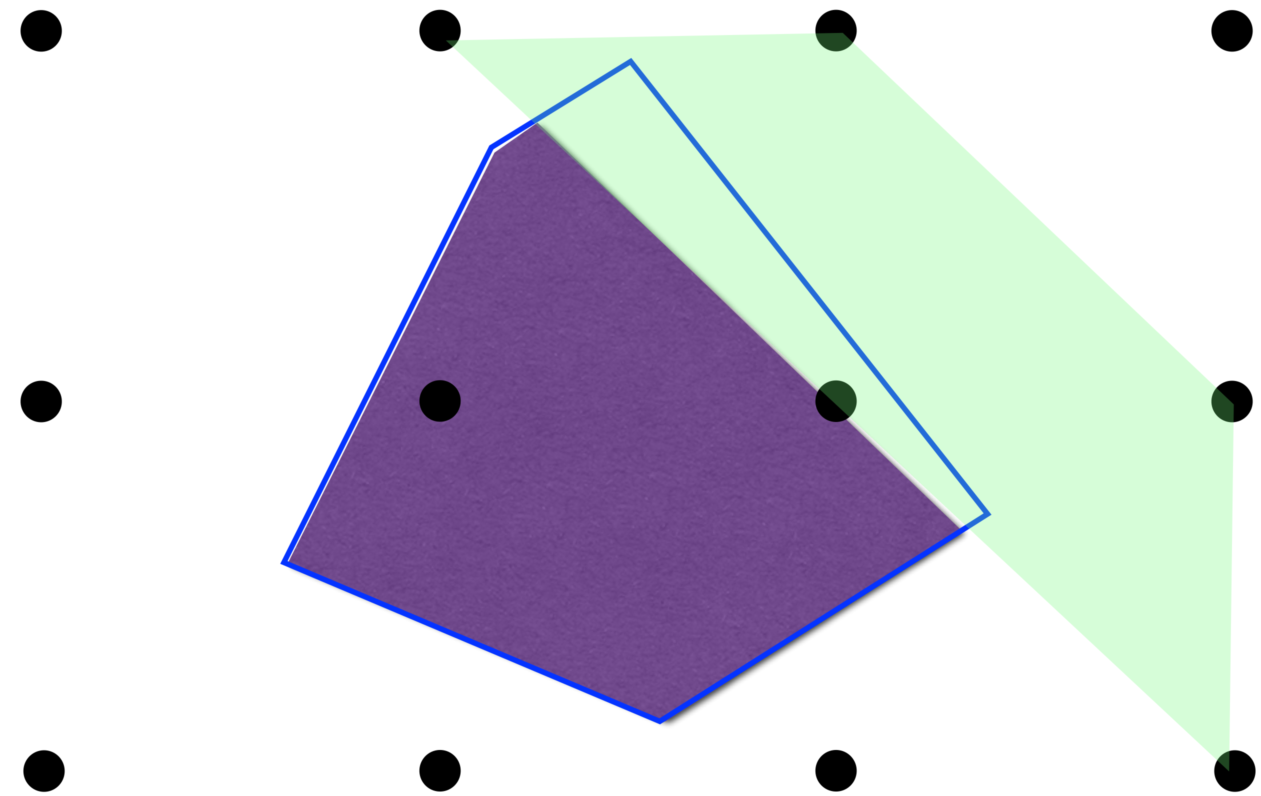

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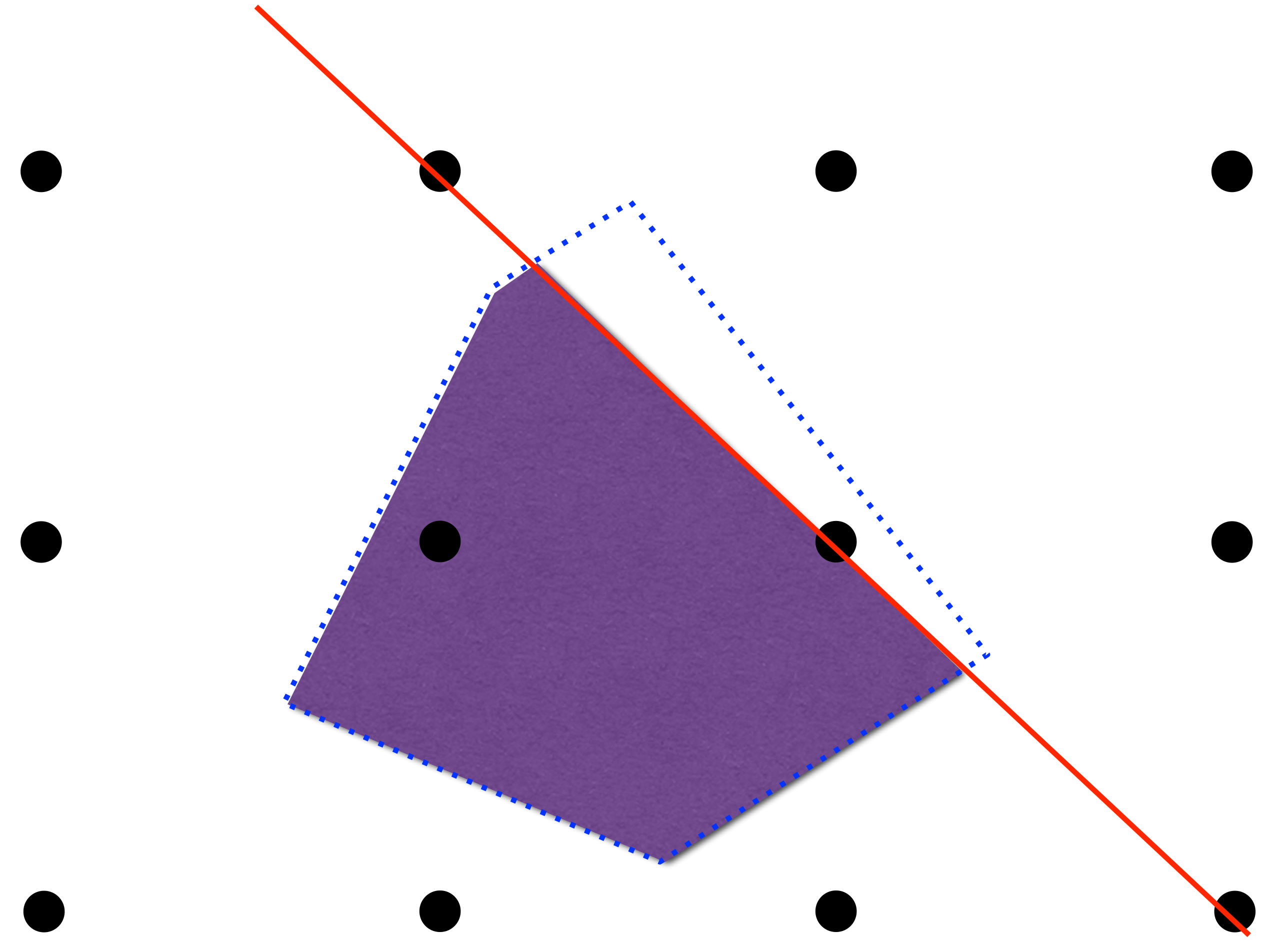

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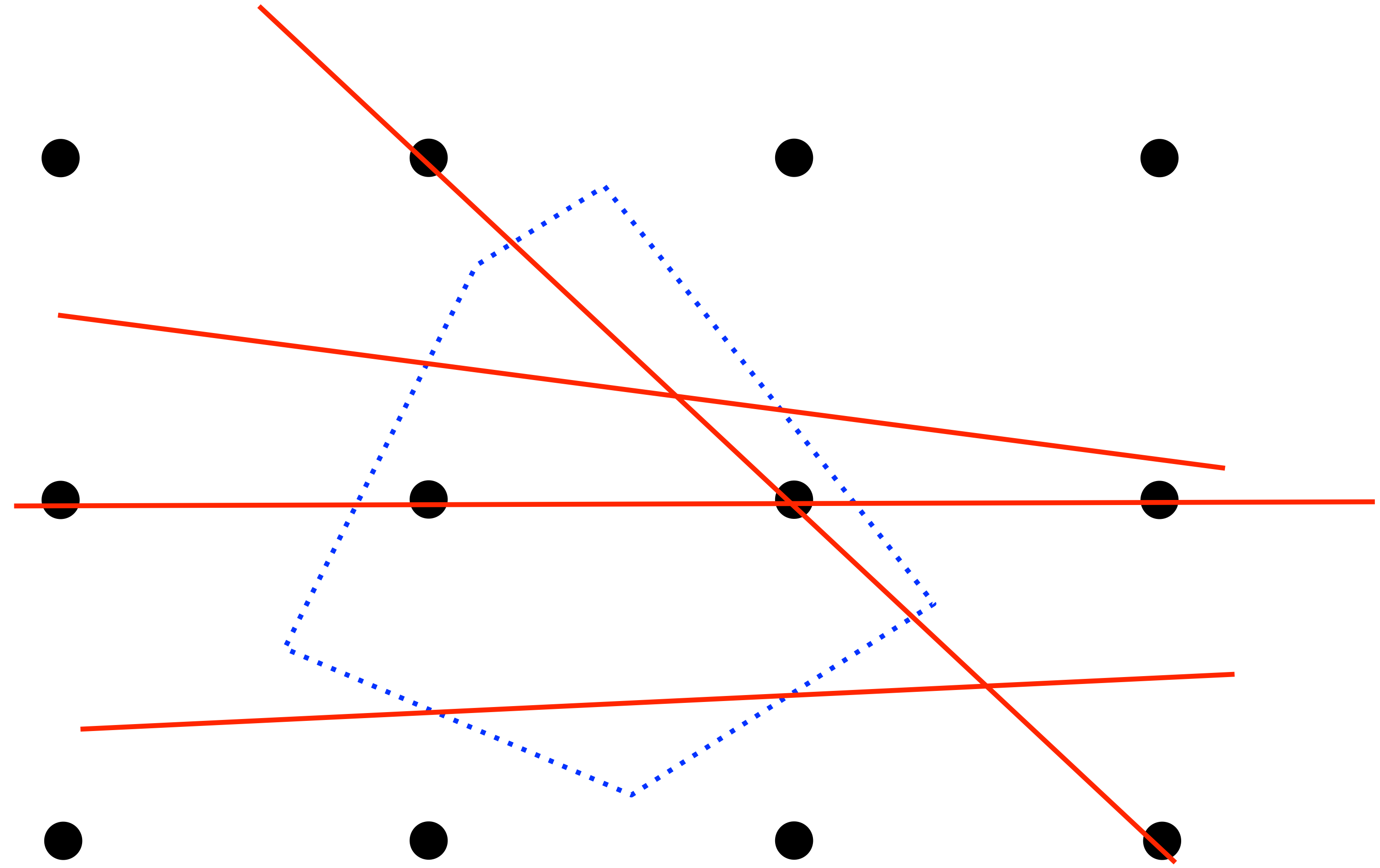

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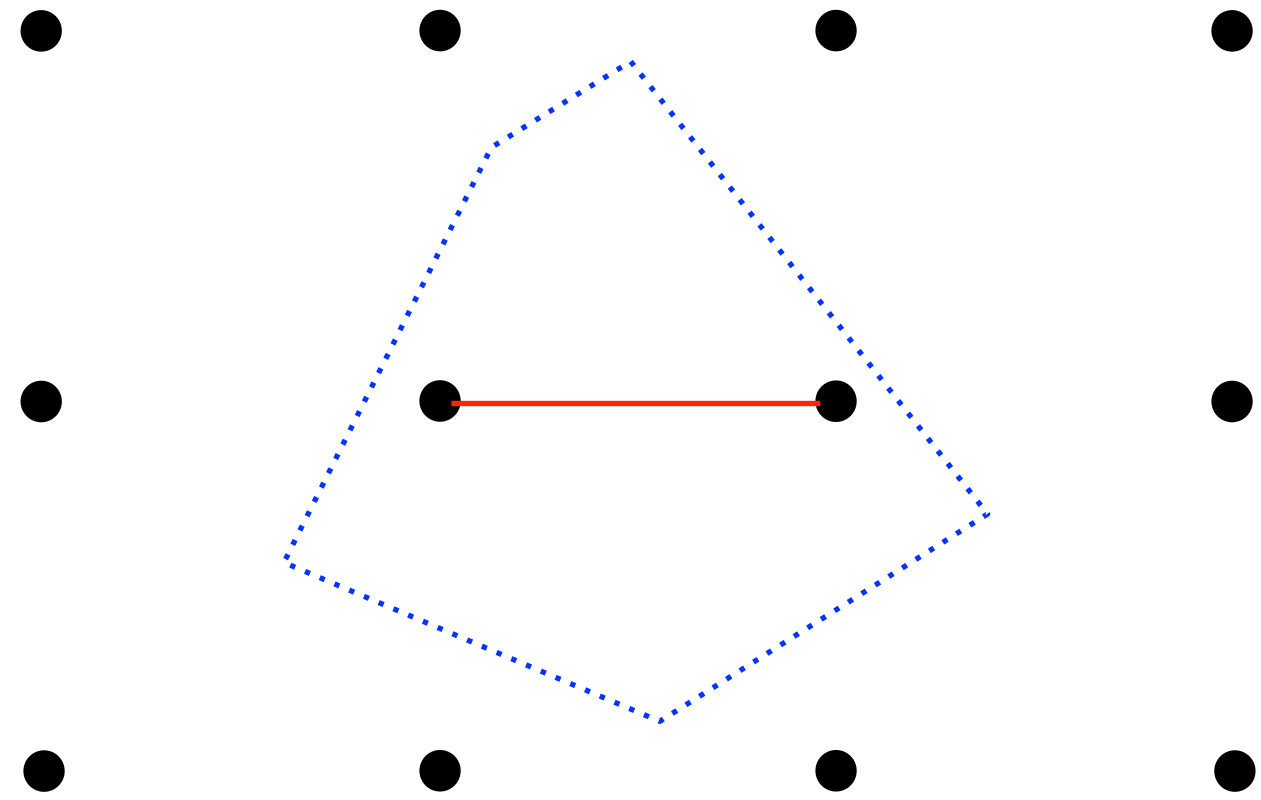

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Are binarizations stronger?

Example: A *single cut* in the extended formulation is stronger than any in the original space.

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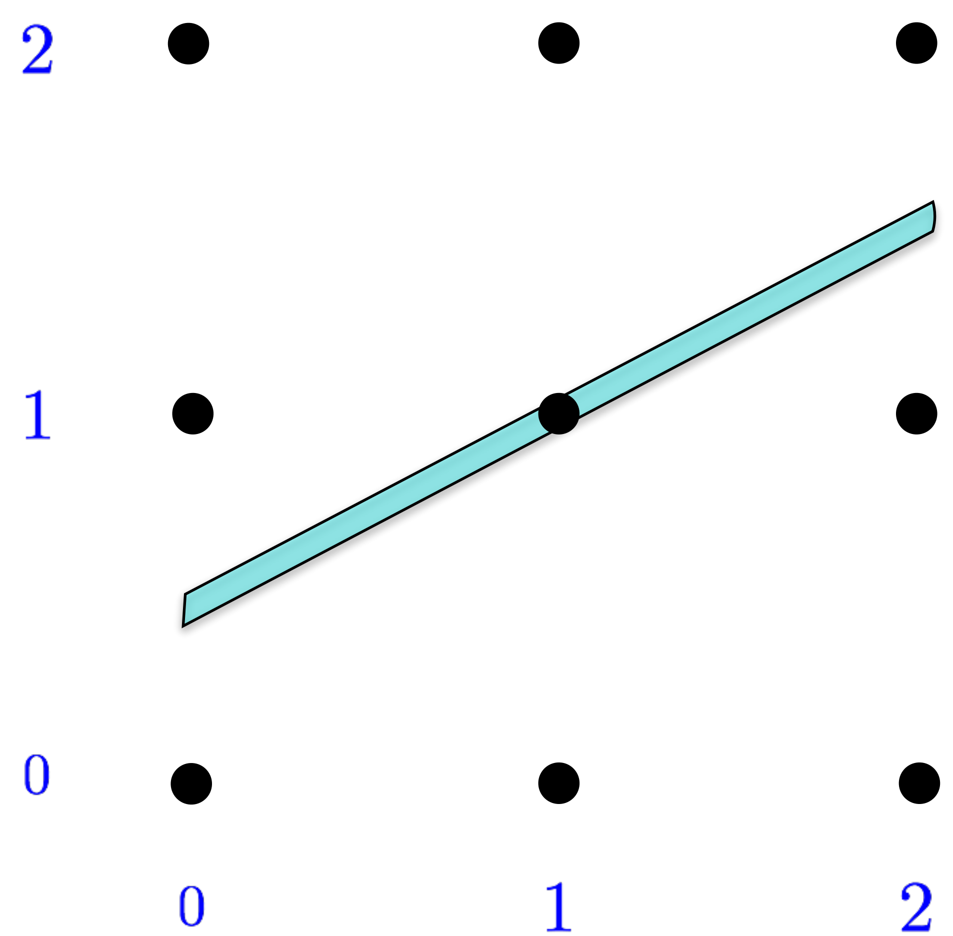
$$P = \{x \in [0,2]^2 : x_2 = \tfrac{1}{2}x_1 + \tfrac{1}{2}\}$$



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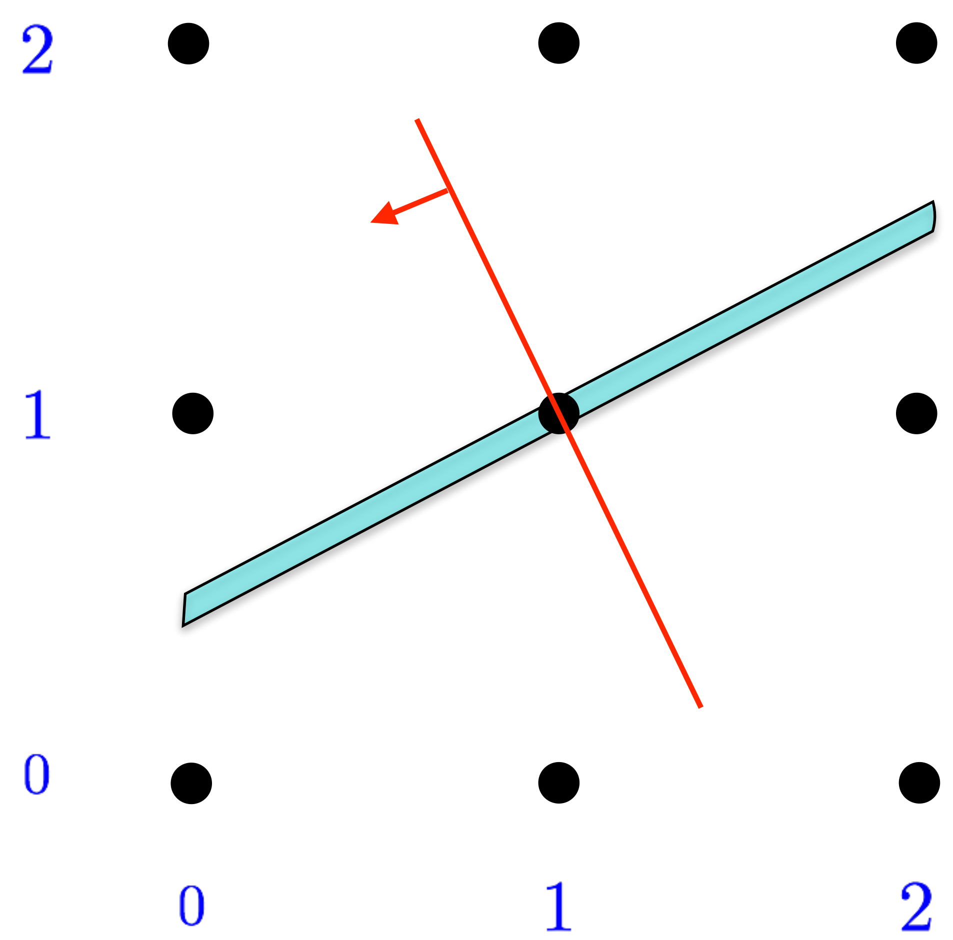

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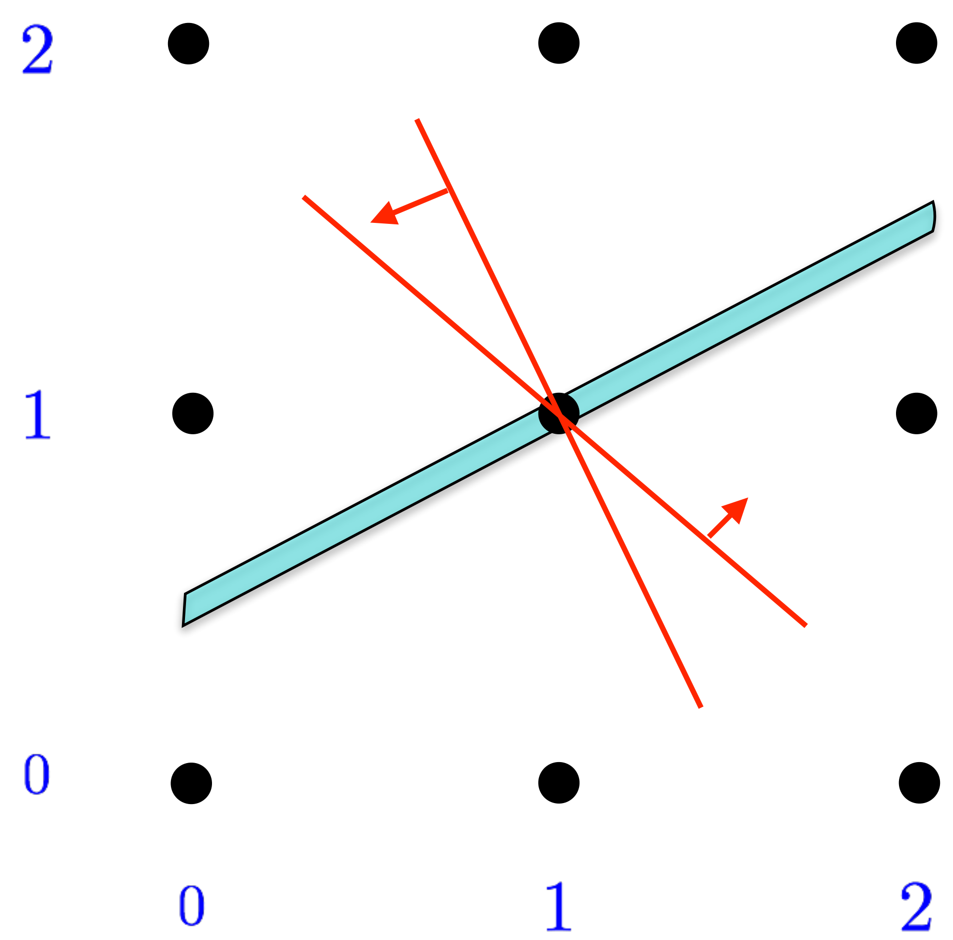

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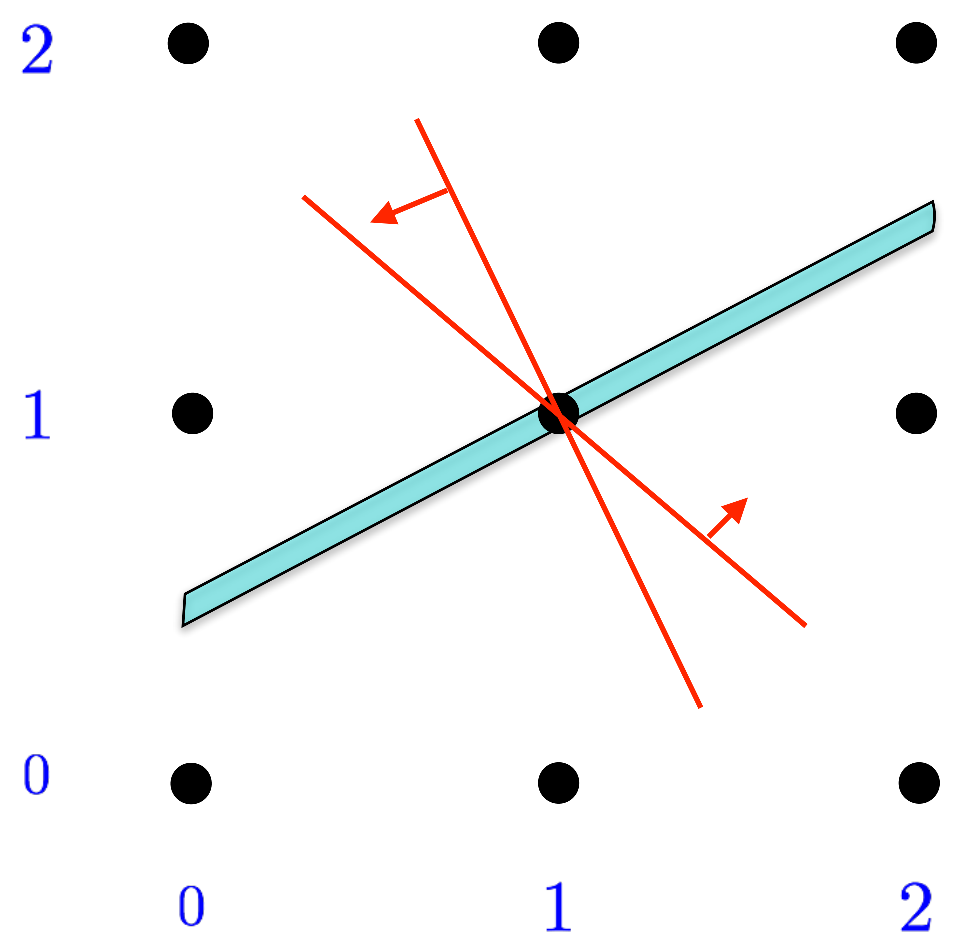

$$B_1 = \{(x_1, z^1) \in \mathbb{R} \times [0,1]^2 : x_1 = z_1^1 + 2z_2^1, z_1^1 + z_2^1 \le 1\}$$

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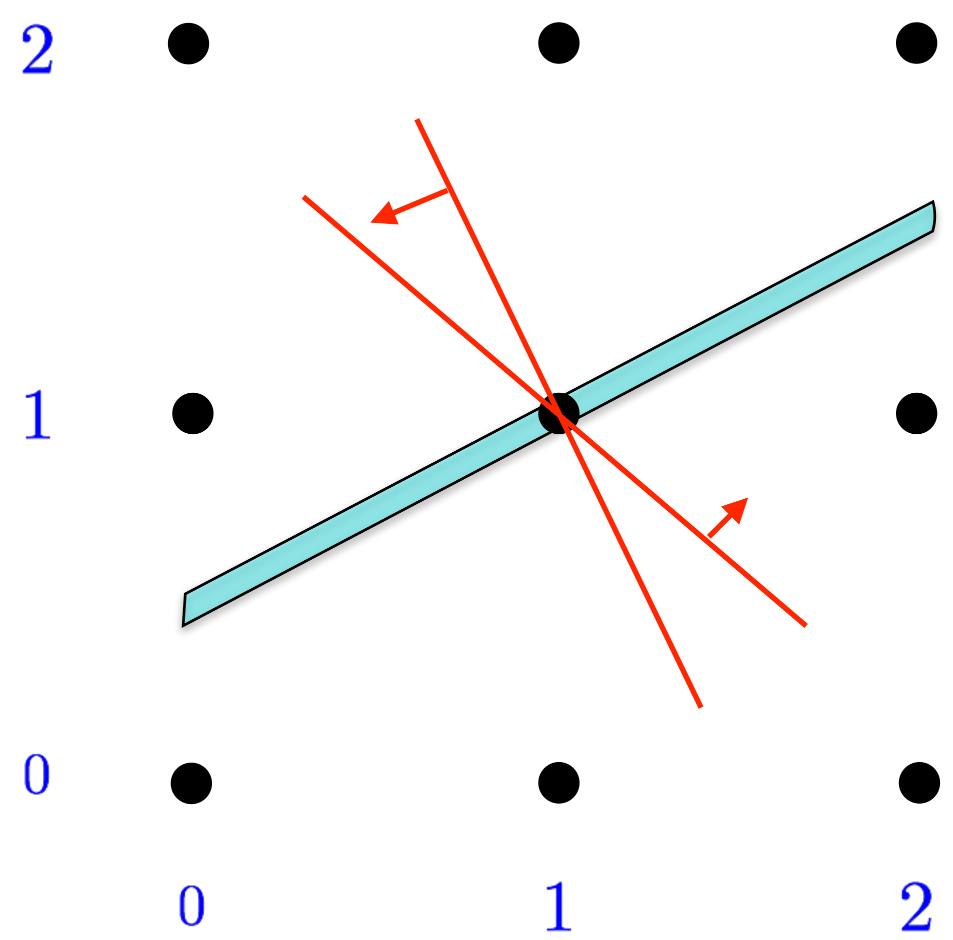

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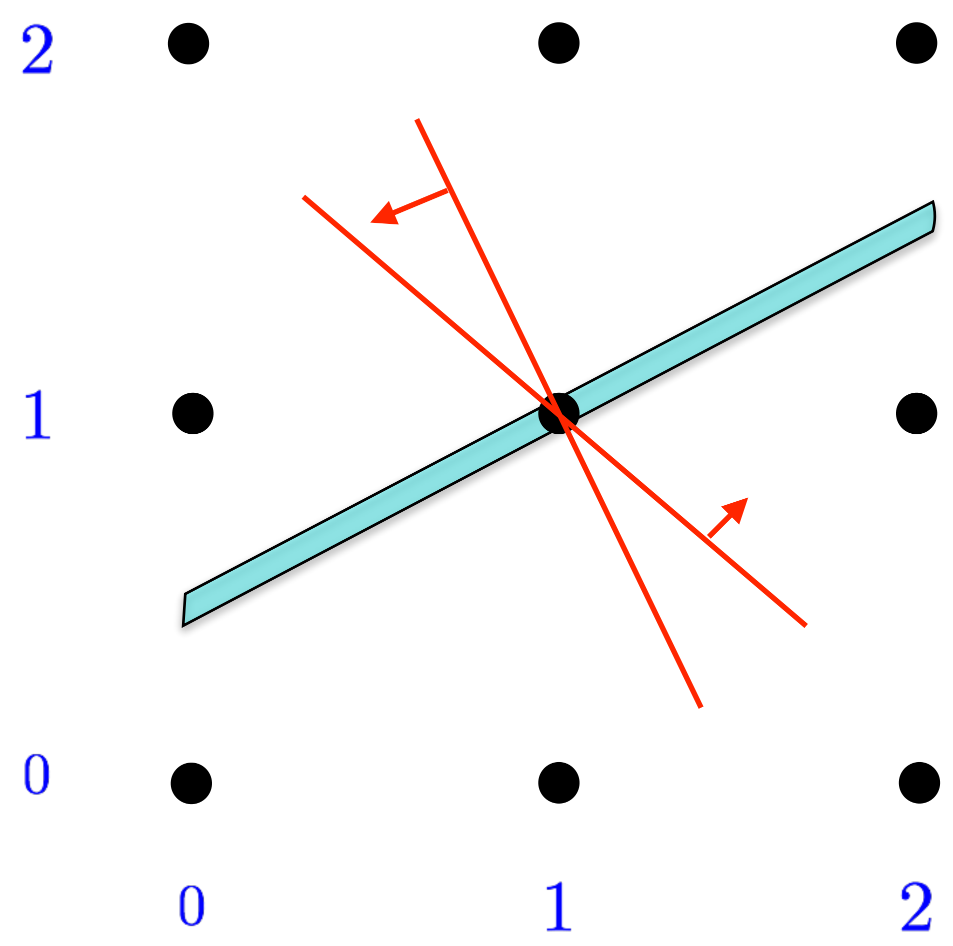

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$$P_{\mathcal{B}} = \{(x,z) : x \in P, (x,z) \in \mathcal{B}\}$$

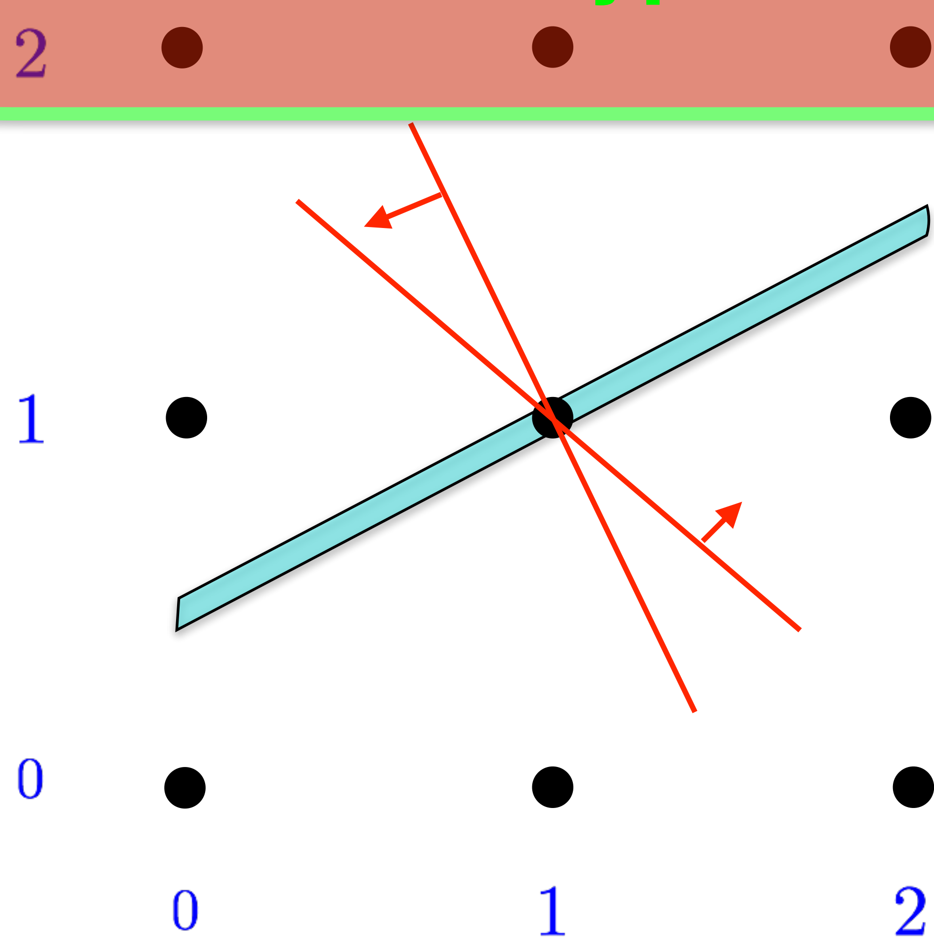
$$x_2 - z_2^1 \geq 1$$

$$\text{proj}_x(P_{\mathcal{B}}^+) = P^I$$

Example: A *single cut* in the extended formulation is stronger than any in the original space.

A SINGLE CUT can only be stronger when mixing variables of different types

$$P = \{x \in [0,2]^2 : x_2 = \tfrac{1}{2}x_1 + \tfrac{1}{2}\}$$



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$$x_2 + z_2^1 \ge 1$$

$$\mathrm{proj}_x(P_{\mathcal{B}}^+) = P^I$$

Proposition: Split cuts on variables *from one original variable* are not stronger.

Proposition 10 (Single variable extended splits dominated in original space)**.** *Let $P \subseteq \mathbb{R}^n$ and $I = [l]$ with $1 \leq l \leq n$. And suppose $0 \leq x_1 \leq u$ for all $x \in P$. Let $P_{\mathcal{B}} \subseteq \mathbb{R}^n \times \mathbb{R}^q$ be a binary extended formulation of P with $\mathcal{B} = (B_1, \ldots, B_l)$ and $B_1 \in \Gamma_u^{q_1}$. Let W be the associated binarization matrix with B_1, that is, $\{(i, w^i), i = 1, \ldots, u\} = B \cap (\mathbb{Z} \times \{0,1\}^{q_1}$. Let $S \subseteq \mathbb{R}^{n+q}$ on only the variables z^1, that is*

$$S = \{(x, z) : \pi_0 < \pi^T z^1 < \pi_0 + 1\}.$$

for some integral vector $\pi \in \mathbb{Z}^{q_1}$ and integer $\pi_0 \in \mathbb{Z}$.

Then there exists a split $S' \subseteq \mathbb{R}^n$ given by

$$S' = \{x : \pi_0' < x_1 < \pi_0' + 1\}$$

for some integer π_0', such that

$$\operatorname{proj}_x(\operatorname{conv}(P_{\mathcal{B}} \setminus S)) \supseteq \operatorname{conv}(P \setminus S').$$

Proposition: Split cuts on variables *from one original variable* are not stronger.

$$B_1 = \{(x_1, z^1) \in \mathbb{R} \times [0,1]^2 : x_1 = z^1_1 + 2z^1_2,\ z^1_1 + z^1_2 \leq 1\}$$

$$B_2 = \{(x_2, z^2) \in \mathbb{R} \times [0,1]^2 : x_2 = z^2_1 + 2z^2_2,\ z^2_1 + z^2_2 \leq 1\}$$

$$\mathcal{B} = B_1 \times B_2$$

$$P_{\mathcal{B}} = \{(x, z) : x \in P, (x, z) \in \mathcal{B}\}$$

If $\alpha^\top z^1 \leq \alpha_0$ is a split cut for $P_{\mathcal{B}}$, then there exists a dominating inequality $\beta^\top x \leq \beta_0$ that is a split cut for P, i.e.,

$$P^+ \subseteq \text{proj}_x(P^+_{\mathcal{B}})$$

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$$P = \left\{(x, y) \in \mathbb{R}^3 \times [0,1]^3 \;:\; \sum_{i=1}^3 x_i = 3,\; x_i \le 3y_i,\; 0 \le x_i \le 2, \;\; \text{for } i = 1,2,3\right\},$$

$$B_i = \{(x_i, z_i) \in \mathbb{R} \times [0,1]^2 \;:\; x_i = z_{i1} + 2z_{i2}\} \text{ for } i = 1,2,3,$$

$$P_{\mathcal{B}_{LG}} = \left\{(x, y, z) \in \mathbb{R}^{3+3+6} \;:\; (x, y) \in P, (x_i, z_i) \in B_i \;\; \text{for } i = 1,2,3\right\}.$$

Lemma 17. *The point* $\bar{p} = (\bar{x}, \bar{y}) = [(1,1,1), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})]$ *belongs to the split closure of* P.

Lemma 18. *The inequality* $y_1 + y_2 + y_3 \ge 2$ *is a valid inequality for the split closure of* $P_{\mathcal{B}}$ *and therefore the point* $\bar{p}$ *defined in Lemma 17 is not contained in* $\text{proj}_{x,y} SC(P_{\mathcal{B}_{LG}})$.

Comparing Split Closures

Given binarizations $\mathcal{B}_1, \mathcal{B}_2$,

$\mathcal{B}_1$ is *better* than $\mathcal{B}_2$ if...

1. $\text{proj}_x(SC(P_{\mathcal{B}_1})) \subseteq \text{proj}_x(SC(P_{\mathcal{B}_2}))$
2. $\mathcal{B}_1 \subseteq \mathcal{B}_2$
3. $f(\mathcal{B}_1) \subseteq \mathcal{B}_2$
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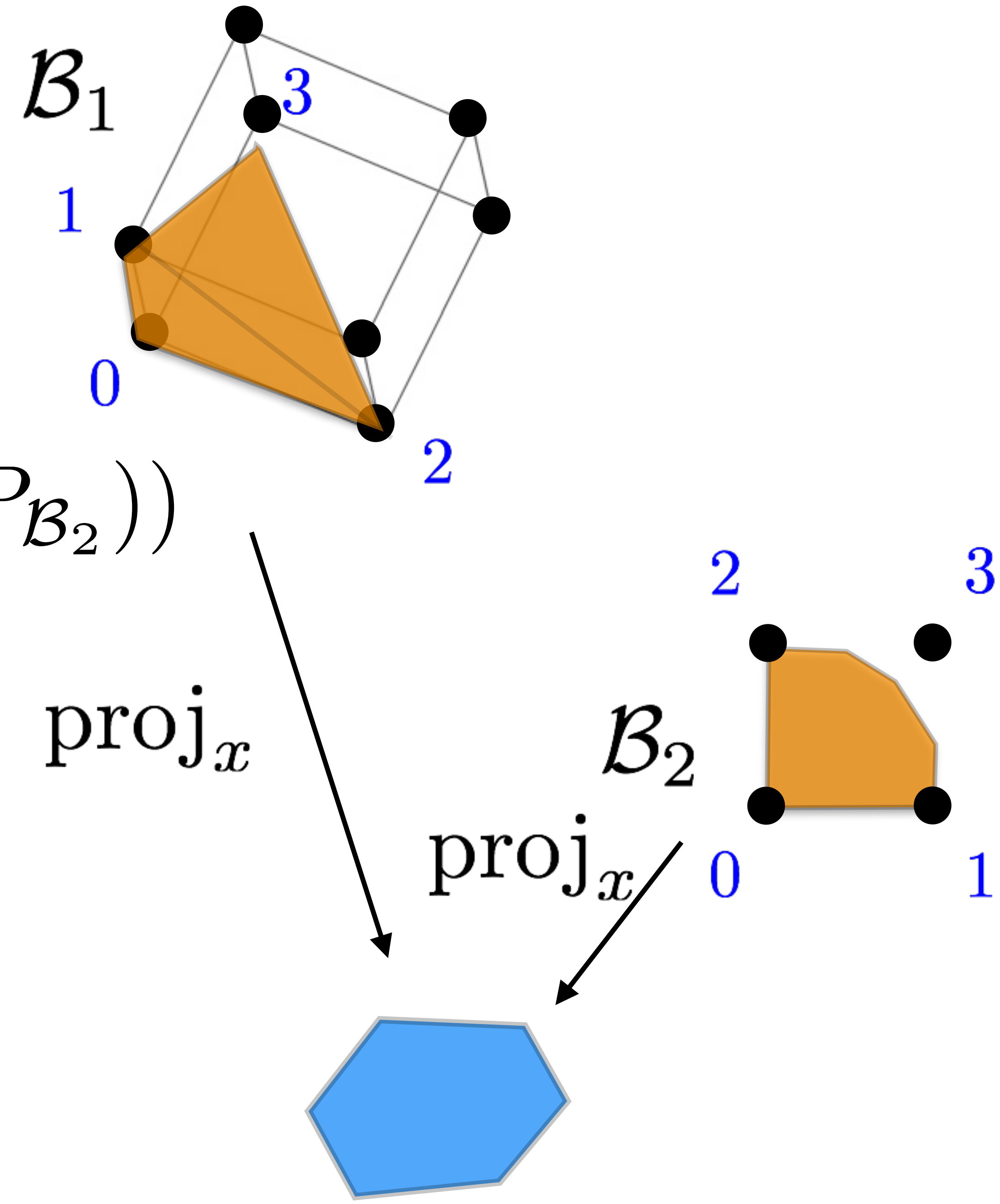
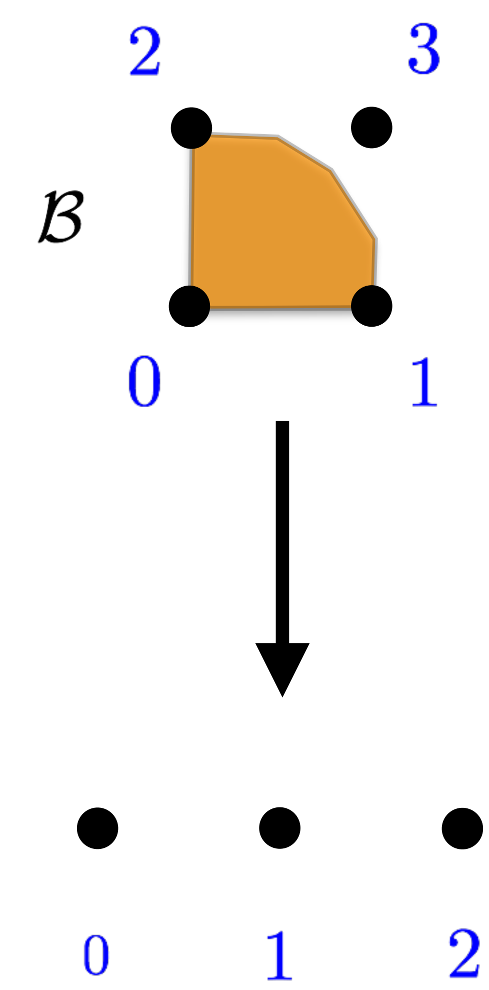


Binarization

- A binarization is *exact* if $\text{proj}_x : \mathcal{B} \cap (\mathbb{R}^n \times \{0,1\}^q \to U \cap \mathbb{Z}^n$ is a bijection $\mathcal{B} \cap (\mathbb{R}^n \times \{0,1\}^q$
- A binarization is *perfect* if it is exact and $\mathcal{B} = \text{conv}(\mathcal{B} \cap (\mathbb{R}^n \times \{0,1\}^q))$



- $P = \{x \in \mathbb{R}^n : Ax \leq b,\ 0 \leq x \leq u\}$.
- $P^I = P \cap \mathbb{Z}^n$
- $U = \{x \in \mathbb{R}^n : 0 \leq x \leq u\}$.
- A *valid binarization* for U is a polytope $\mathcal{B} \subseteq \mathbb{R}^n \times [0,1]^q$ such that

  $U \cap \mathbb{Z}^n \subseteq \text{proj}_x(\mathcal{B} \cap (\mathbb{R}^n \times \{0,1\}^q))$.
- A *coordinate-wise binarization* of U is a polytope $\mathcal{B} = B^1 \times \ldots \times B^n$ where B^i is a binarization of $[0, u_i]$.

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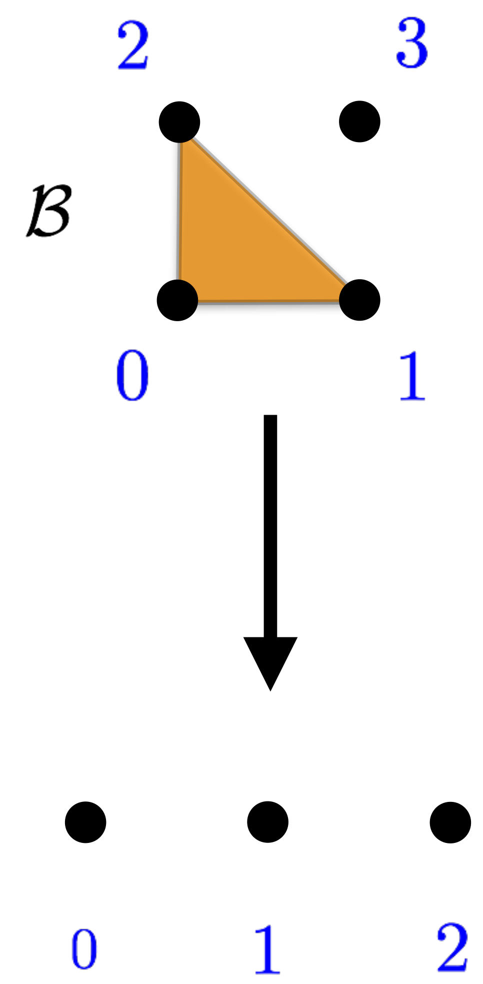

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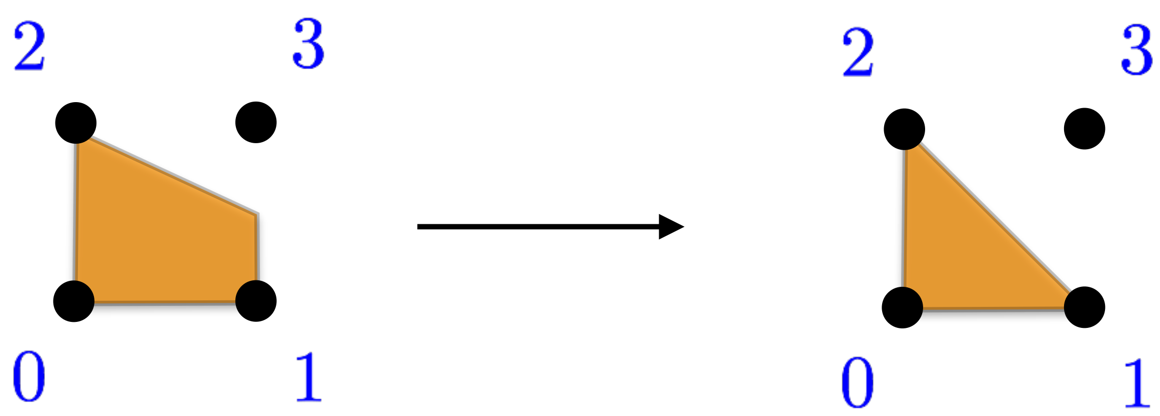
Log can be made perfect



$$0 \leq x \leq 2$$

Log

$$x = \sum_{j=0}^{\lfloor \log(u+1) \rfloor} 2^j z_j$$

$$z \in \{0,1\}^{\lfloor \log(u+1) \rfloor}$$

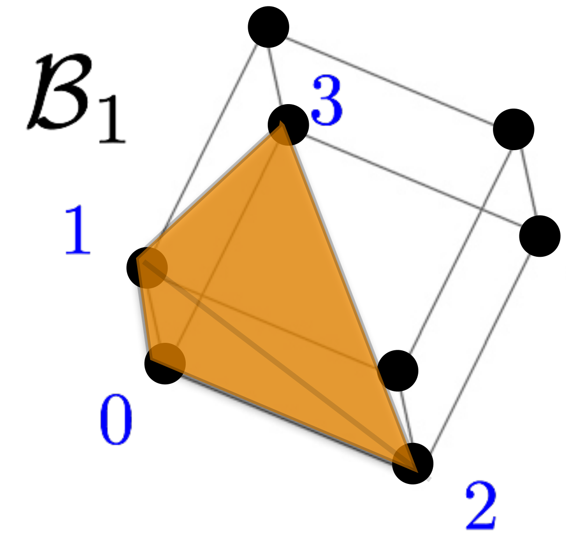
Theorem:
Let $B = \{(x,z) : x = \sum_{j=0}^{\lfloor \log(u+1) \rfloor} 2^j z_j,\ 0 \leq z \leq 1,\ 0 \leq x \leq u\}$.

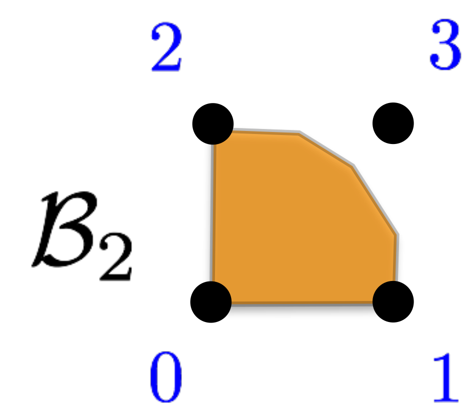
Then $B^+ := \text{conv}(B \cap \mathbb{Z} \times \{0,1\}^q) = \{(x,z) \in B : {a_i}^\top z \leq 1,\ \ i \in J\}$

with $|J| \leq q$.

Given binarizations $\mathcal{B}_1, \mathcal{B}_2$,

$\mathcal{B}_1$ is *better* than $\mathcal{B}_2$ if...





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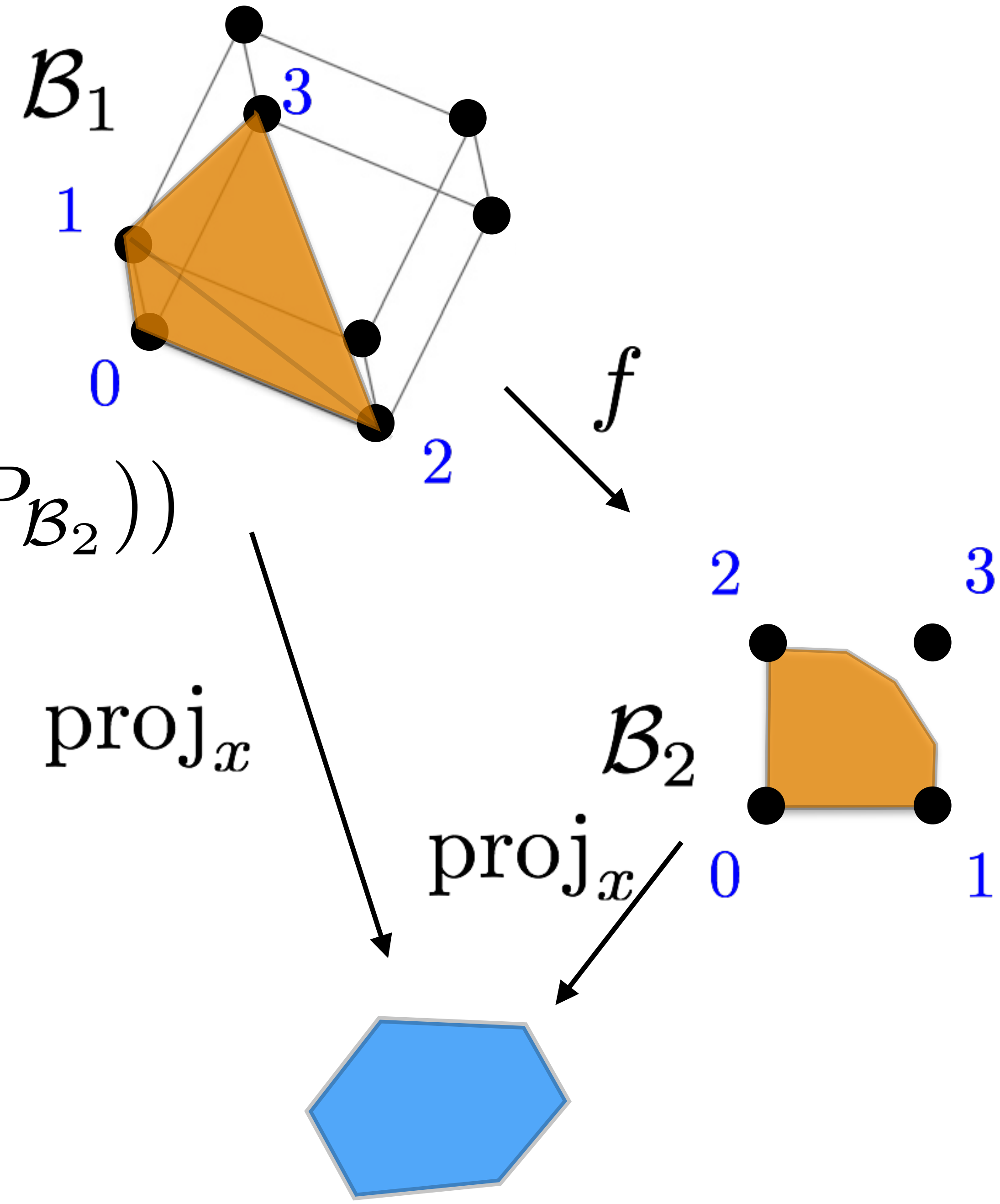

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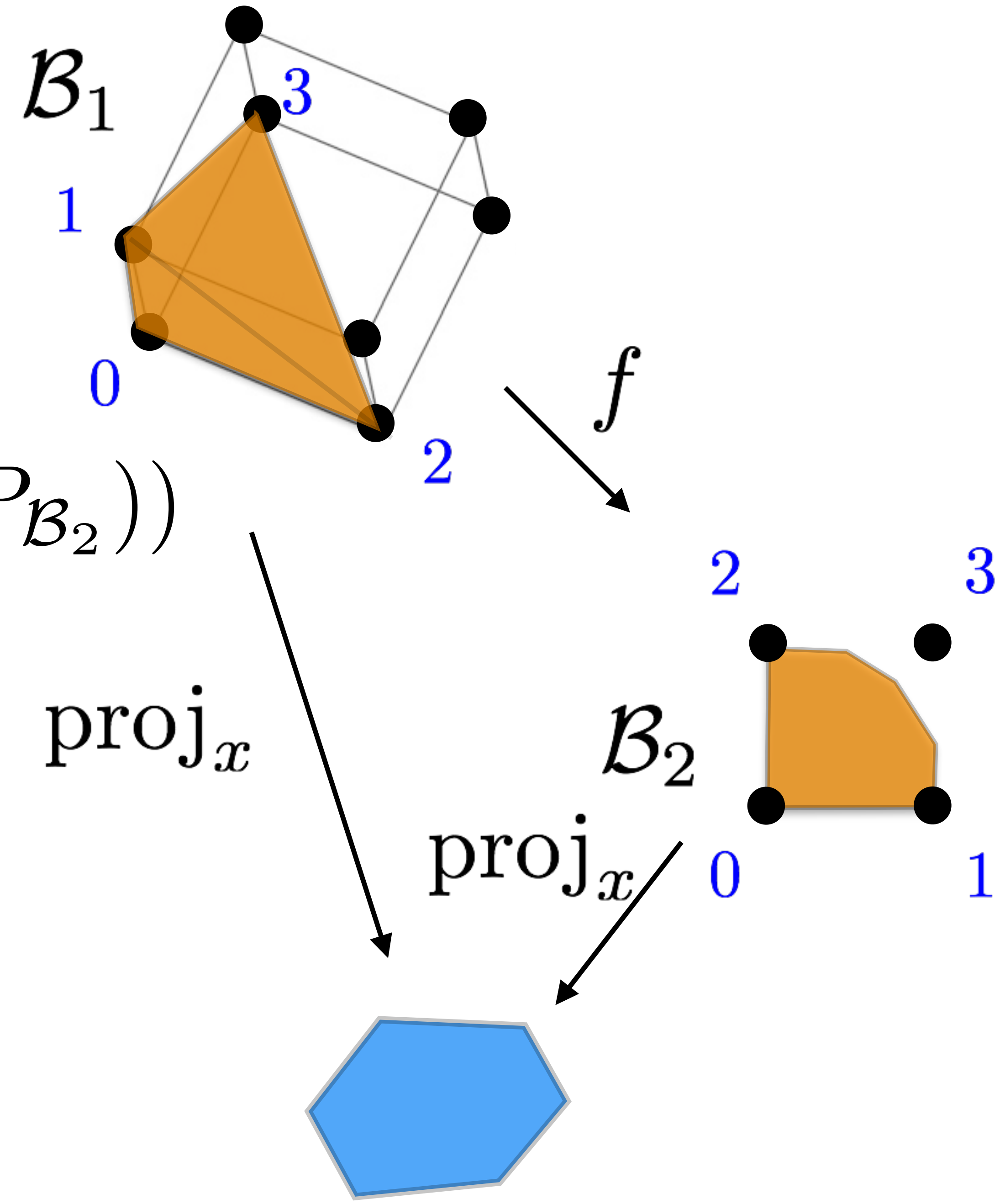

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Lemma: If $f(x, z)$ is an integral affine transformation on z and identity on x, and $f(\mathcal{B}_1) \subseteq \mathcal{B}_2$, then

$f(SC(\mathcal{B}_1)) \subseteq SC(\mathcal{B}_2)$

and

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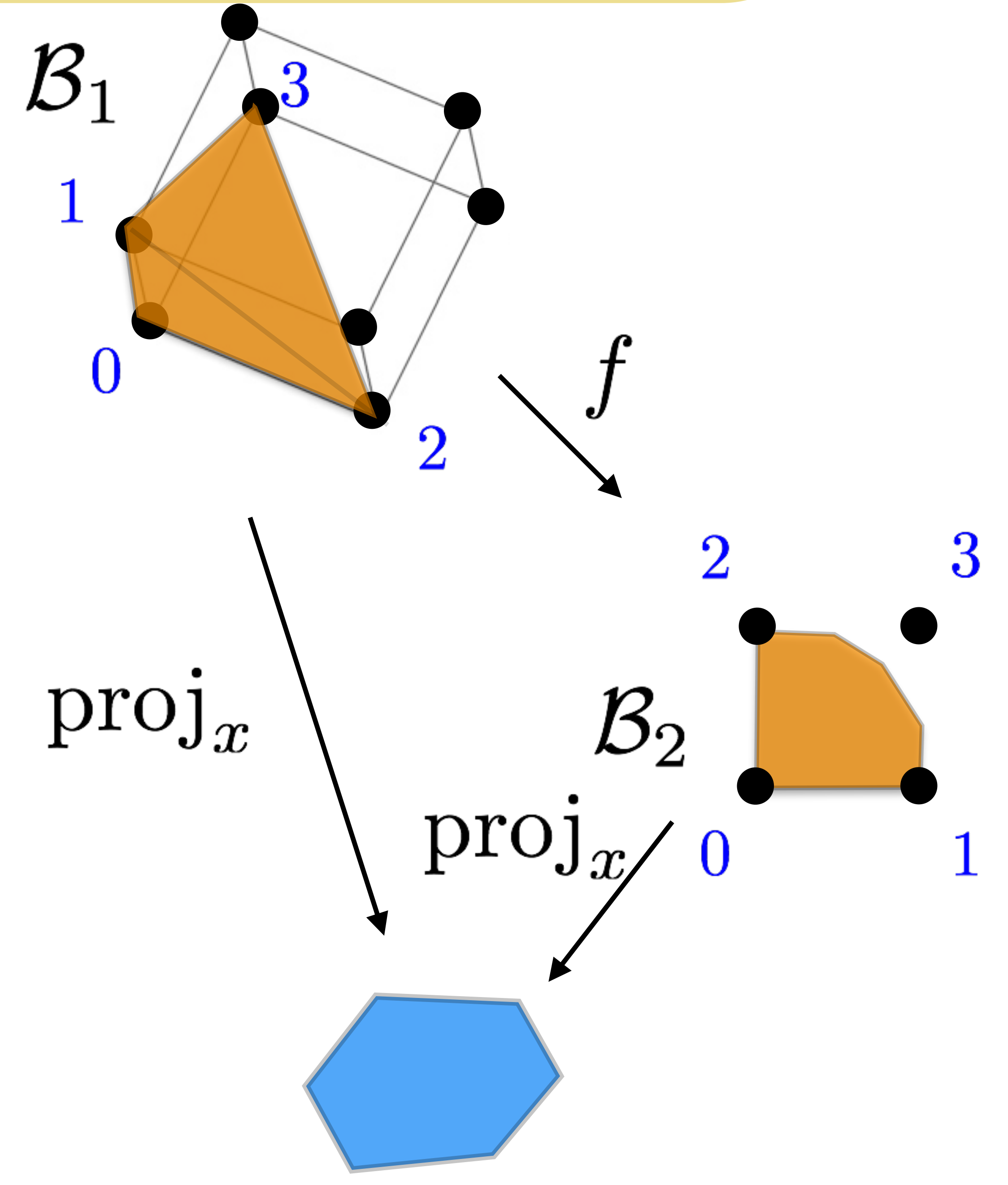

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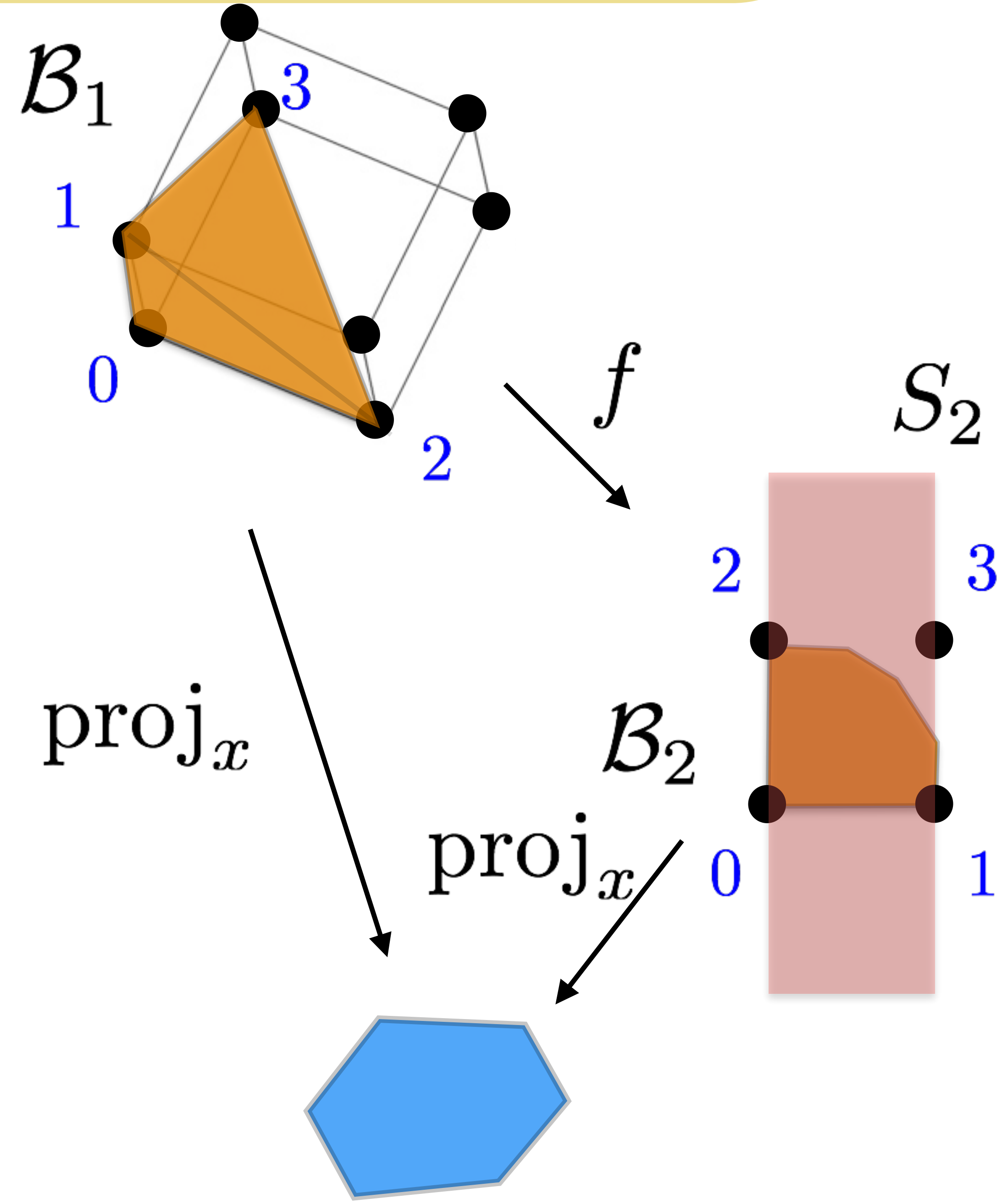
Proof.

- Let S_2 be a split for $\mathcal{B}_2$.



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- Let S_2 be a split for $\mathcal{B}_2$.
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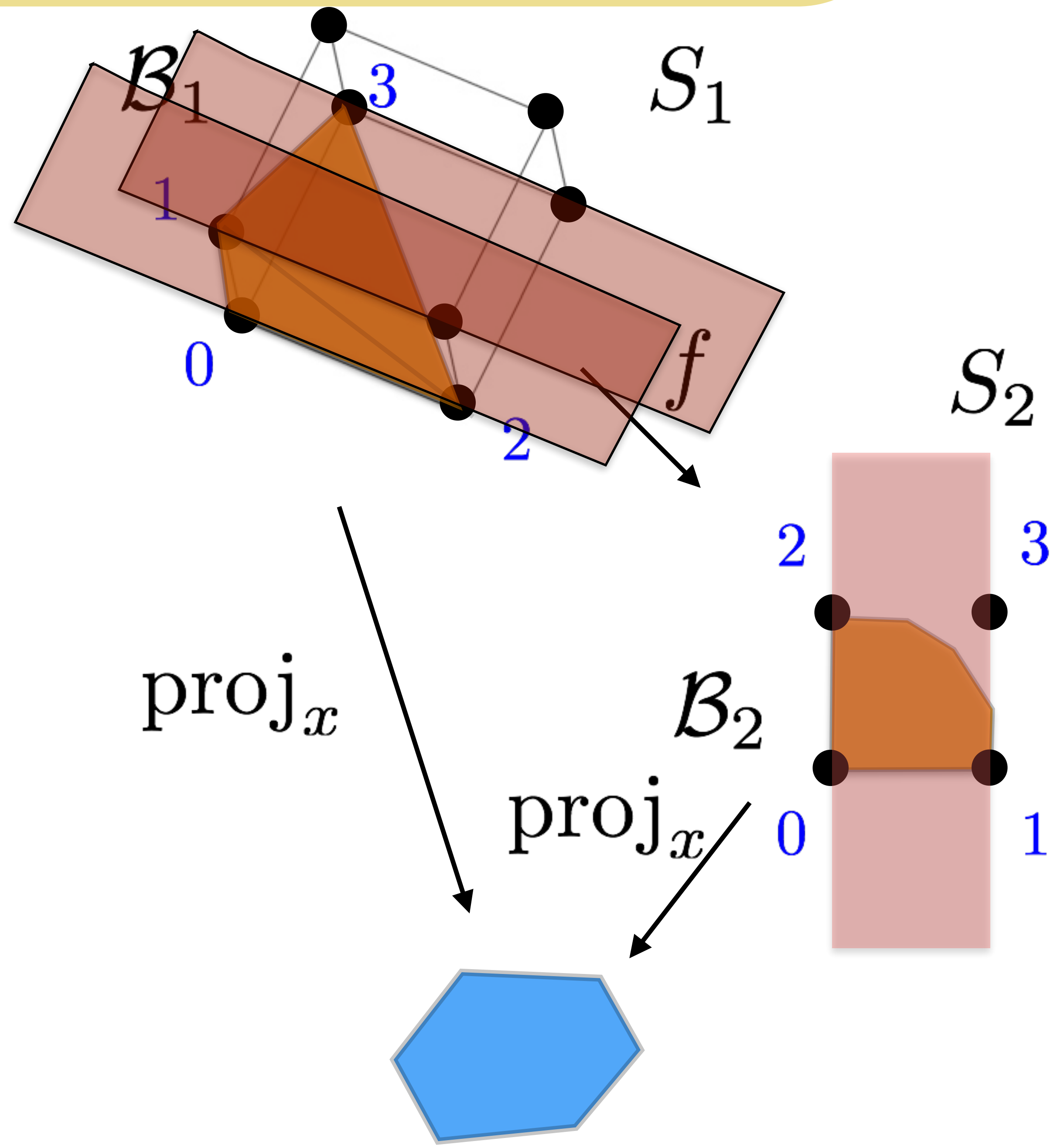

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- Let S_2 be a split for $\mathcal{B}_2$.
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- Show $f(\text{conv}(P_{\mathcal{B}_1} \setminus S_1)) \subseteq \text{conv}(P_{\mathcal{B}_2} \setminus S_2)$
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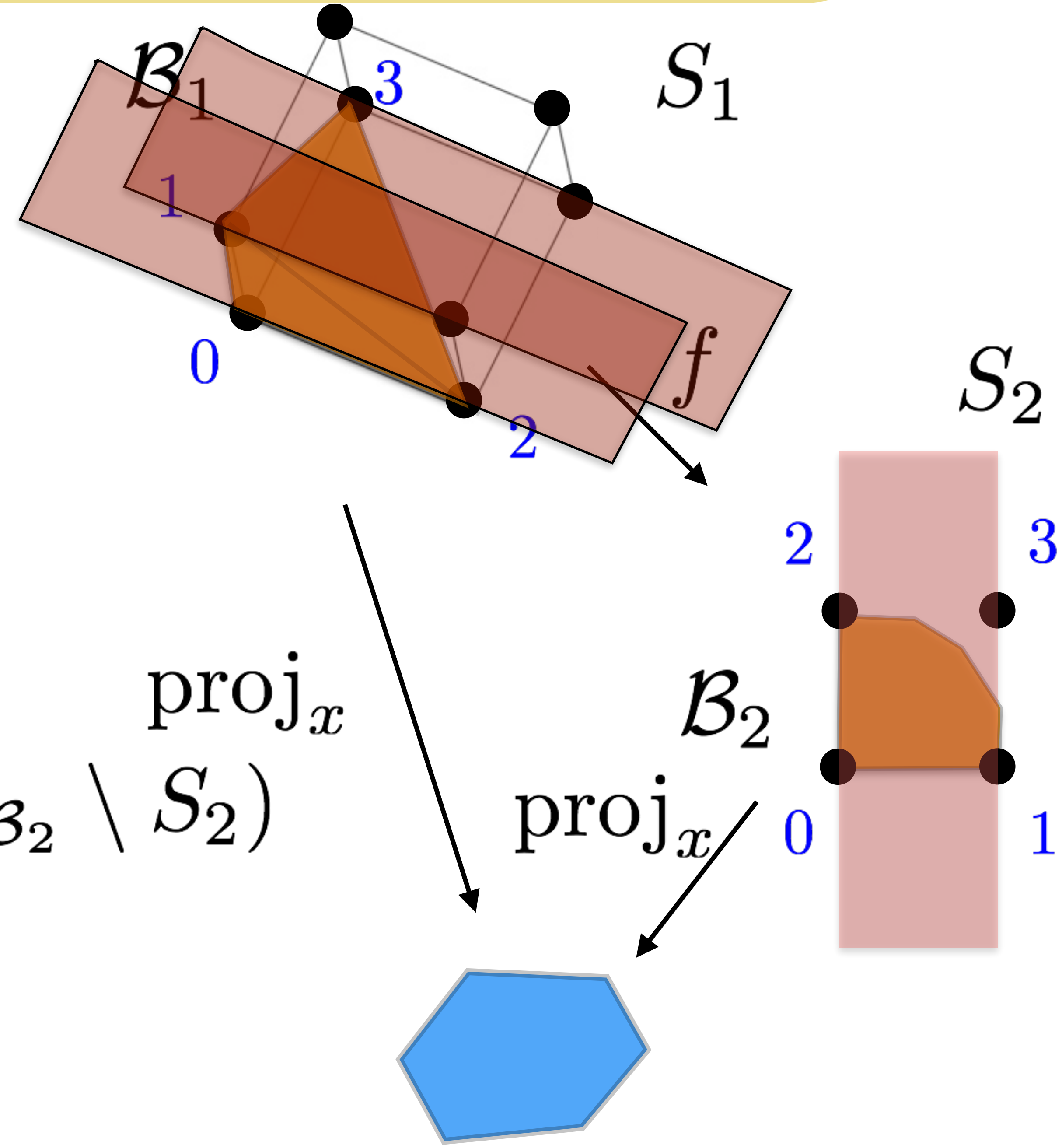

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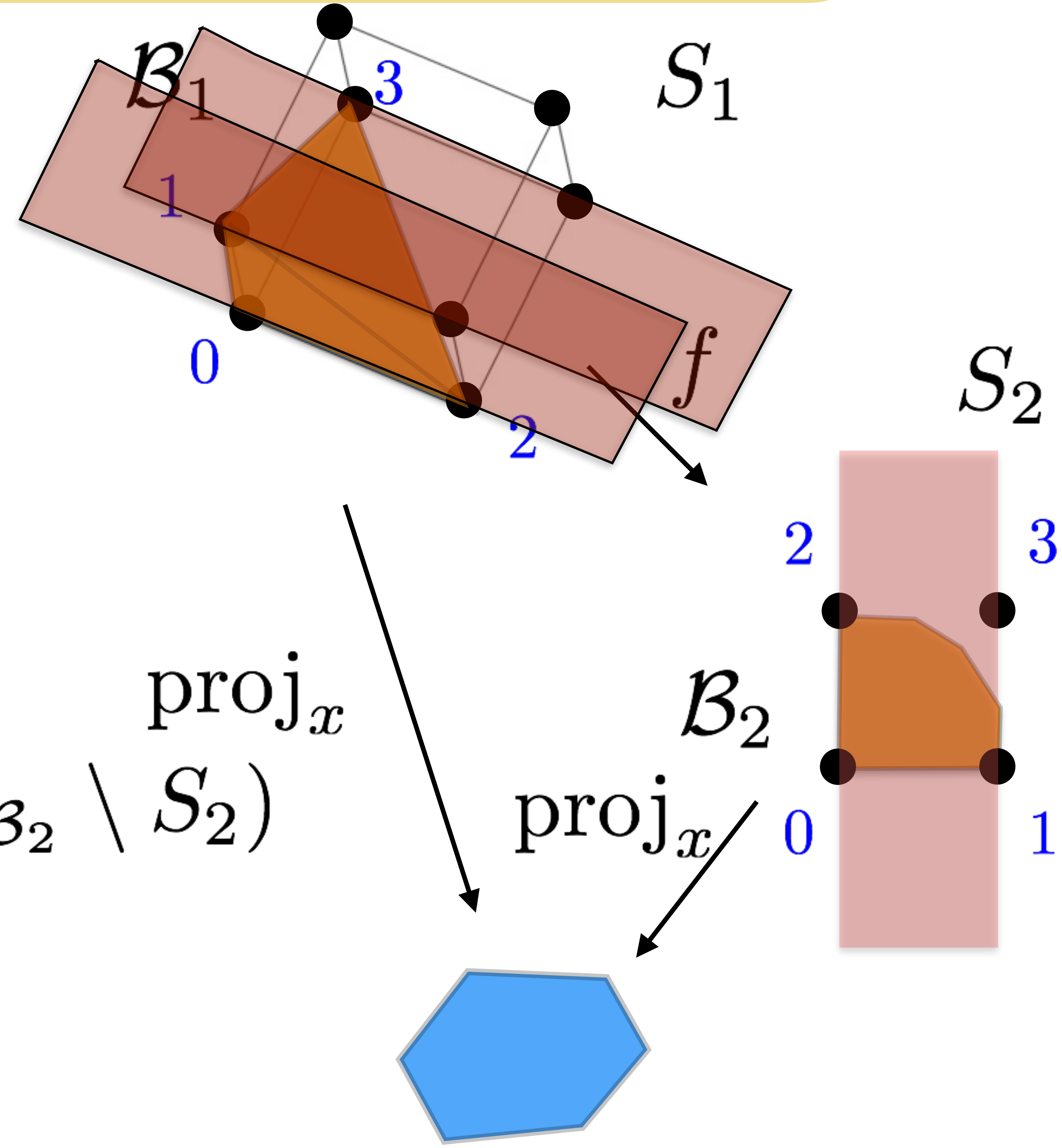

Unary

$$x = \sum_{j=1}^{u} z_j$$

$$z_1 \geq z_2 \geq \ldots \geq z_u$$

$$z \in \{0,1\}^u$$

Full

$$x = \sum_{j=1}^{u} j z_j$$

$$z_1 + z_2 + \cdots + z_u \leq 1$$

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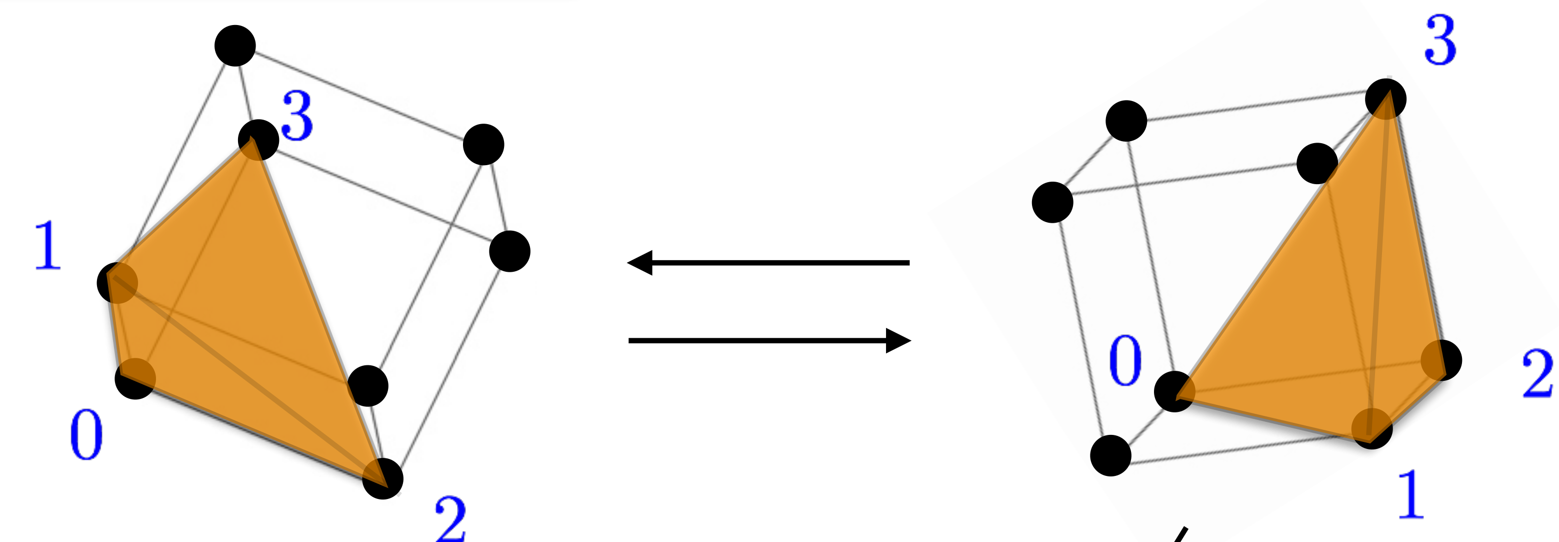
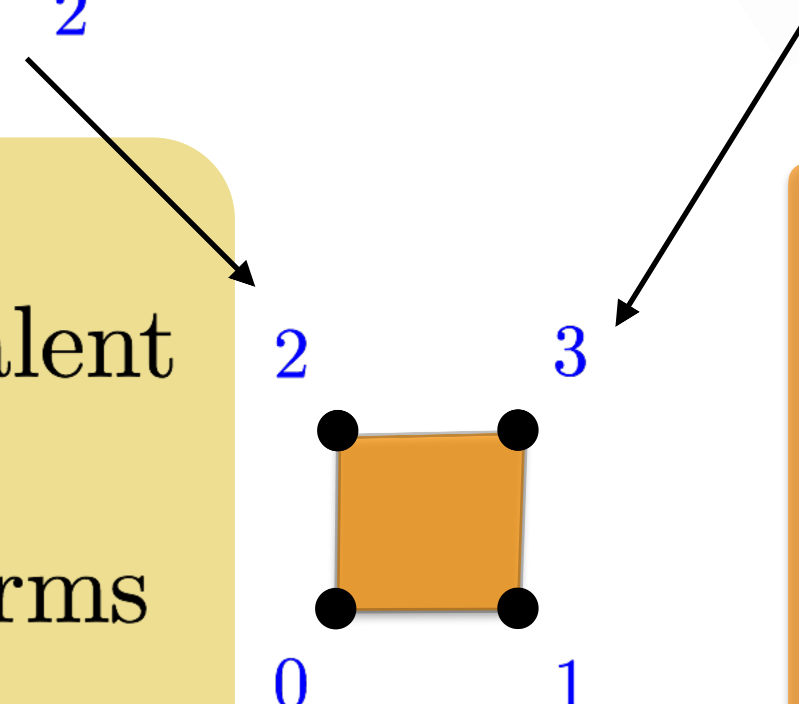


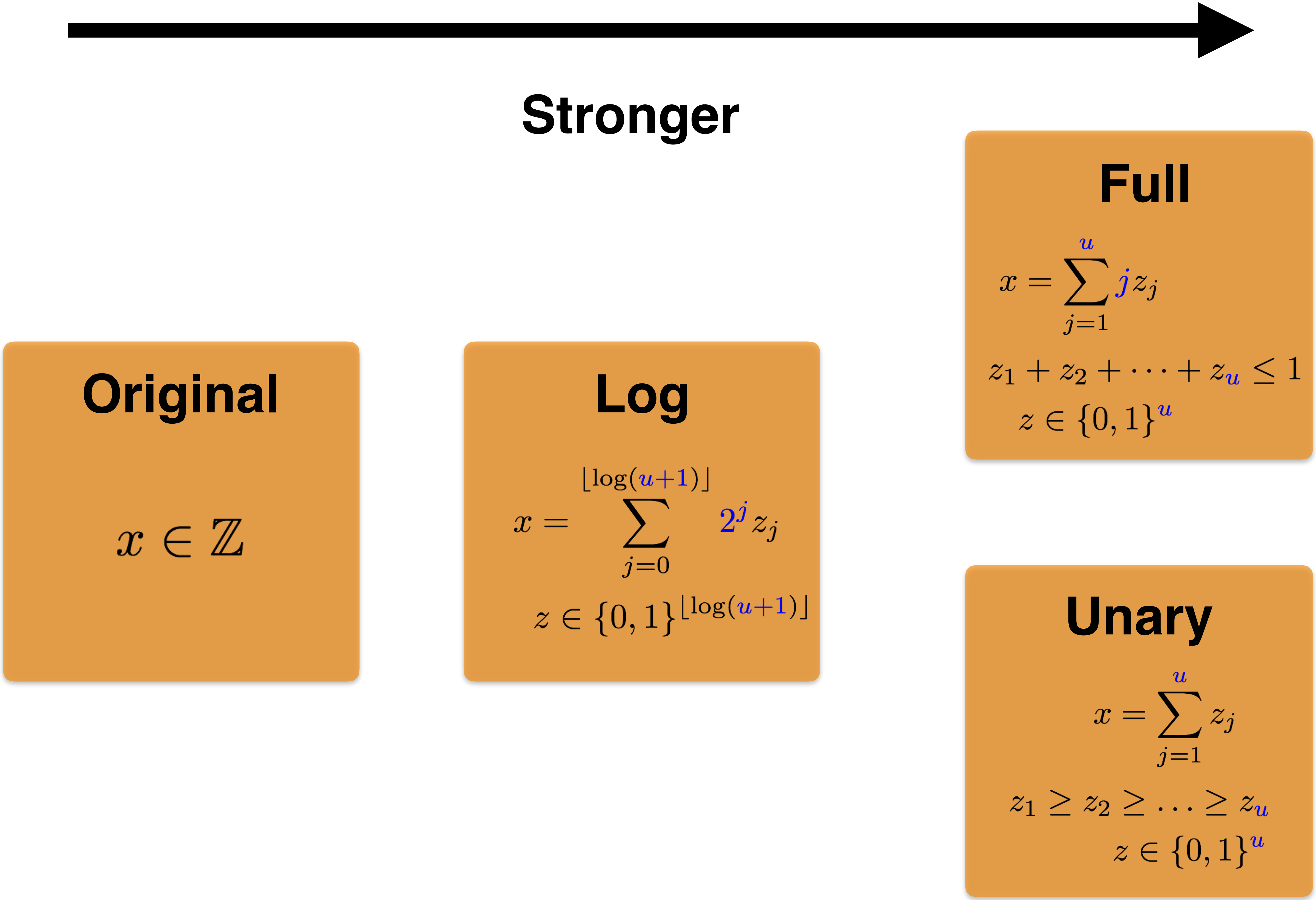


Log

$$x = \sum_{j=0}^{\lfloor \log(u+1) \rfloor} 2^j z_j$$

$$z \in \{0,1\}^{\lfloor \log(u+1) \rfloor}$$

Theorem:
Unary and Full are equivalent and dominate all other coordinate-wise bin. in terms of split closure.



Stronger
Original
$x \in \mathbb{Z}$
Log
$x = \sum_{j=0}^{\lfloor \log(u+1) \rfloor} 2^j z_j$
$z \in \{0,1\}^{\lfloor \log(u+1) \rfloor}$
Full
$x = \sum_{j=1}^{u} j z_j$
$z_1 + z_2 + \cdots + z_u \leq 1$
$z \in \{0,1\}^u$
Unary
$x = \sum_{j=1}^{u} z_j$
$z_1 \geq z_2 \geq \ldots \geq z_u$
$z \in \{0,1\}^u$

Example

Angulo, Van Vyve '17:
Fixed Charged Problem

$$
\begin{aligned}
\min \quad & \sum_{ij} q_{ij} y_{ij} \\
& \sum_i x_{ij} = d_j \quad \forall j \\
& \sum_j x_{ij} \le c_i \quad \forall i \\
& y_{ij} \le x_{ij} \le a_{ij} y_{ij} \quad \forall i < j \\
& x_{ij} \in \mathbb{Z}, y_{ij} \in \{0,1\} \quad \forall i < j
\end{aligned}
$$

Geometric mean of running times for 5 instances with CPLEX 12.6.1
(600 sec time limit)

Before Substitution:

Problem $n, \max\{c_i\}$	P	F(P)	U(P)	AvV
30, 10	4.9	12.0	23.0	10.1
40, 10	75.2	77.0	261.5	48.1
40, 20	469.0	471.1	600	434.1

After Substitution:

Problem $n, \max\{c_i\}$	P	F(P)	U(P)	AvV
30, 10	4.9	2.5	600	0.9
40, 10	75.2	7.1	600	2.4
40, 20	469.0	23.8	600	6.3

*Computational analysis by Bonami, Dash, Lodi, Tramontani

Branching

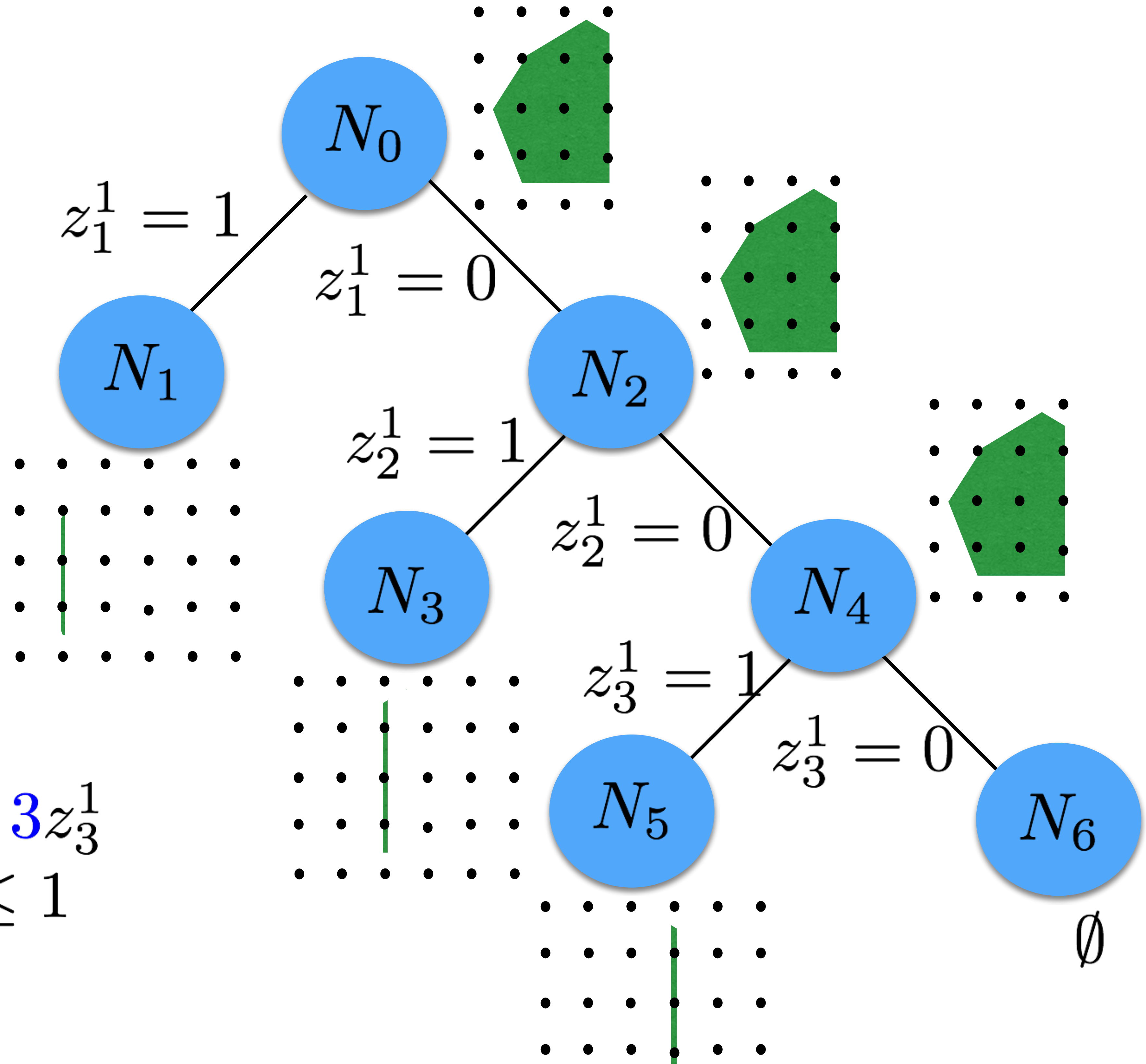
Branching on binary variables

Full

$$x = \sum_{j=1}^{u} j z_j$$

$$z_1 + z_2 + \cdots + z_u \le 1$$

$$z \in \{0,1\}^u$$



$$0 \le x_1 \le 3$$

$$x_1 = z_1^1 + 2z_2^1 + 3z_3^1$$

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Theorem: [Owen–Mehrotra '02] (paraphrased) Unless the top coefficient variable is branched on, the domain of the branching tree is unchanged.

Log

$$x = \sum_{j=0}^{\lfloor \log(u+1) \rfloor} 2^j z_j$$

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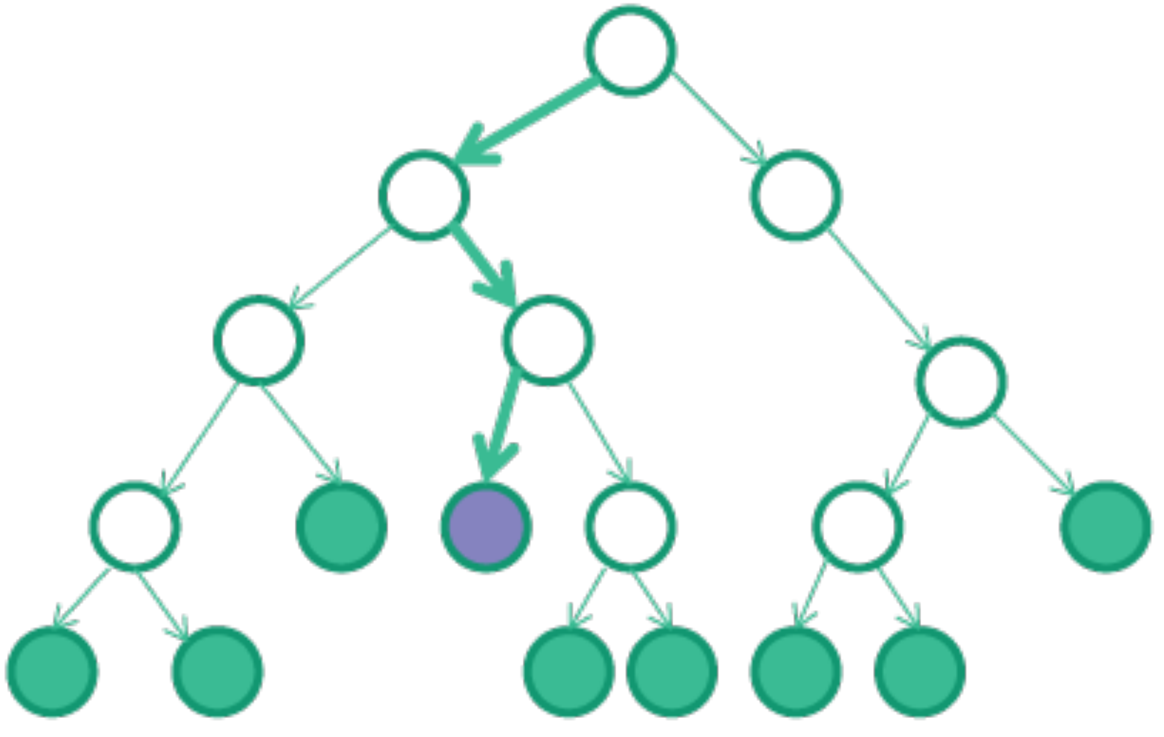
"remodeling of mixed-integer programs by binary variables should be avoided in practice unless special techniques are used to handle these variables."

Branch and Bound



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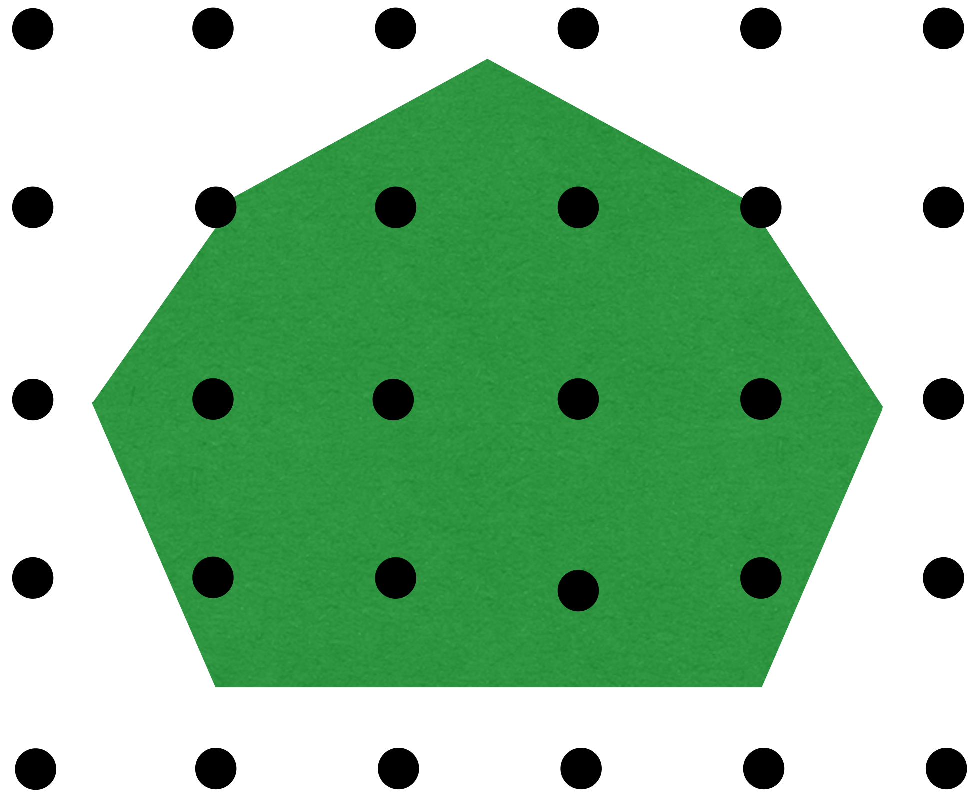
Branching



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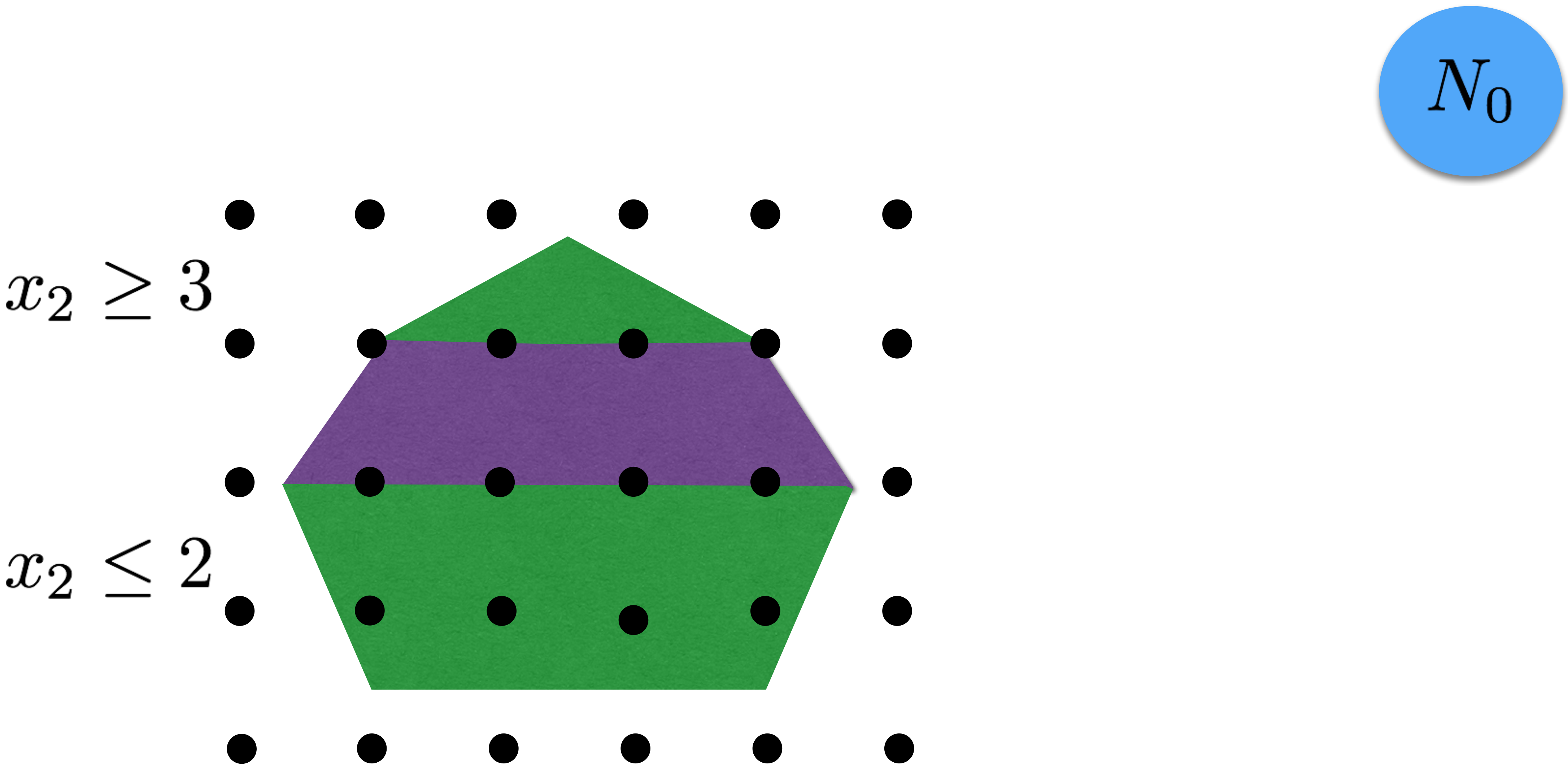

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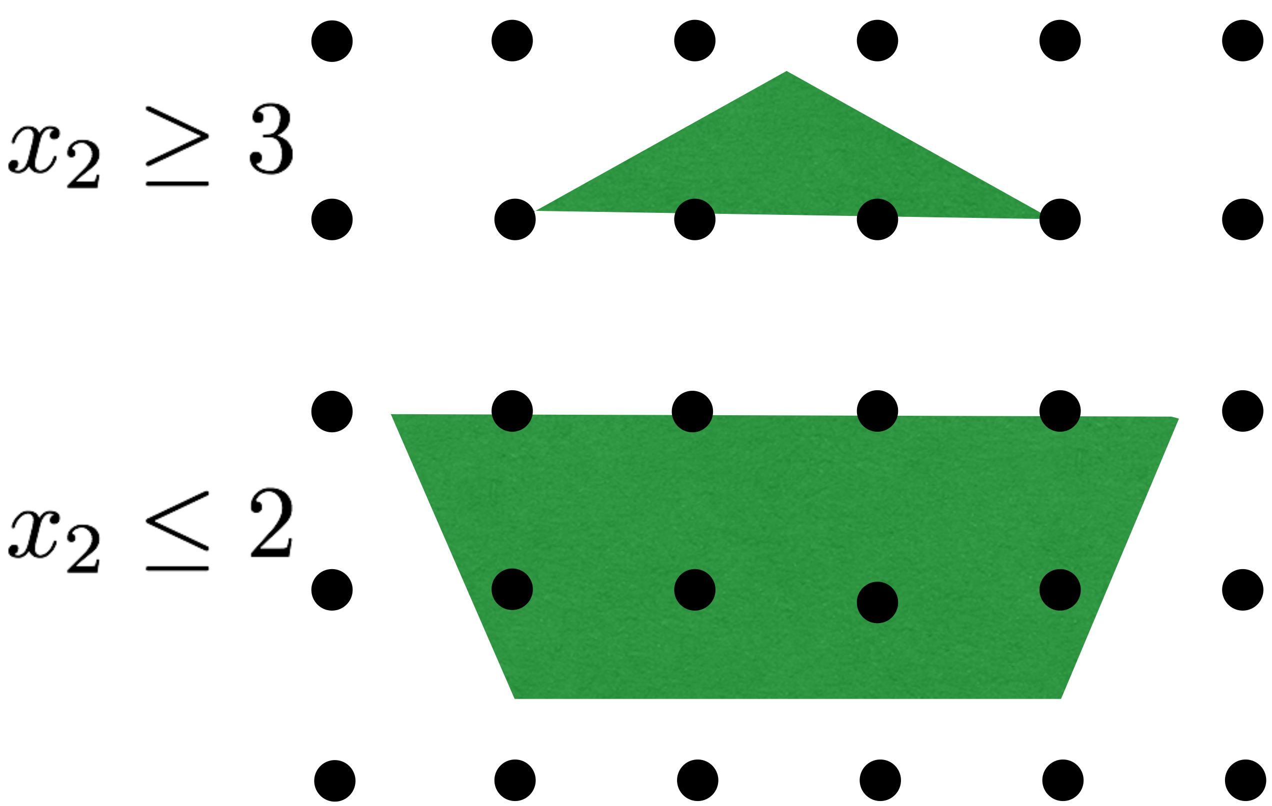

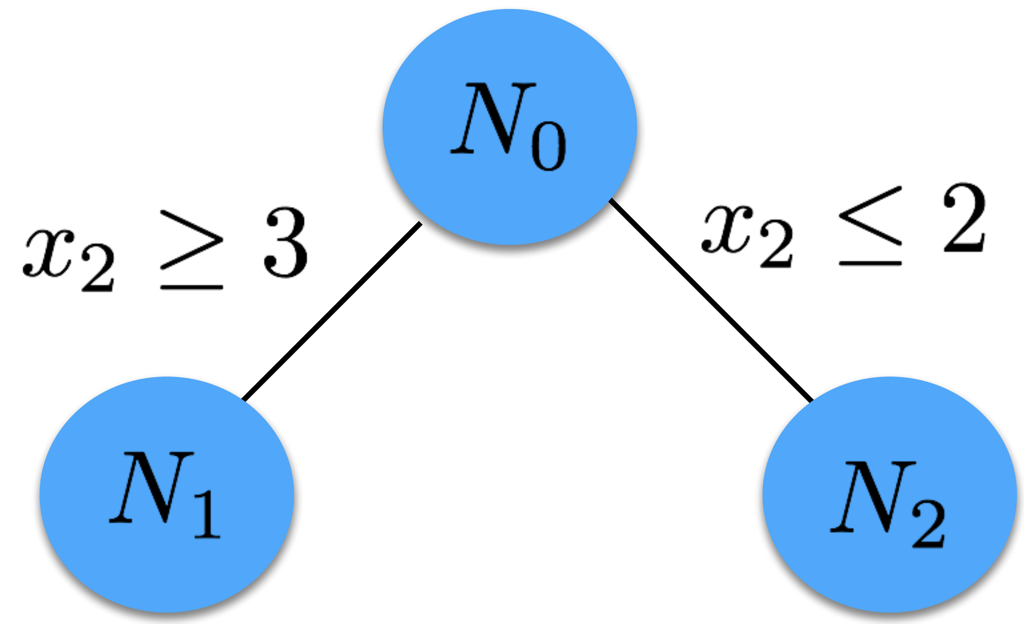

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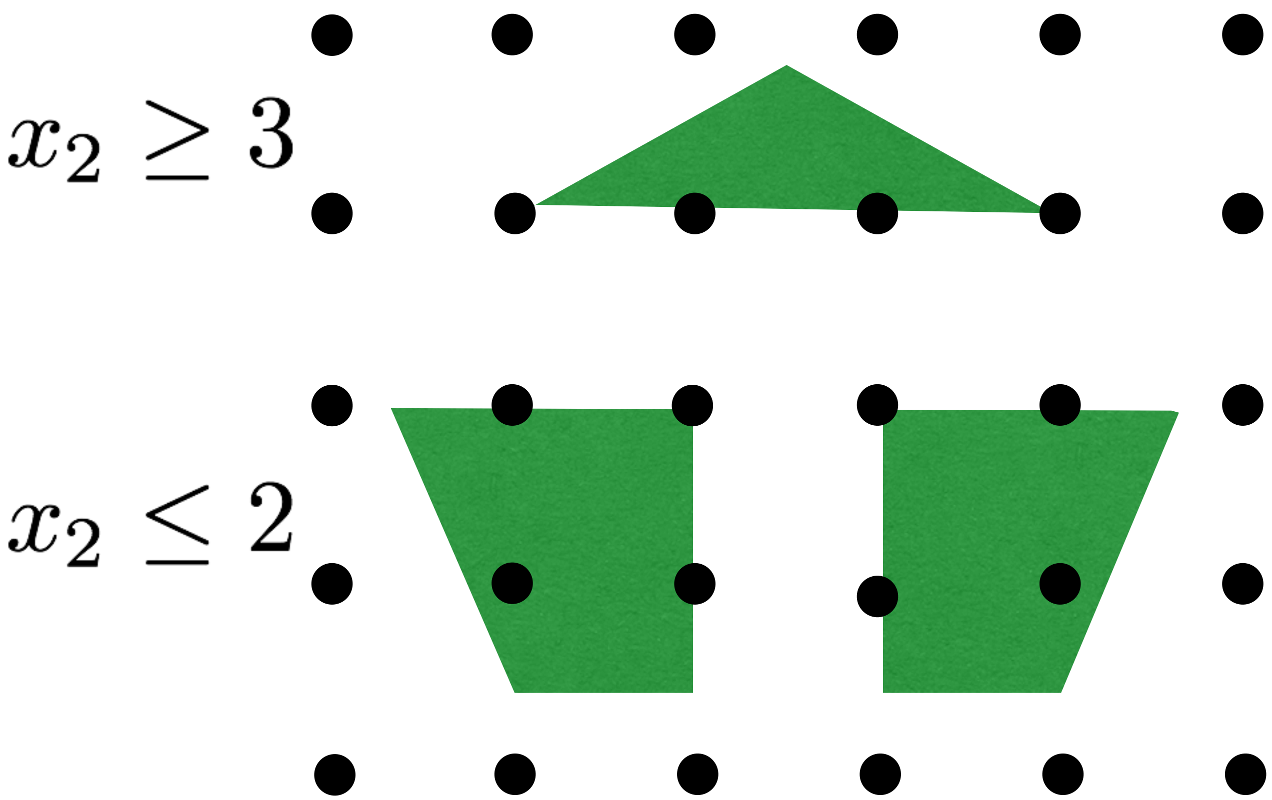

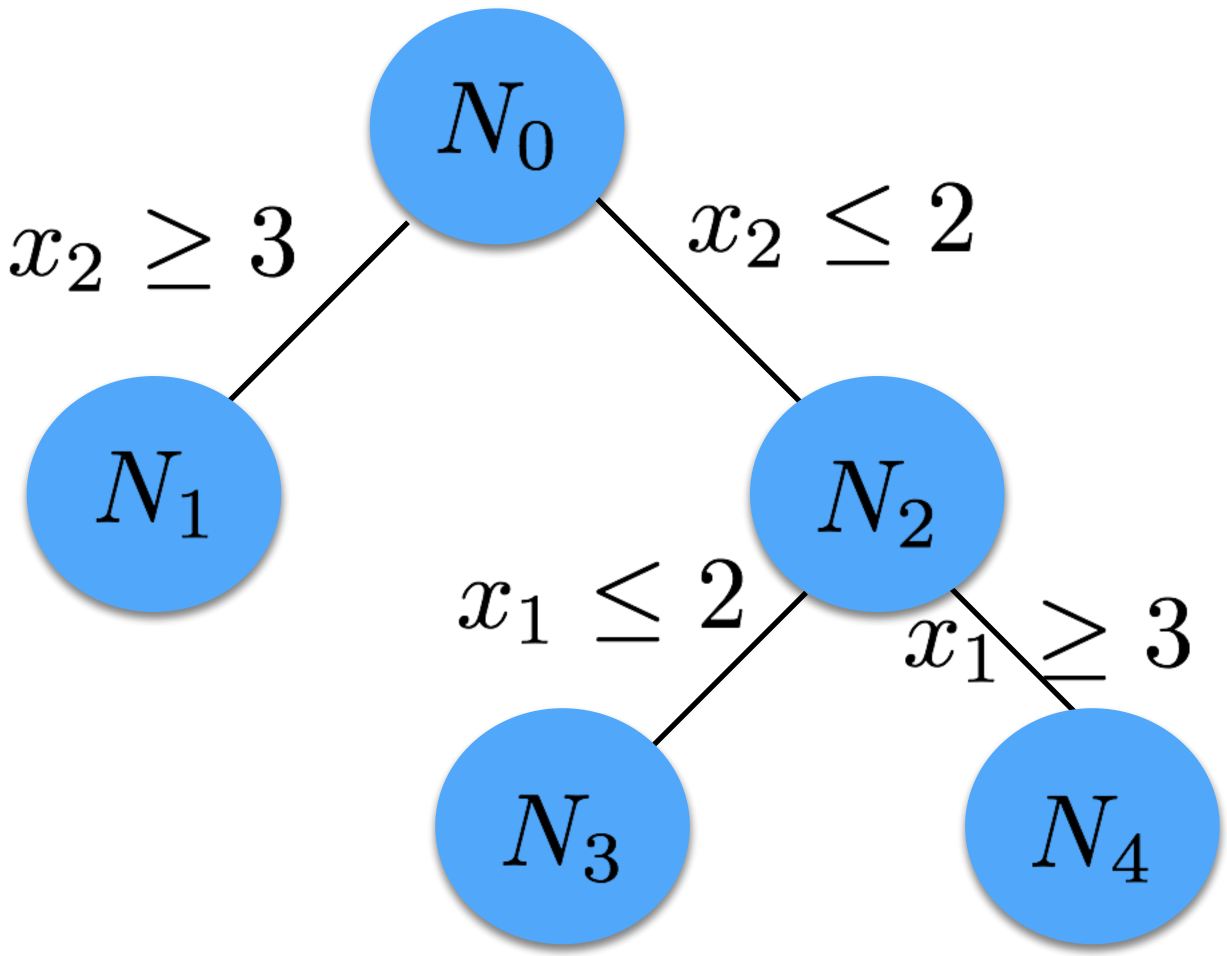

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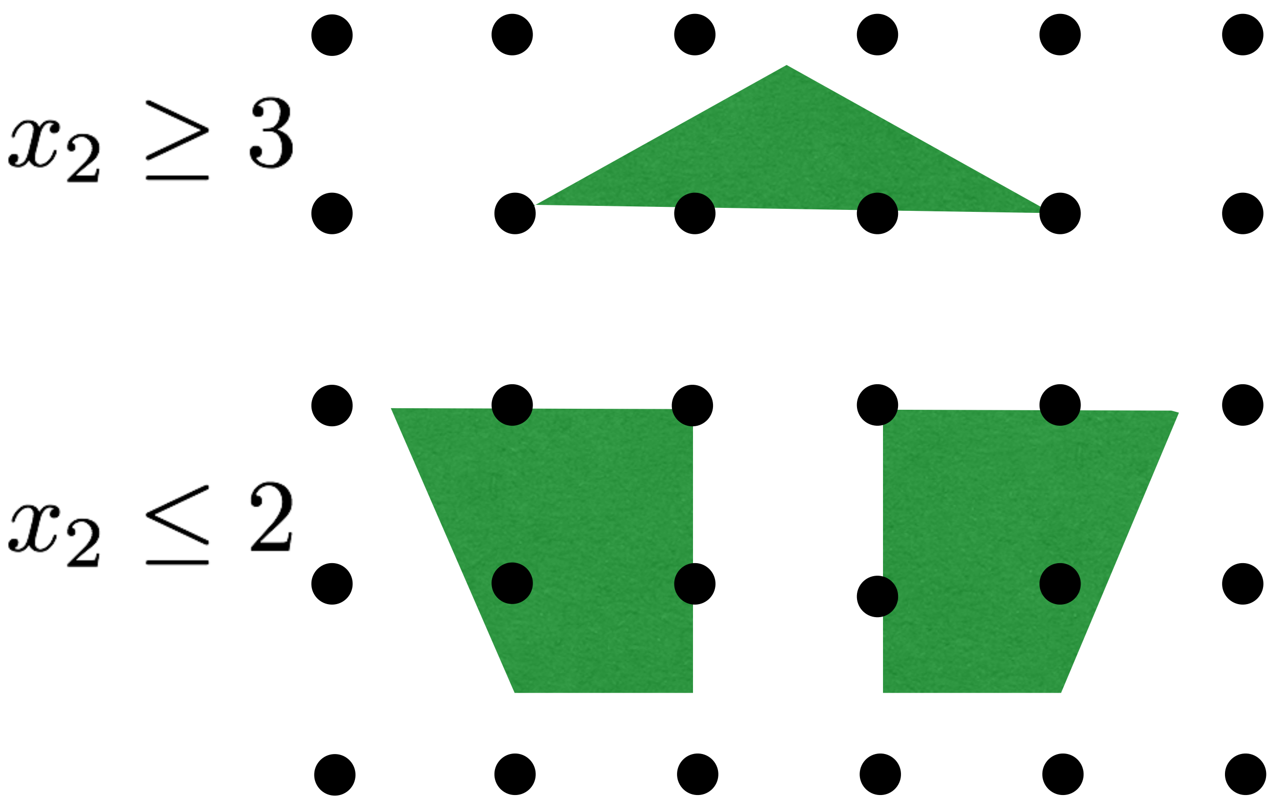
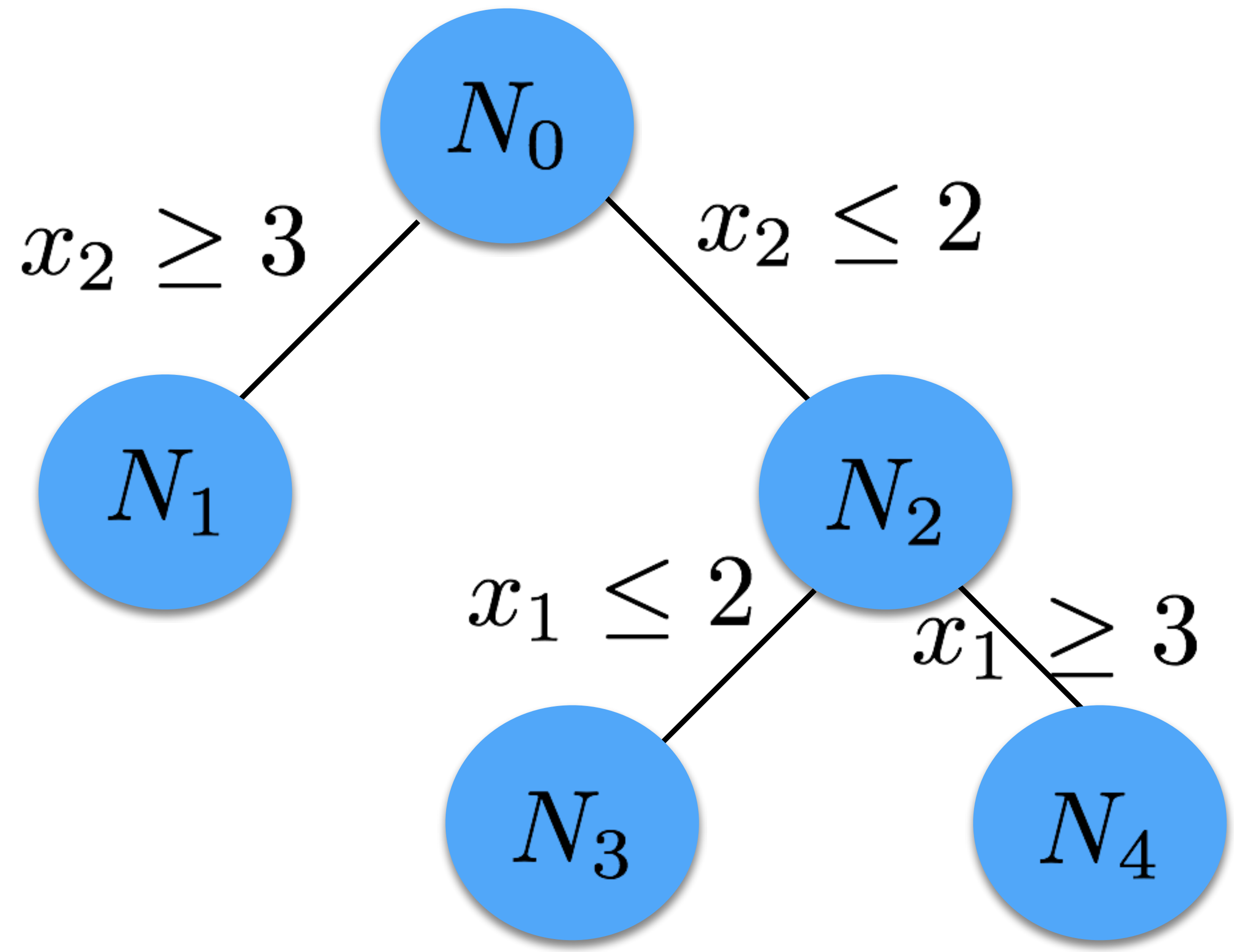




Given a branching tree $\mathcal{T}$,
the *size* of $\mathcal{T}$ is the number of leaves in the tree,
and the *domain* of $\mathcal{T}$ is the union of the leaf nodes.

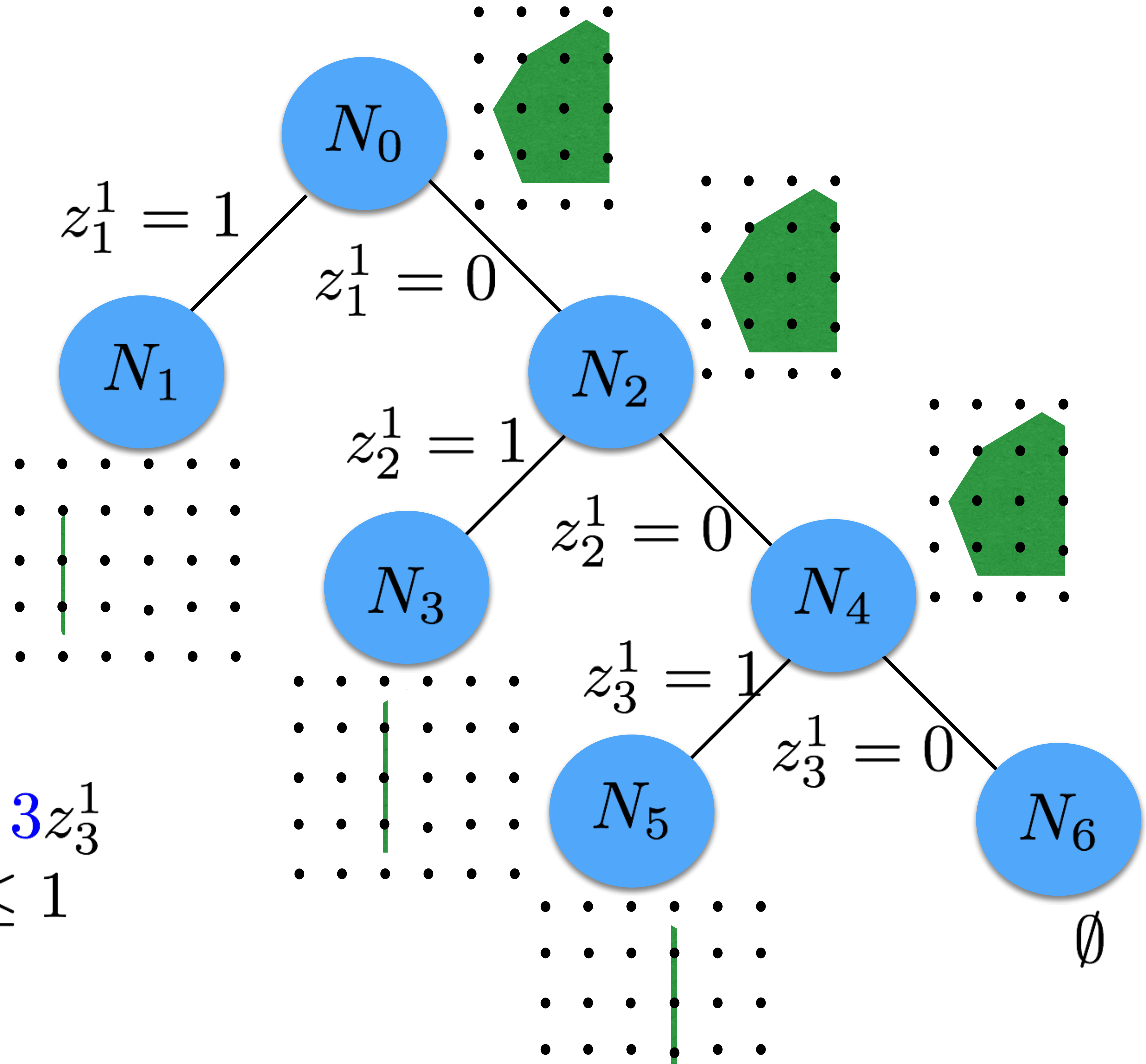
Branching on binary variables

Full

$$x = \sum_{j=1}^{u} j z_j$$

$$z_1 + z_2 + \cdots + z_u \leq 1$$

$$z \in \{0,1\}^u$$



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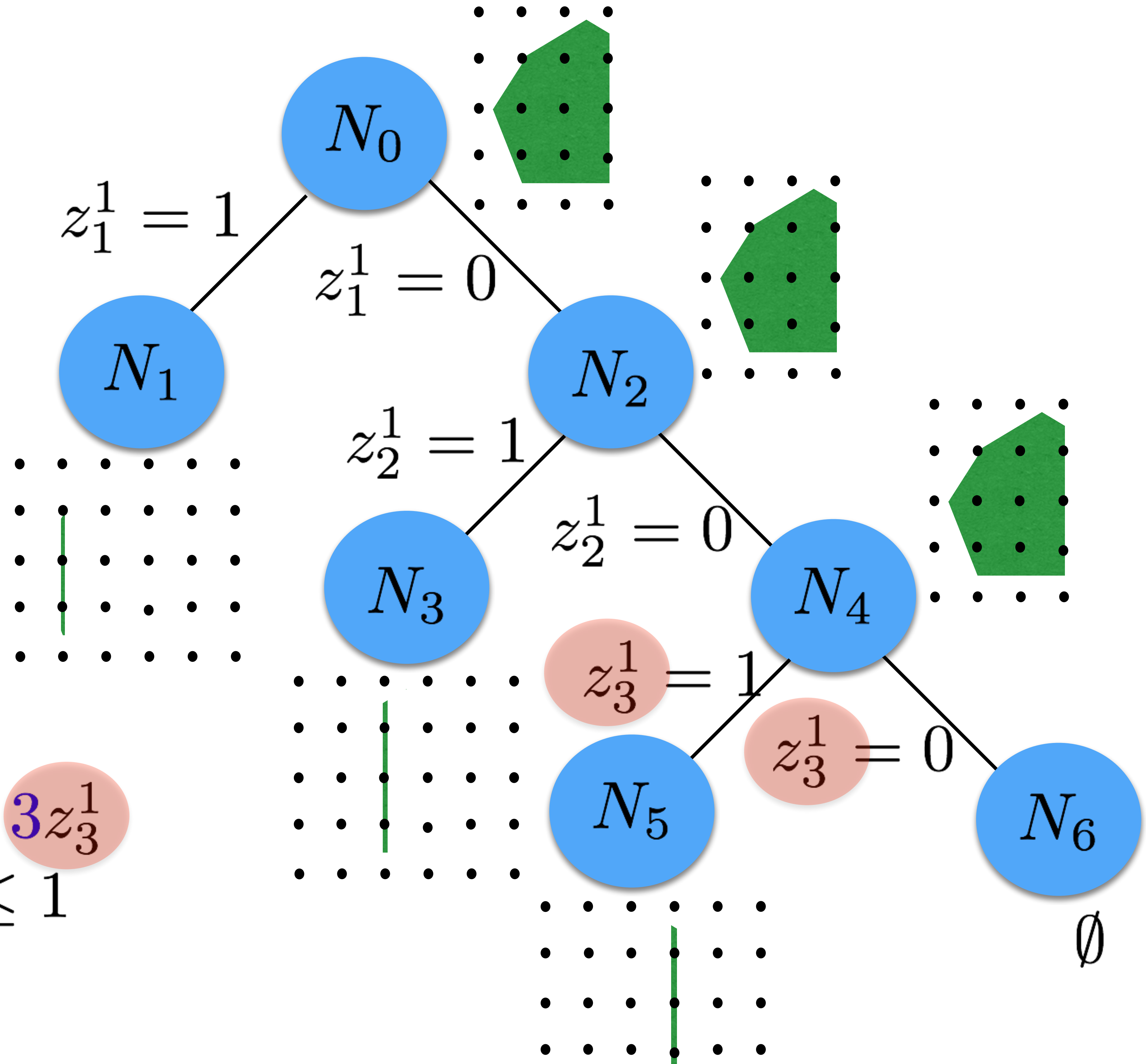

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Domain of Alternative Bin.

Alt

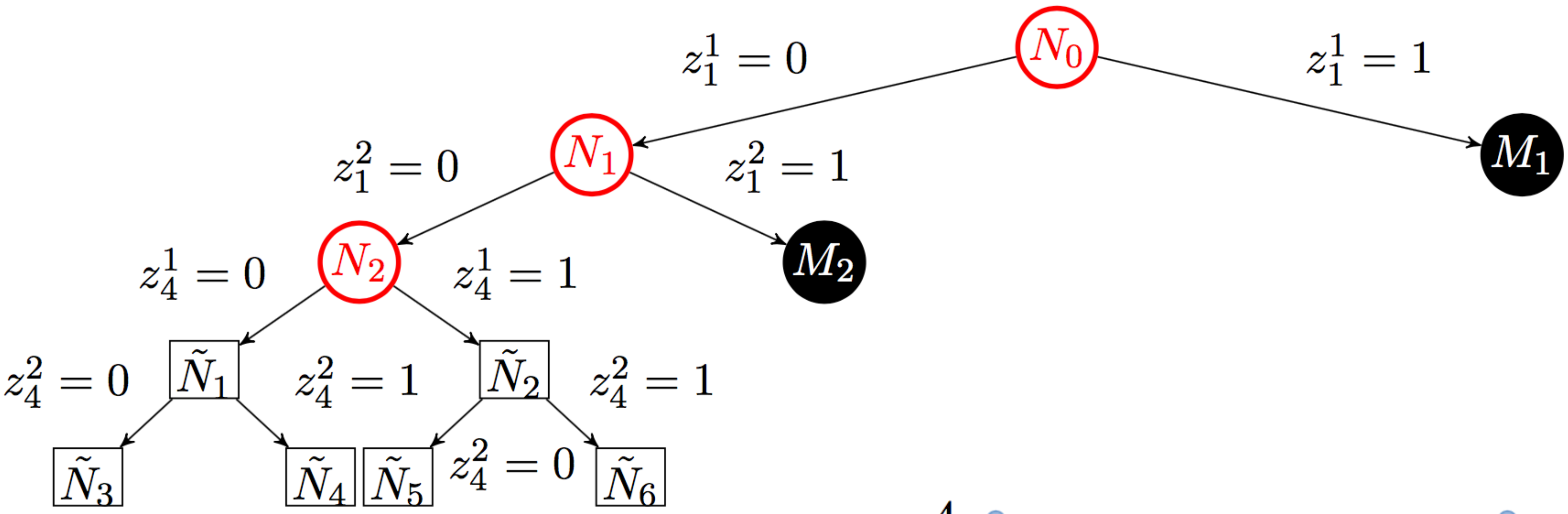
$$x = 4z_4 + \sum_{j=1}^{3} z_j$$

$$0 \leq z_3 \leq z_2 \leq z_1$$

$$z_1 + z_4 \leq 1$$

$$z \in \{0,1\}^4$$

Domain of Alternative Bin.



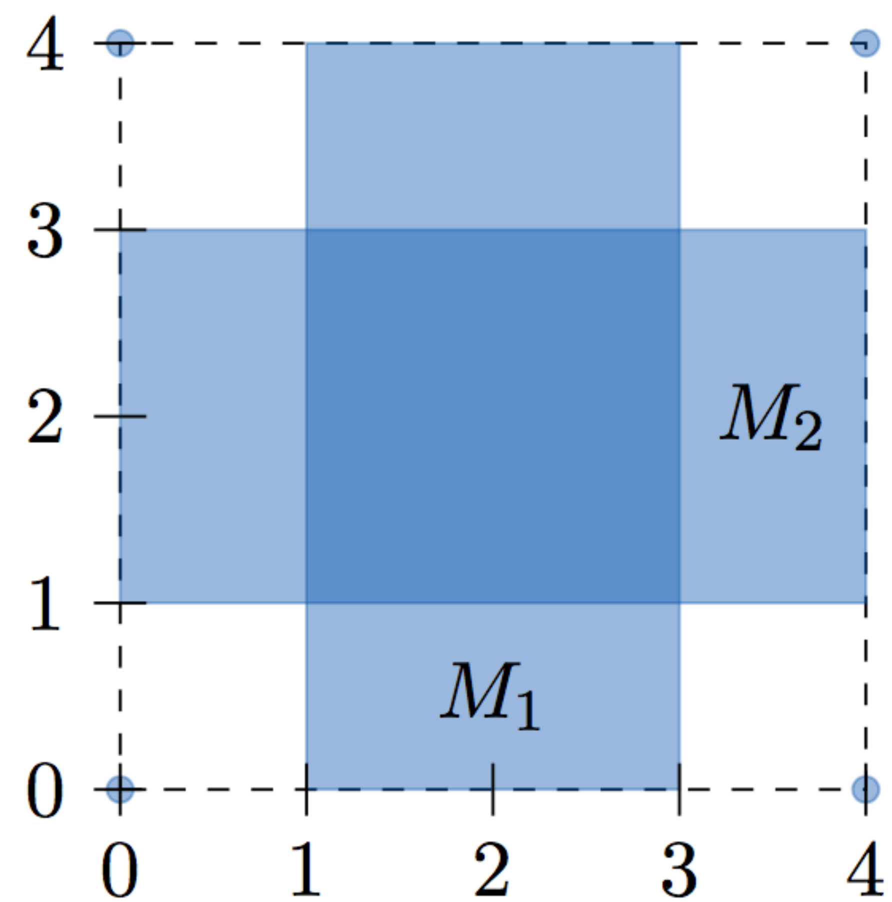
Alt

$$x = 4z_4 + \sum_{j=1}^{3} z_j$$

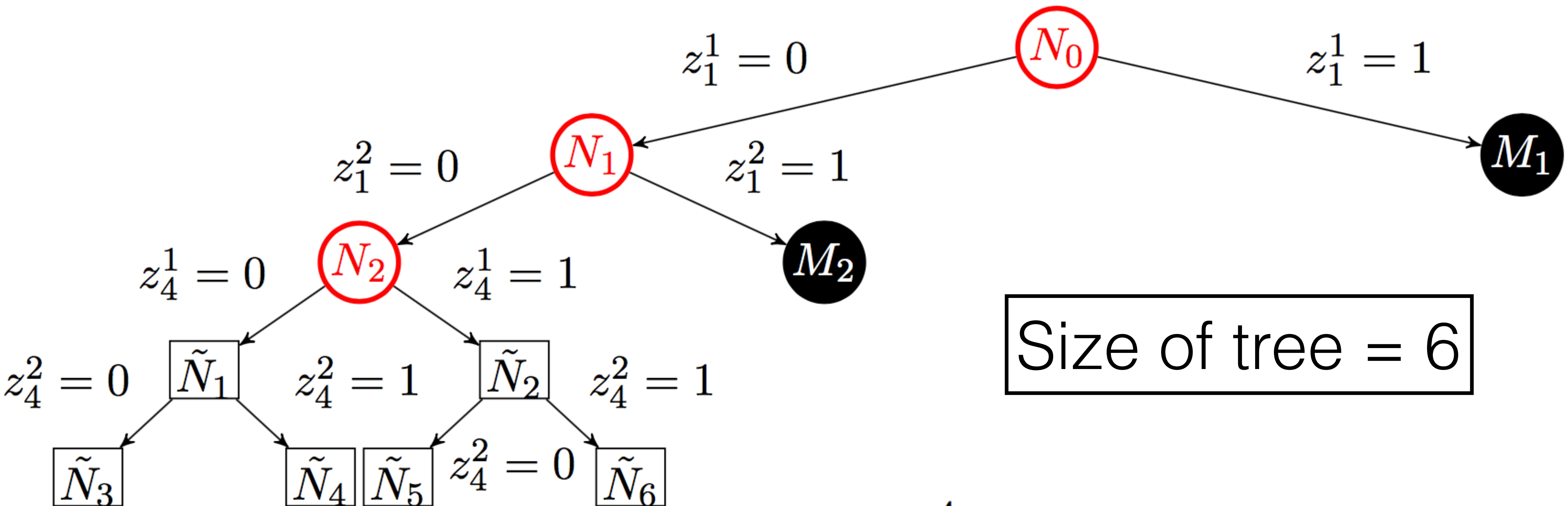
$$0 \leq z_3 \leq z_2 \leq z_1$$

$$z_1 + z_4 \leq 1$$

$$z \in \{0,1\}^4$$



Domain of Alternative Bin.



Size of tree = 6

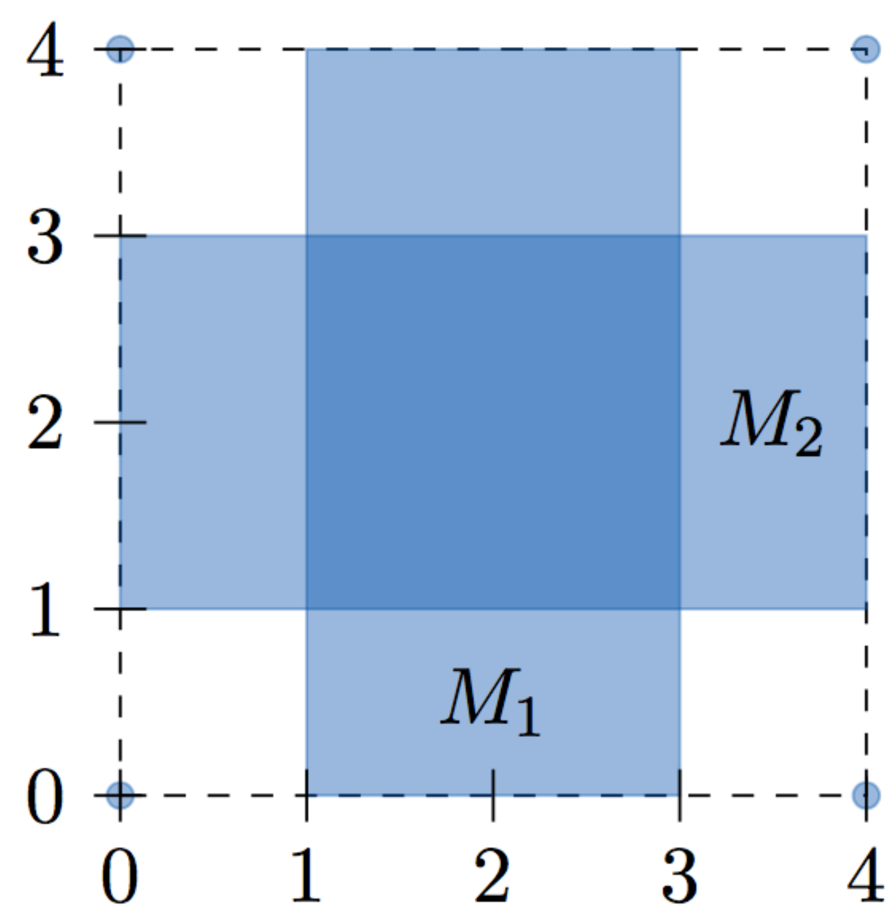
Alt

$$x = 4z_4 + \sum_{j=1}^{3} z_j$$

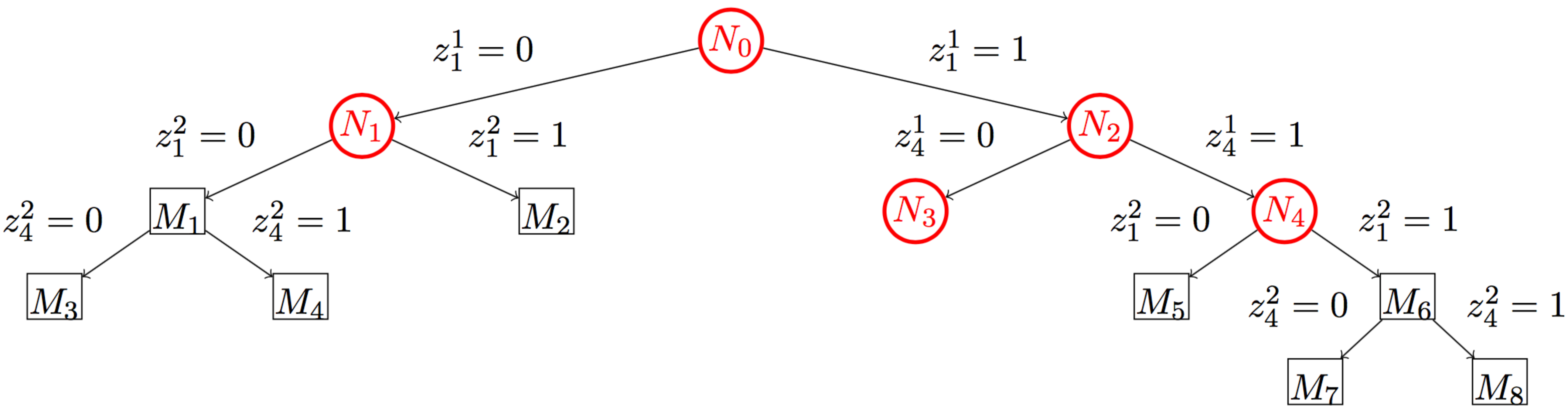
$$0 \leq z_3 \leq z_2 \leq z_1$$

$$z_1 + z_4 \leq 1$$

$$z \in \{0,1\}^4$$



Domain of unary tree

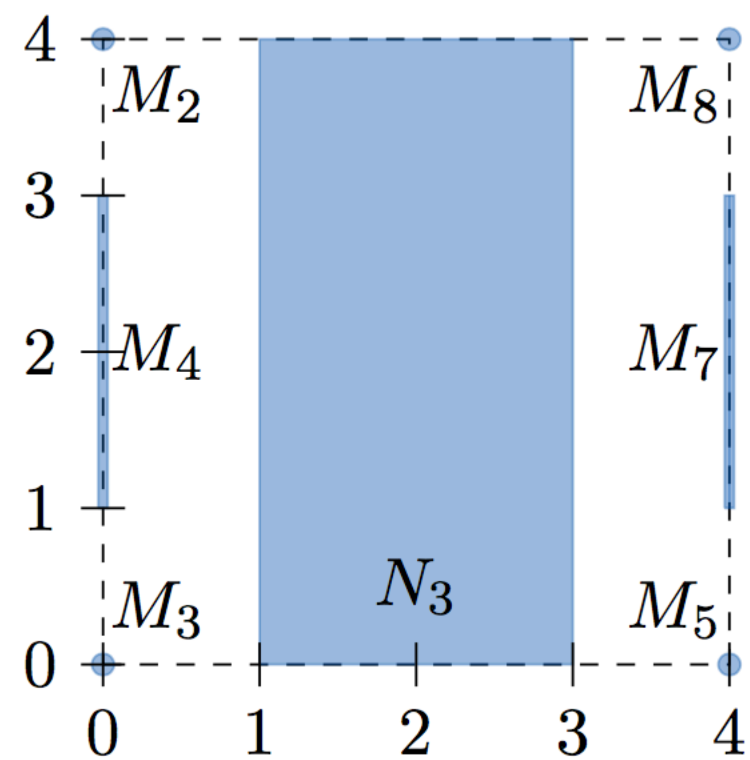


Unary

$$x = \sum_{j=1}^{u} z_j$$

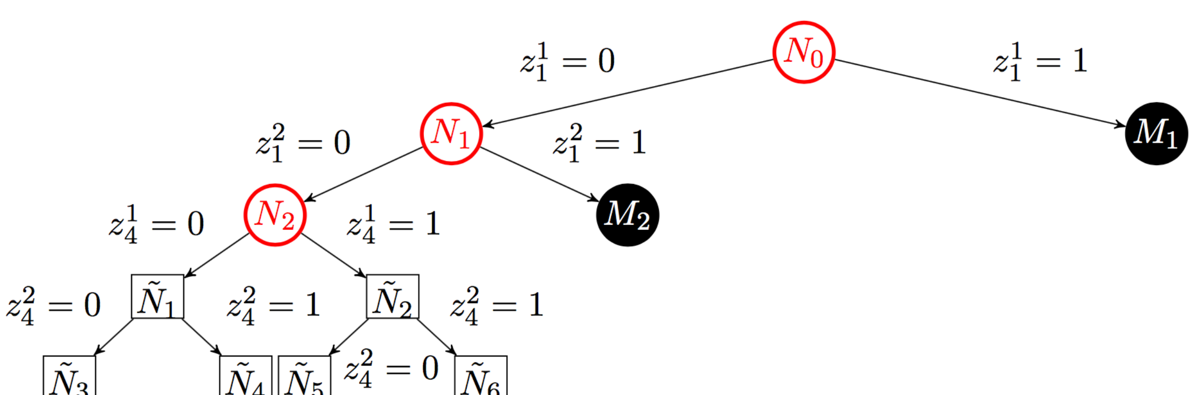
$$z_1 \geq z_2 \geq \ldots \geq z_u$$

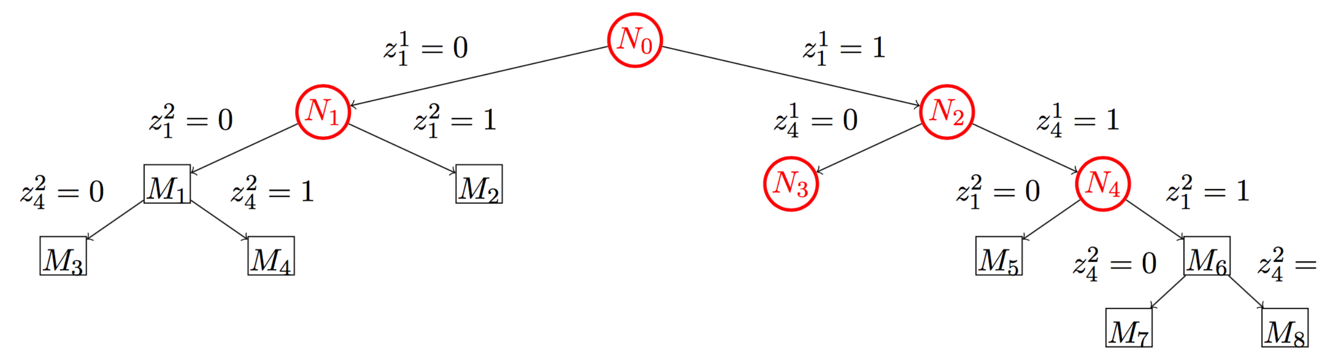
$$z \in \{0,1\}^u$$



Improving domain

Theorem: There exists a branching tree $\mathcal{T}_{\text{ALT}}$ of size $2^n + n$ with domain D_{ALT} such that for any tree $\mathcal{T}_{\text{unary}}$ with domain $D_{\text{unary}} \subseteq D_{\text{ALT}}$ must have size at least $2 \cdot 2^n - 1$.





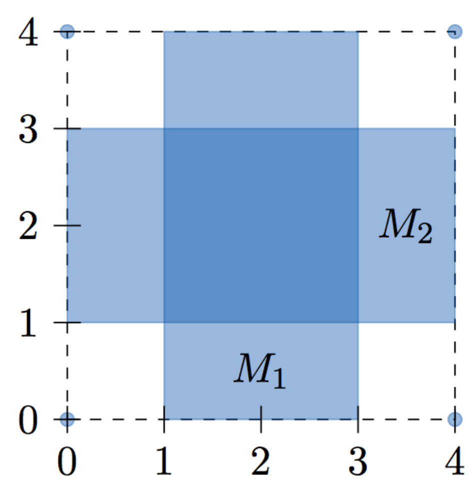
Alt

$$x = 4z_4 + \sum_{j=1}^{3} z_j$$

$$0 \leq z_3 \leq z_2 \leq z_1$$

$$z_1 + z_4 \leq 1$$

$$z \in \{0,1\}^4$$

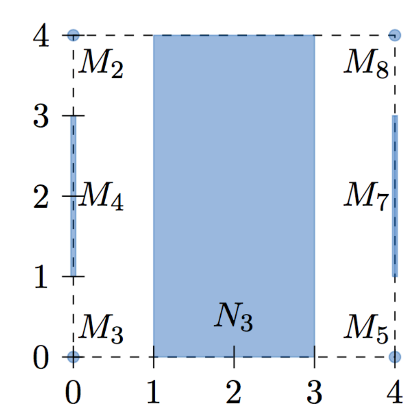


Unar

$$x = \sum_{j=1}^{u} z_j$$

$$z_1 \geq z_2 \geq \ldots \geq z_u$$

$$z \in \{0,1\}^u$$



Conclusions

(or at least, things to consider)

#1. If you can afford it, use full

Strong

Original

$$x \in \mathbb{Z}$$

Log

$$x = \sum_{j=0}^{\lfloor \log(u+1) \rfloor} 2^j z_j$$

$$z \in \{0,1\}^{\lfloor \log(u+1) \rfloor}$$

Full

$$x = \sum_{j=1}^{u} j z_j$$

$$z_1 + z_2 + \cdots + z_u \leq 1$$

$$z \in \{0,1\}^u$$

Unary

$$x = \sum_{j=1}^{u} z_j$$

$$z_1 \geq z_2 \geq \ldots \geq z_u$$

$$z \in \{0,1\}^u$$

#2. SUBSTITUTE!

$$\begin{aligned} \min \quad & p^\top x + q^\top y \\ & \sum_i x_{ij} = d_j \quad \forall j \\ & \sum_j x_{ij} \leq c_i \quad \forall i \\ & y_{ij} \leq x_{ij} \leq a_{ij} y_{ij} \\ & x_{ij} = \sum_{k=1}^{a_{ij}} k z_{i,j,k}, \\ & \sum_k z_{i,j,k} \leq 1 \\ & z_{i,j,k} \in \{0,1\} \ \forall \ k \\ & y_{ij} \in \{0,1\} \end{aligned}$$

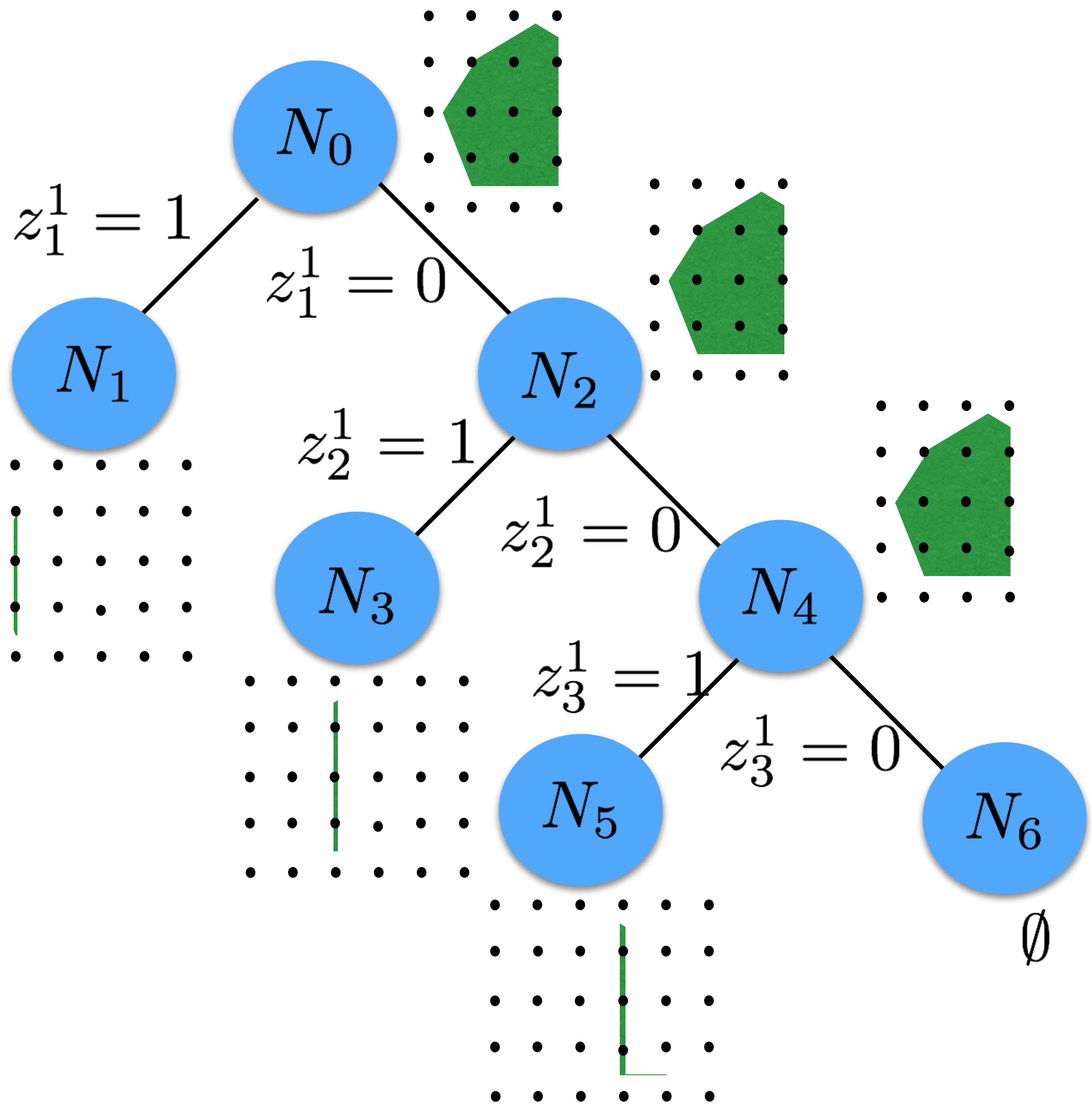
Substitute

$$\begin{aligned} \min \quad & \hat{p}^\top z + q^\top y \\ & \sum_i \sum_{k=1}^{a_{ij}} k z_{i,j,k} = d_j \quad \forall j \\ & \sum_j \sum_{k=1}^{a_{ij}} k z_{i,j,k} \leq c_i \quad \forall i \\ & y_{ij} \leq \sum_{k=1}^{a_{ij}} k z_{i,j,k} \leq a_{ij} y_{ij} \\ & \sum_k z_{i,j,k} \leq 1 \\ & z_{i,j,k} \in \{0,1\} \ \forall \ k \\ & y_{ij} \in \{0,1\} \end{aligned}$$

#3. BRANCHING

branch on original variables

NOT on binary variables



Reference

S. Dash, R. Hildebrand and O. Günlük. Binary extended formulations of polyhedral mixed-integer sets. Mathematical Programming 170 (1), 207-236. 2018,

https://sites.google.com/site/robertdhildebrand/